A CONNECTEDNESS-TYPE PROPERTY OF CLOSED SETS

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Abstract. In this article we characterize closed sets with a connectedness property and apply their properties in olympiad problems. Most of the problems also have different solutions, but the following results lead to more straightforward ones, using similar ideas throughout their proofs.

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Introduction

For a subset $A \subseteq \mathbb{R}$ denote by A' the set of all its real limit points, i.e. the set of real numbers x for which there exists a sequence $(x_n)_{n\geqslant 1}$ of numbers from A such that $\lim\limits_{n\to\infty} x_n = x$. The set A is called *closed* if $A' \subseteq A$. For instance, any interval $[a, b]$ is closed.

Whenever dealing with continuous functions, it is natural to consider the set A', where A is a set that is easy to work with. A well-known and very important result that we will use throughout this paper is that the intersection of two closed sets is also closed. Moreover, we will use the infimum and supremum of a set. It is easy to prove that $\inf A$ and $\sup A$ belong to A', provided that they are real numbers.

Next, we will present the statements of the main results and prove them, followed by some examples of olympiad problems.

The two main properties

Lemma 1 (bounded version). Consider a nonempty bounded set $M \subseteq \mathbb{R}$ with the following two properties:

i) M is closed;

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ii) For each two distinct numbers $x, y \in M$ there exists $\lambda \in M$ such that $x < \lambda < y$.

Then M is a closed interval $[a, b]$ of $\mathbb{R}$.

Proof. Consider $a = \inf M$ and $b = \sup M$. Since M is bounded, then a and b are real numbers and $a \leqslant b$. We will show that $M = [a, b]$.

First, since M is closed then $a, b \in M$, then $M \subseteq [a, b]$.

Next, consider an arbitrary c from $[a, b]$ and the following two sets:

$$U = [a, c] \cap M, \qquad V = [c, b] \cap M.$$

We will show that $c \in M$. Note that $a \in U$ and $b \in V$, hence U and V are not empty sets.

If $c = a$ or $c = b$, then $c \in M$, hence we can assume the contrary.

Consider $\sup U =: \alpha \leqslant c \leqslant \beta := \inf V$. Since M is closed, then both U and V are closed, as each of them is the intersection of two closed sets. Then $\alpha \in U$ and $\beta \in V$.

If $c = \alpha$, then, since U is closed, $c \in U$, hence $c \in M$. Similarly, if $c = \beta$, then $c \in M$.

Finally, assume that $\alpha < c < \beta$. Using property (ii) of M, there exists $\lambda \in M$ such that $\alpha < \lambda < \beta$.

If $c < \lambda$, then $c < \lambda < \beta$, hence λ belongs to V and is smaller than $\inf V$, which is a contradiction. Similarly, if $c > \lambda$ we get a contradiction with the definition of $\alpha = \sup U$. Thus, the only possible case is when $\lambda = c$, so $c \in M$.

Since c was randomly chosen from $[a, b]$, $[a, b] \subseteq M \subseteq [a, b]$ and the lemma is proved. $\square$

Lemma 2 (unbounded version). Consider an unbounded set $M \subseteq \mathbb{R}$ with the following two properties:

i) M is closed;

ii) For each two distinct numbers $x, y \in M$ there exists $\lambda \in M$ such that $x < \lambda < y$.

Then M is of the form $[a, \infty)$, or $(-\infty, b]$, or $M = \mathbb{R}$.

Proof. Assume that M is unbounded from above and bounded from below. Consider $\inf M = a \in M$ to infer that $M \subseteq [a, \infty)$. Since M is unbounded from above, there exist a sequence $(x_n)_{n \geqslant 1}$ of numbers from M such that $x_n \geqslant n$.

Consider the sets $M_n = [a, x_n] \cap M$, where $n \geqslant a$.

Since $M_n \subseteq M$ and M_n is the intersection of two closed sets, then M_n has both properties (i) and (ii), hence we can use Lemma 1 to infer that $M_n = [a, x_n]$, for all $n \geqslant a$. Consider an arbitrary c greater than a and $p = \lceil c \rceil$.

Thus, since $x_p \geqslant p \geqslant c$, $[a, c] \subseteq [a, p] \subseteq [a, x_p] = M_p \subseteq M$, hence $c \in M$. Since c can be any number greater than a, then $[a, \infty) \subseteq M \subseteq [a, \infty)$ and the conclusion follows.

In a similar fashion we can handle the case when M is bounded from above and unbounded from below.

The third case, when M is unbounded in both directions, follows by a similar reasoning as above by choosing two sequences of numbers from M, say $(x_n)_{n\geqslant 1}$ and $(y_n)_{n\geqslant 1}$ such that $x_n \geqslant n$ and $y_n \leqslant -n$ and using again Lemma 1 to infer that $M_n = [y_n, x_n] \cap M = [y_n, x_n]$. Since x_n and y_n can be simultaneously arbitrarily large and arbitrarily small, respectively, and $M_n \subseteq M$, the lemma is now proved.

Olympiad Problems

Now, we are ready to show how to use these two properties for contest-type problems. The main trick for each problem is to choose a set for which one of the two lemmas from above can be applied.

1. (Mihai Piticari, Romanian Mathematical Olympiad 2005, District Round) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for any $a, b \in \mathbb{R}$, with $a < b$ and $f(a) = f(b)$, there exists some $c \in (a, b)$ such that $f(a) = f(b) = f(c)$. Prove that f is monotonic over $\mathbb{R}$.

Solution. Assume that f is not monotonic. Then it is easy to prove that there exist $a < b < c$ such that

$$f(a) < f(b) \text{ and } f(b) > f(c)$$

or

$$f(a) > f(b) \text{ and } f(b) < f(c).$$

Assume we are in the first case, as the second one can be treated in a similar way. Choose λ such that

$$f(b) < \lambda < \max\{f(a), f(c)\}$$

Since f is continuous, it follows from the intermediate value theorem that there exist x, y such that $a < x < b < y < c$ and $f(x) = f(y) = \lambda$. Hence, if we consider the set

$$M = \{t \mid t \in [x, y] \text{ and } f(t) = \lambda\},$$

then $x, y \in M$ and M is clearly bounded. From the continuity of f it follows that M is closed. Moreover, from the problem hypothesis, it is clear that for each $u < v$ from M there exists $w \in (u, v)$ such that $w \in M$.

So we can apply Lemma 1, hence M is a closed interval. Since $x, y \in M$ and $x < b < y$, then $b \in M$, which is a contradiction since $f(b) < \lambda$. The conclusion follows. $\square$

2. (Marius Cavachi, Romanian Mathematical Olympiad 2020, District Round, 2020) Determine the continuous functions $f : \mathbb{R} \to \mathbb{R}$ having the property that, for all $x, y \in \mathbb{R}$, there exists $t \in (0, 1)$ such that

$$f((1-t)x + ty) = (1-t)f(x) + tf(y).$$

Solution. We will prove that only affine functions work.

First of all, it is easy to see that they do satisfy the condition.

Conversely, consider two real numbers $a < b$. We will take M as the set of points $t \in [a,b]$ such that the point $(t, f(t))$ lies on the line determined by $(a, f(a))$ and $(b, f(b))$. Formally speaking, consider the set

$$M = \bigl\{x \in [a,b] \mid \bigl(f(x) - f(a)\bigr)(x-b) = \bigl(f(x) - f(b)\bigr)(x-a)\bigr\}.$$

It is clear that M is bounded, $a, b \in M$ and M is closed since f is continuous. Moreover, the hypothesis implies that between any two numbers u and v from M there exists $w = (1-t)u + tv$, $t \in (0,1)$, so that $f(w) = (1-t)f(u) + tf(v)$, which can be rewritten as

$$\frac{f(w) - f(u)}{w-u} = \frac{f(w) - f(v)}{w-v}$$

or, equivalently, $w \in M$. Thus M has also property (ii) so, from Lemma 1, $M = [a,b]$, hence f is affine on the whole interval $[a,b]$. This shows that f is affine on any closed interval.

In order to finish the solution, assume that the graph of f is not a line. Hence, there exist three real numbers $x < y < z$ such that the points of coordinates $(x, f(x))$, $(y, f(y))$, and $(z, f(z))$ are not collinear. This contradicts the fact that f is affine on $[x,z]$ and concludes the solution. $\square$

3. (Nicolae Bourbăcuţ) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for every $x, y \in \mathbb{R}$, $x < y$, there exists $z \in (x,y)$ so that

$$(y-x)f(z) \leqslant (y-z)f(x) + (z-x)f(y).$$

a) Give an example of a non-convex function f with the given property.

b) Prove that a continuous function with the given property is convex.

Solution. We will say that a function which fulfills the condition from the statement has property $\mathcal{P}$.

a) Consider the real function

$$f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$$

Clearly this function works, since any $z \in (x,y)$, with $z \neq 0$ satisfies

$$(y-x)f(z) = (y-x) \cdot 0 = 0 \leqslant (y-z)f(x) + (z-x)f(y).$$

Finally, note than f is not convex since $f(0) > \frac{f(1)+f(-1)}{2}$.

b) Suppose that f is not a convex function. Then, there exist three real numbers x, y, λ, with $x < y$ and $\lambda \in (0,1)$, such that

$$f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y).$$

Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined as $g(t) = f(t) - mt - n$, where the coefficients m and n are taken so that $g(x) = g(y) = 0$, that is $m = \frac{f(y)-f(x)}{y-x}$

and $n = \frac{yf(x)-xf(y)}{y-x}$. Define the set

$$M = \{t \in [x, y] \mid g(t) \leqslant 0\}$$

and note that $x, y \in M$, since $g(x) = g(y) = 0$.

Since f is continuous, then g is also continuous. Moreover, it is easy to check that g has property $\mathcal{P}$, since the affine function is canceled out from both sides. From the continuity of g, it follows that M is closed. Choose $u < v$ from M. Since g has property $\mathcal{P}$, then there exists $w \in (u, v)$ such that

$$(v-u)g(w) \leqslant (v-w)g(u) + (w-u)g(v).$$

Since $v > u$ and the right hand side term is nonpositive, then $w \in M$, which shows that M has property (ii). As M is clearly bounded, we can apply Lemma 1 to infer that $M = [x, y]$.

So, for every $z \in (x, y)$, $f(z) \leqslant mz + n$. Substituting the values of m, n in this inequality easily leads to

$$(y-x)f(z) \leqslant (y-z)f(x) + (z-x)f(y).$$

In particular, for $z = \lambda x + (1-\lambda)y$ this yields

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y),$$

which is a contradiction with the initial assumption on x, y and λ. This proves that f must be convex on $\mathbb{R}$. □

4. (after a problem of Dan Marinescu) A function $f : \mathbb{R} \to \mathbb{R}$ is said to have property $\mathcal{P}$ if for every $a, b \in \mathbb{R}$ there exists $c \in (a, b)$ such that $f(c) \in \{f(a), f(b)\}$, where the brackets can also denote a multiset.

a) Give an example of a non-constant function possessing $\mathcal{P}$.

b) If f has $\mathcal{P}$ and is continuous, prove that it is constant.

Solution. a) The Dirichlet function

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is not constant and property $\mathcal{P}$ follows from the density of both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$.

b) Assume that f is not constant and pick real numbers $a < b$, such that $f(a) \neq f(b)$. Define the set

$$M = \{x \in [a, b] \mid f(x) = f(a) \text{ or } f(x) = f(b)\}.$$

It is clear that $a, b \in M$. Choose $u < v$ from M. Since f has property P, then there exists $w \in (u, v)$ such that $f(w) = f(u)$ or $f(w) = f(v)$. Since $u, v \in M$ then

$$f(u), f(v) \in \{f(a), f(b)\} \Rightarrow f(w) \in \{f(a), f(b)\}$$

hence $w \in M$.

Next, we proceed to show that M is closed. Consider a sequence $(x_n)_{n\geqslant 1}$ of real numbers from M such that $\lim\limits_{n\to\infty} x_n = x$. Since f is continuous, then $\lim\limits_{n\to\infty} f(x_n) = f(x)$. Since $f(x_n)$ can take only two values, that is $f(a)$ and $f(b)$, and the sequence is convergent, then it must become eventually stationary. Thus, the limit $f(x)$ is either $f(a)$ or $f(b)$, therefore $x \in M$.

Hence, we can apply Lemma 1 to infer that $M = [a, b]$ i.e.

$$f\big([a,b]\big) \subseteq \{f(a), f(b)\}.$$

Since f is continuous, the image of f is an interval. From the last relation it is clear that the interval must be reduced to one point, which means that $f(a) = f(b)$, a contradiction.

Further study

We invite the reader to solve the following problems with the help of the aforementioned properties and ideas.

Problem 1. (Nicolae Bourbăcuț, Romanian Mathematical Olympiad 2012, District Round) We will say that a function $f : \mathbb{R} \to \mathbb{R}$ has property $\mathcal{F}$ if for each real number a there exists $b < a$ such that $f(x) \leqslant f(a)$, for all $x \in (b, a)$.

a) Give an example of a function with property $\mathcal{F}$ that is not monotonic on $\mathbb{R}$.

b) Prove that a continuous function that has property $\mathcal{F}$ is nondecreasing.

Problem 2. (Radu Gologan) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that, on each non degenerated interval I, the function reaches its maximum or its minimum in an interior point of I. Prove that f is a constant.

Problem 3. (Dorin Andrica and Mihai Piticari, Romanian Mathematical Olympiad 2007) A $P-$*function* is a differentiable function $f : \mathbb{R} \to \mathbb{R}$ with a continuous derivative on $\mathbb{R}$, such that $f(x + f'(x)) = f(x)$ for all x in $\mathbb{R}$.

a) Prove that the derivative of a $P-$function has at least one zero.

b) Provide an example of a non-constant $P-$function.

c) Prove that a $P-$function whose derivative has at least two distinct zeros is constant.

References

[1] The Romanian Mathematical Competitions Collection 2004-2020.