A QUESTION RELATED TO THE 4-TH PROBLEM FROM IMO 2018

MARIUS BECEANU[1)]

Abstract. Although the 4^{th} problem from IMO 2018 was rated 'easy', it still may lead to other questions, some of them similar to the one which is treated in the sequel

Keywords: path in graph

MSC: 05C10

The fourth problem from IMO 2018 was the following.

Queenie and Horst play a game on a 20×20 *chessboard. In the beginning the board is empty. In every turn, Horst places a black knight on an empty square in such a way that his new knight does not attack any previous knights. Then Queenie places a white queen on an empty square. The game gets finished when somebody cannot move.*

Find the maximal positive K *such that, regardless of the strategy of Queenie, Horst can put at least* K *knights on the board.*

Starting from here, one might ask the following question.

Problem

Ana and Bogdan play a game on a rectangular 20×20 *board made of squares of unit length. They alternately place pieces in free squares on the board. Ana plays first, but is forbidden from placing her pieces at a distance of* $\sqrt{5}$ *from any pieces placed by Bogdan (a knight's move). Bogdan can place his pieces in any free square.*

The game ends when Ana has no more allowed moves.

Show that, regardless of Ana's moves, Bogdan can make the game end in at most 59 *moves.*

[1)] Assistant Professor dr., University at Albany, State University of New York

Solution. Consider this 20×20 game board and represent it as a graph with 400 vertices, with an edge between two vertices if and only if one can be reached from the other by a knight's move.

Call a vertex red if Ana played there and blue if Bogdan did.

We say that an edge is blocked when it runs into either a blue, or a red vertex, or both.

Initially the graph has $(256 \cdot 8 + 64 \cdot 6 + 68 \cdot 4 + 8 \cdot 3 + 4 \cdot 2)/2 = 1368$ edges, for an average of $(1368/400) \cdot 2 = 6.84$ edges per vertex.

Suppose that the blue player adopts the strategy of always moving into a vertex with the largest remaining number of empty neighbors.

Let n be the maximum number of turns during which blue can find vertices with at least k empty neighbors.

By the end of the nth turn, the blue player will have blocked $N \geq kn$ edges. Since each edge newly blocked by blue corresponds to a blocked vertex, $N \geq kn$ vertices are now forbidden to the red player. Then at least $(k+2)n$ vertices are unavailable to the red player after n moves, so $n(k+2) \leq 400$ or else the game ends.

Since n is the maximum, $N \geq kn$ neighboring spots have been blocked by blue after n moves, but at most $N+k-1$ neighboring spots can be blocked after $n+1$ moves by blue, hence at most $2(N+k-1)$ edges can be blocked by both.

Then the largest number of unblocked edges any vertex can have after n turns is $k-1$, so we have blocked all but at most $200(k-1)$ edges. So we will have blocked at least $1368 - 200(k-1)$ edges. But each turn the blue player blocks at least as many edges as the red player does next turn and the first turn the red player can only block 8 edges. So at least around $680 - 100(k-1)$ edges will have been blocked by the blue player, so

$$680 - 100(k-1) \leq N,$$

so $N \geq 780 - 100k$.

For the game to continue, it is necessary that $N + 2n \leq 400$. So

$$780 - 100k \leq 400, \ (k+2)n \leq 400.$$

So $100k \geq 380$.

So the game will end before blue runs out of 4-moves, using this strategy, which imposes an upper bound of at most $\left[\dfrac{400}{6}\right] + 1 = 67$ moves.

How many moves better than 4-moves does blue necessarily have? Suppose red sets out to minimize blue's number of 8-moves. There are $16^2 = 256$ squares in the middle that could potentially be 8-moves, but, as soon as either red or blue makes a play for one of them or for a neighboring spot, it can no longer be an 8-move.

Supposing that both red and blue keep making 8-moves that always block 8 other squares from being future 8-moves (though in at least 4 cases one can block at most 6 other squares). Then all future 8-moves can be blocked in p moves (by either player) only if

$$(p-2)\times 9+2\times 7\geq 256.$$

So $9p\geq 260$, so $p\geq 29$. So there are at least 29 8-moves, of which blue will get at least 14 if both players take turns.

This leaves $260=400-14\times(8+2)$ squares. Even if blue is left with using 4-moves for the remainder of the game, the game will end in at most $[260/6]+1=44$ more moves, for a total of $14+44=58$ moves.

How many ≥ 7 moves can blue make? The total number of possible such moves is 256. Each of these points needs at least two adjacent moves (or to be moved onto) to no longer be a ≥ 7 move. Each move can block at most 8 such points (not including itself) and in the corners only 6. If p moves by either player have been made, all future ≥ 7-moves can be blocked if

$$2[(p-2)\times 8+2\times 6]+p\times 2\geq 2\times 256.$$

So $18p\geq 520$, so $p\geq 29$. Of these, blue will get at least 14.

But these can all be 8-moves, so this is no better than the previous result.