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AN AREA LOWER BOUND FOR CONVEX POLYGONS WITH LARGE PERIPHERAL TRIANGLES

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Abstract. This note gives a lower bound for the area of a convex polygon whose 'frontier' triangles (triangles spanned by three consecutive vertices) have area at least 1.

Keywords: convex polygon, area, lower bound

MSC : 51M25

Question 4 of the 2021 Stars of Mathematics Competition, Senior Grade, asked one to show that, if every three consecutive vertices of a convex n-gon span a triangle of area at least 1, then the area of the n-gon is (strictly) greater than $\frac{1}{4}n\log_2 n - \frac{1}{2}$.

Our purpose here is to improve this order $n\log_2 n$ area lower bound to one of order $n^{\log_3 4}$ for all but finitely many positive integers n.

For convenience, a convex n-gon every three consecutive vertices of which span a triangle of area at least 1 will be referred to as a *suitable* n-gon. This area condition is vacuously true if $n = 2$, so segments are suitable 2-gons of area zero.

Claim. *The area of a suitable n-gon is greater than*

$$\frac{5}{32}n^{\log_3 4}$$

for all but finitely many positive integers n.

Moreover, the lower bound

$$\frac{1}{8}\left(n - \frac{3}{2}\right)^{\log_3 4}$$

holds for all integers $n \geqslant 3$.

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The argument hinges on the lemma below.

Lemma. *For any non-negative integer k, the area of a suitable n-gon is at least*

$$\left(\frac{4}{3}\right)^k \left(n - \frac{3^k+3}{2}\right).$$

Remark. Only finitely many of these area lower bounds are non-negative, namely, those in the range of non-negative integers $k \leqslant \log_3(2n-3)$; to be non-empty, this requires $n \geqslant 2$. Furthermore, an inspection of the derivative shows that these area lower bounds increase in the range of non-negative integers $k \leqslant \log_3(2n-3) - 2$; to be non-empty, this requires $n \geqslant 6$.

Assume the lemma for the moment to prove the claim. To deal with the first statement in the claim, write $\alpha = \log_3 4$ and $c_0 = \frac{1}{8}\left(\frac{2}{\alpha}\right)^\alpha (\alpha-1)^{\alpha-1}$.

We first show that, given any positive real constant $c < c_0$, the area of a suitable n-gon is (strictly) greater than cn^α for all but finitely many positive integers n. Then we show that $c = \frac{5}{32} < c_0$ and the conclusion follows.

Consider a positive real number $a < 2$, to be chosen later on, and a positive real number $c < \frac{1}{8}a^{\alpha-1}(2-a)$. Consider further an integer $n \geqslant \max\left(3, \frac{1}{a}\right)$, satisfying $\lfloor \log_3(an) \rfloor \geqslant 1 - \log_3\left(\frac{2}{a} - 1 - \frac{8c}{a^\alpha}\right)$. Notice that $k =: \lfloor \log_3(an) \rfloor \geqslant 0$ and $\frac{1}{3^{k-1}} \leqslant \frac{2}{a} - 1 - \frac{8c}{a^\alpha}$. By the lemma, the area of a suitable n-gon is then at least

$$\left(\frac{4}{3}\right)^k \left(n - \frac{3^k+3}{2}\right) = 4^k\left(\frac{n}{3^k} - \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{3^{k-1}}\right)$$

$$\geqslant 4^k\left(\frac{1}{a} - \frac{1}{2} - \frac{1}{2}\left(\frac{2}{a} - 1 - \frac{8c}{a^\alpha}\right)\right) = 4^{k+1}\frac{c}{a^\alpha} = 3^{(k+1)\alpha}\frac{c}{a^\alpha} > (an)^\alpha \frac{c}{a^\alpha} = cn^\alpha.$$

To choose a, maximise $f(a) = \frac{1}{8}a^{\alpha-1}(2-a)$ over the open interval $(0, 2)$. It is easily seen that f is maximised at $a_0 = 2\left(1 - \frac{1}{\alpha}\right)$ alone, where it achieves the value c_0.

Consequently, given any positive real constant $c < c_0$, the area of a suitable n-gon is (strictly) greater than cn^α for all but finitely many positive integers n, namely, for all integers $n \geqslant \max\left(3, \frac{1}{a_0}\right) = 3$ satisfying $\lfloor \log_3(a_0 n) \rfloor \geqslant 1 - \log_3 \frac{8(c_0-c)}{a_0^\alpha}$.

To show that $c = \frac{5}{32} < c_0$, notice that f is strictly increasing on the half-open interval $(0, a_0]$. Since $a_0 > \frac{1}{3}$, it follows that $c = \frac{5}{32} = f\left(\frac{1}{3}\right) < f(a_0) = c_0$. This establishes the first statement in the claim.

To prove the second, consider an integer $n \geqslant 3$ and define the number $k = \left\lfloor \log_3\left(n - \frac{3}{2}\right) \right\rfloor$; clearly, k is a non-negative integer. With reference again to the lemma, the area of a suitable n-gon is at least

$$\left(\frac{4}{3}\right)^k \left(n - \frac{3^k+3}{2}\right) = 4^k \left(\frac{1}{3^k}\left(n-\frac{3}{2}\right) - \frac{1}{2}\right) \geqslant 4^k \left(1 - \frac{1}{2}\right)$$
$$= \frac{1}{8} \cdot 4^{k+1} = \frac{1}{8} \cdot \left(3^{k+1}\right)^{\log_3 4} > \frac{1}{8}\left(n - \frac{3}{2}\right)^{\log_3 4}.$$

This establishes the second statement and concludes the proof of the claim.

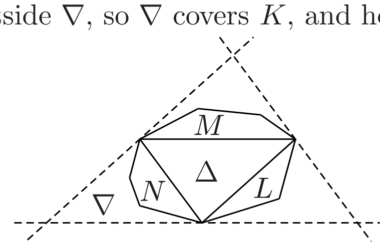
We now turn to prove the lemma. The area of a polygon K will be denoted by $[K]$. Before proceeding to the proof proper, notice that, if a and b are consecutive vertices of a suitable polygon, and c is any other vertex, then $[abc] \geqslant 1$. Indeed, if x and y are the other boundary neighbours of a and b, respectively, then $[abc] \geqslant \min([abx], [aby]) \geqslant 1$, by convexity.

Consequently, any diagonal splits a suitable polygon into suitable polygons. (It then also follows that any three vertices of a suitable polygon span an area of at least 1, but this will not be needed in the sequel.)

For convenience, write $a_k = \left(\frac{4}{3}\right)^k$ and $b_k = \frac{1}{2}(3^k + 3)$, so $a_{k+1} = \frac{4}{3}a_k$ and $b_{k+1} = 3b_k - 3$.

Let K be a suitable n-gon. We are to show that $[K] \geqslant a_k(n - b_k)$ for all non-negative integers k. Induct on k. The base case, $k = 0$, follows by the preceding: The $n-3$ diagonals from some vertex tile K by $n-2$ triangles of area at least 1 each, so $[K] \geqslant n - 2 = a_0(n - b_0)$.

To perform the induction step, consider three vertices of K spanning a triangle Δ of maximal area. Through each vertex of Δ draw a parallel to the opposite side to form a triangle ∇ of area $[\nabla] = 4[\Delta]$. By maximality of $[\Delta]$, no vertex of K lies outside ∇, so ∇ covers K, and hence $[K] \leqslant [\nabla] = 4[\Delta]$.



Notice further that K is tiled by Δ and three other (possibly degenerate) polygons, say, a p-gon L, a q-gon M and an r-gon N, where p, q and r are all at least 2 and $p + q + r = n + 3$. Then $[K] = [\Delta] + [L] + [M] + [N] \geqslant \frac{1}{4}[K] + [L] + [M] + [N]$, so $[K] \geqslant \frac{4}{3}([L] + [M] + [N])$.

Finally, by the remark preceding the proof proper, L, M and N are all suitable, so

$$\geqslant \frac{4}{3}([L] + [M] + [N]) \geqslant \frac{4}{3}\big(a_k(p - b_k) + a_k(q - b_k) + a_k(r - b_k)\big)$$
$$= \frac{4}{3}a_k(p + q + r - 3b_k) = \frac{4}{3}a_k(n + 3 - 3b_k) = a_{k+1}(n - b_{k+1}).$$

This completes the induction and concludes the proof of the lemma.