AN ELEMENTARY SOLUTION TO A CONTEST PROBLEM

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Abstract. The geometry problem from the second IMAR test, October 2003, involved a configuration including a triangle, its incenter, its circumcenter and one of the excircles. The pupose of this note is to provide a 'pure' solution to this problem.

Keywords: incircle, excircle, circumcircle

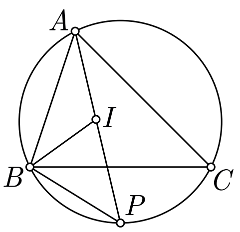
MSC: 51M04

This mathematical note provides a simple solution to a geometry problem, illustrating the power of auxiliary constructions over algebraic manipulations.

We will make use of the following two well-known lemmas.

Lemma 1. *Let I be the incenter of the triangle ABC and P be the second intersection of the bisector AI of the angle $\sphericalangle BAC$ and the circumcircle of the triangle ABC. Then $BP = PI$.*

Proof. From ΔABI, $\sphericalangle BIP = \sphericalangle ABI + \sphericalangle BAI$. Then
$\sphericalangle IBP = \sphericalangle CBI + \sphericalangle CBP = \sphericalangle ABI + \sphericalangle CBP$,
as BI is the bisector of $\sphericalangle ABC$. Since $\sphericalangle PBC = \sphericalangle PAC$ and $\sphericalangle PAC = \sphericalangle BAI$, $\sphericalangle IBP = \sphericalangle ABI + \sphericalangle BAI$.
It follows that $\sphericalangle BIP = \sphericalangle IBP$ and so $BP = PI$.



□

Lemma 2. *Let I be the incenter of the triangle ABC and I_a be the center of its excircle corresponding to A. Let P be the intersection between the bisector AI of $\sphericalangle BAC$ and the circumcircle of ABC. Then P is the midpoint of the segment II_a.*

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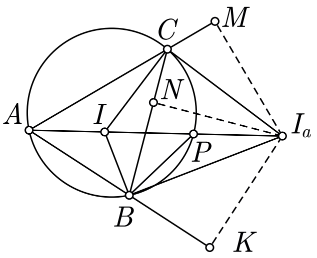
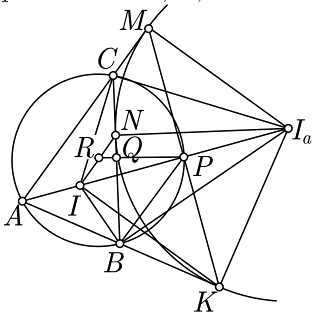


Proof. Let K, M, N be the contacts of the $A-$ excircle with AB, AC, respectively BC. Then $\sphericalangle I_aBP = \sphericalangle I_aBN - \sphericalangle PBN$. Since $[BI_a$ is the bisector of $\sphericalangle KBC$, $\sphericalangle KBI_a = \sphericalangle I_aBN$. Hence $\sphericalangle I_aBP = \sphericalangle KBI_a - \sphericalangle PBN = \sphericalangle KBI_a - \sphericalangle BAP$. Since $\sphericalangle KBI_a = \sphericalangle BAI_a + \sphericalangle BI_aI$ and $\sphericalangle BAI_a = \sphericalangle BAP$, $\sphericalangle BI_aI = \sphericalangle KBI_a - \sphericalangle BAP$. Therefore $\sphericalangle I_aBP = \sphericalangle BI_aP$, $BP = IP$, and because $BP = I_aP$, P is the midpoint of the segment II_a. □

Problem. *Let I and O be the incenter and circumcenter, respectively, of the triangle ABC. The excircle ω_A is tangent to AB, AC and BC in K, M, N respectively. If the midpoint P of KM is on the circumcircle of ABC, prove that O, I, N are collinear.*



Solution. Let Q be the midpoint of BC and R be the intersection of lines PQ and IN. Since both lines are perpendicular to BC, $PR \parallel I_aN$. Therefore $\dfrac{PR}{NI_a} = \dfrac{IP}{II_a}$, hence $\dfrac{PR}{IP} = \dfrac{NI_a}{II_a}$. But $IP \equiv BP$ (by Lemma 1) and $NI_a \equiv KI_a$ (both are radii of the excircle), so

$$\frac{PR}{BP} = \frac{KI_a}{II_a}. \tag{1}$$

Now

$$\sphericalangle BPR = 90^\circ - \sphericalangle PBQ = 90^\circ - \frac{\sphericalangle A}{2}$$

and

$$\sphericalangle KI_aI = 90^\circ - \sphericalangle KAI_a = 90^\circ - \frac{\sphericalangle A}{2}.$$

So $\sphericalangle BPR = \sphericalangle KI_aI$ and, by (1), $\triangle PRB \sim \triangle I_aKI$. It follows that $\sphericalangle BRP = \sphericalangle IKI_a$.

By Lemma 2, P is the midpoint of $[II_a]$. We also know that $AI_a \perp KM$, as PI_a is the median of the isosceles triangle KMI_a. Therefore $\triangle KII_a$ is isosceles and

$$\sphericalangle IKI_a = 2\sphericalangle PKI_a = 2\sphericalangle KAI_a.$$

Therefore $\sphericalangle BRP = 2\sphericalangle KAI_a = \sphericalangle A$. Since $\triangle RBC$ is isosceles, we have $\sphericalangle BRC = 2\sphericalangle BRP = 2\sphericalangle A$. We can conclude that R is the circumcenter of triangle ABC, so I, O, N are collinear.

References

[1] Andrei Neguţ, Problems For the Mathematical Olympiads, Editura Gil, Zalău, 2005.