GAZETA MATEMATICĂ

SERIA B

PUBLICAŢIE LUNARĂ PENTRU TINERET

Fondată în anul 1895

Anul CXXI nr. 9 **septembrie 2016**

ARTICOLE ŞI NOTE MATEMATICE

A RATIONAL REFINEMENT OF YUN'S INEQUALITY IN BICENTRIC QUADRILATERALS

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Abstract. In this paper we shall find the best minimal and maximal rational bounds of form $\frac{\alpha R+\beta r}{R+\gamma r}$ with $\alpha,\beta\in\mathbb{R}$ and $\gamma>-\sqrt{2}$, for the sum $\sum\limits_{\text{cyclic}}\sin\frac{A}{2}\cos\frac{B}{2}$ where A,B,C,D represent the angles of a bicentric quadrilateral with circumradius R, inradius r and semiperimeter s.

Keywords: bounds for cyclic sums of trigonometric functions of the angles of a bicentric quadrilateral

MSC: 51M16, 26D05

1. Introduction

In [6], *Zhang Yun* established the following inequality.

1. *In every bicentric quadrilateral holds the inequality*

$$\frac{r\sqrt{2}}{R}\le\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right)\le 1.$$

Another proof for this inequality, given by *Martin Josefsson*, can be found in [5].

A refinement of *Yun*'s inequality is given by *Vasile Jiglău* in [4].

In [2] appears a refinement of *Yun*'s inequality of the type

$$f(R,r)\le\frac{1}{2}\sum_{\text{cyclic}}\sin\frac{A}{2}\cos\frac{B}{2}\le g(R,r)$$

where $f(r,R)$, $g(r,R)$ represent the best minimal and maximal homogenous

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functions for the sum $\frac{1}{2}\sum_{\text{cyclic}} \sin\frac{A}{2}\cos\frac{B}{2}$; $g(R,r)$ is determined also in [4], where it is stated that

2. *In every bicentric quadrilateral holds*

$$\frac{1}{2}+\frac{1}{2}\sqrt{\frac{r(r+\sqrt{4R^2+r^2})}{2r^2}} \le \frac{1}{2}\sum_{\text{cyclic}} \sin\frac{A}{2}\cos\frac{B}{2} \le \frac{\sqrt{4R^2+r^2}+r}{2\sqrt{2}R}. \qquad (1)$$

The proof of this theorem is based on the monotony of the function (see [2])

$$E:[S_1,S_2]\to\mathbb{R}, E(S)=\frac{1}{2}\sum_{\text{cyclic}} \sin\frac{A}{2}\cos\frac{B}{2}=\frac{1}{2}\sqrt{1+\frac{x_3}{4R^2}+\frac{x_3}{8R^2r}S}. \qquad (2)$$

We remember the known facts that

$$\sin\frac{A}{2}=\cos\frac{C}{2}=\sqrt{\frac{bc}{ad+bc}}$$

$$x_3=ac+bd=2r\left(r+\sqrt{4R^2+r^2}\right),$$

(see [1]) and

$$S_1=\sqrt{8r(\sqrt{4R^2+r^2}-r)},\ S_2=r+\sqrt{4R^2+r^2}$$

are the semiperimeters of the bicentric quadrilaterals $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ which make up the minimal and maximal semiperimeter from *Blundon-Eddy* inequality $S_1\le S\le S_2$ (see [3]).

2. Main results

In the following we find the best real constants α,β and $\gamma>-\sqrt{2}$ such that the inequality

$$\frac{\alpha R+\beta r}{R+\gamma r}\le\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right). \qquad (3)$$

is true in every bicentric quadrilateral.

3. In every bicentric quadrilateral holds

$$\frac{R+2\sqrt{2}r}{2R+\sqrt{2}r}\le\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right). \qquad (4)$$

Proof. From (1) we have that

$$\frac{1}{2}+\frac{1}{2}\sqrt{\frac{r(r+\sqrt{4R^2+r^2})}{2R^2}}\le$$
$$\le\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{C}{2}\right).$$

In order to prove (4) it will be sufficient to prove that

$$\frac{R+2\sqrt{2}r}{2R+\sqrt{2}r} \leq \frac{1}{2}+\frac{1}{2}\sqrt{\frac{r(r+\sqrt{4R^2+r^2})}{2R^2}}$$

or, after denoting $x=\frac{R}{r}$ that $\frac{x+2\sqrt{2}}{2x+\sqrt{2}} \leq \frac{1}{2}+\frac{1}{2}\sqrt{\frac{1+\sqrt{4x^2+1}}{2x^2}}$, that is $\frac{3\sqrt{2}}{2x+\sqrt{2}} \leq \sqrt{\frac{1+\sqrt{4x^2+1}}{2x^2}}$, or $\frac{9}{\left(\sqrt{2}x+1\right)^2} \leq \frac{1+\sqrt{4x^2+1}}{2x^2}$, or

$$18x^2 \leq \left(\sqrt{2}x+1\right)^2+\left(\sqrt{2}x+1\right)^2\sqrt{4x^2+1},$$

or $(16x^2-2\sqrt{2}x-1)^2 \leq (4x^2+1)(2x^2+2\sqrt{2}x+1)^2$, or, after performing some calculation, that $4x^2(x-\sqrt{2})^2(4x^2+16\sqrt{2}x+5) \geq 0$, which is true.

Next, we shall prove that the inequality (4) is the best of type (3).

We suppose that other constants $\alpha_0, \beta_0 \in \mathbb{R}$, $\gamma_0 > -\sqrt{2}$ exist, such that

$$\frac{\alpha_0 R+\beta_0 r}{R+\gamma_0 r} \leq \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right)$$

is true in every bicentric quadrilateral and this inequality is the best inequality of type (3). So we have that

$$\begin{aligned}&\frac{R+2\sqrt{2}}{2R+\sqrt{2}r} \leq \frac{\alpha_0 R+\beta_0 r}{R+\gamma_0 r} \leq \\ &\leq \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right)\end{aligned} \tag{5}$$

is true in every bicentric quadrilateral.

If we consider the case of bicentric quadrilateral $A_1B_1D_1C_1$ which makes up the minimal semiperimeter $S_1=\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}$, from (5) it results that the inequality

$$\begin{aligned}&\frac{R+2\sqrt{2}r}{2R+\sqrt{2}r} \leq \frac{\alpha_0 R+\beta_0 r}{R+\gamma_0 r} \leq \frac{1}{2}+\frac{1}{2}\sqrt{\frac{r\left(r+\sqrt{4R^2+r^2}\right)}{2R^2}} = \\ &=\frac{1}{2}\left(\sin\frac{A_1}{2}\cos\frac{B_1}{2}+\sin\frac{B_1}{2}\cos\frac{C_1}{2}+\sin\frac{C_1}{2}\cos\frac{D_1}{2}+\sin\frac{D_1}{2}\cos\frac{A_1}{2}\right) \leq \\ &\leq \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right)\end{aligned} \tag{6}$$

is true in every bicentric quadrilateral. Now (2) yields

$$\frac{1}{2}\left(\sin\frac{A_1}{2}\cos\frac{B_1}{2}+\sin\frac{B_1}{2}\cos\frac{C_1}{2}+\sin\frac{C_1}{2}\cos\frac{D_1}{2}+\sin\frac{D_1}{2}\cos\frac{A_1}{2}\right)=$$
$$=\frac{1}{2}\sqrt{1+\frac{x_3}{4R^2}+\frac{x_3}{8R^2r}S_1}\le\frac{1}{2}\sqrt{1+\frac{x_3}{4R^2}+\frac{x_3}{8R^2r}S}=$$
$$=\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right).$$

If we consider the case of square with sides $a=b=c=d=1$, $R=\frac{1}{\sqrt{2}}$, $r=\frac{1}{2}$, if we replace in (6) we obtain

$$1\le\frac{\alpha_0\sqrt{2}+\beta_0}{\sqrt{2}+\gamma_0}\le 1 \quad\text{or}\quad \alpha_0\sqrt{2}+\beta_0=\sqrt{2}+\gamma_0. \tag{7}$$

From (6) we have

$$\frac{x+2\sqrt{2}}{2x+\sqrt{2}}\le\frac{\alpha_0x+\beta_0}{x+\gamma_0}\le\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1+\sqrt{4x^2+1}}{2x^2}} \tag{8}$$

for each $x\ge\sqrt{2}$.

If we take in (8) $x\to\infty$, we obtain $\frac{1}{2}\le\alpha_0\le\frac{1}{2}$ or $\alpha_0=\frac{1}{2}$. From (7) we obtain $\beta_0=\frac{\sqrt{2}}{2}+\gamma_0$. Inequality (8) may be written as

$$\frac{x+2\sqrt{2}}{2x+\sqrt{2}}\le\frac{\frac{1}{2}x+\frac{\sqrt{2}}{2}+\gamma_0}{x+\gamma_0}\le\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1+\sqrt{4x^2+1}}{2x^2}} \tag{9}$$

for each $x\ge\sqrt{2}$.

The left side of (9) is equivalent with $\frac{(\sqrt{2}\gamma_0-1)(x-\sqrt{2})}{\sqrt{2}}\ge 0$ for each $x\ge\sqrt{2}$, or

$$\gamma_0\ge\frac{1}{\sqrt{2}}. \tag{10}$$

The right side of (9) may be written as $\frac{\gamma_0+\sqrt{2}}{x+\gamma_0}\le\sqrt{\frac{1+\sqrt{4x^2+1}}{2x^2}}$ for each $x\ge\sqrt{2}$ or

$$\gamma_0\le\frac{x\left(\sqrt{1+\sqrt{4x^2+1}}-2\right)}{\sqrt{2}x-\sqrt{1+\sqrt{4x^2+1}}} \tag{11}$$

for each $x \geq \sqrt{2}$. We consider the function $f : [\sqrt{2}, +\infty) \to \mathbb{R}$,

$$f(x) = \frac{x\left(\sqrt{1+\sqrt{4x^2+1}} - 2\right)}{\sqrt{2}x - \sqrt{1+\sqrt{4x^2+1}}}.$$

From (11) we have $\gamma_0 \leq \min_{x \geq \sqrt{2}} f(x)$. We compute

$$\lim_{x\to\sqrt{2}} f(x) = \lim_{x\to\sqrt{2}} \frac{x\left(\sqrt{4x^2+1}-3\right)}{\left(2x^2-1-\sqrt{4x^2+1}\right)} \frac{\left(\sqrt{2}x + \sqrt{1+\sqrt{4x^2+1}}\right)}{\left(2+\sqrt{1+\sqrt{4x^2+1}}\right)} =$$
$$= \sqrt{2} \lim_{x\to\sqrt{2}} \frac{4(x^2-2)}{4x^2(x^2-2)} \frac{2x^2-1+\sqrt{4x^2+1}}{\sqrt{4x^2+1}+3} = \frac{1}{\sqrt{2}}.$$

We prove that $\lim_{x\to\sqrt{2}} f(x) = \frac{1}{\sqrt{2}} \leq f(x)$ for each $x \geq \sqrt{2}$ which implies that $\frac{1}{\sqrt{2}} = \lim_{x\to\sqrt{2}} f(x) = \min_{x\geq\sqrt{2}} f(x)$. So

$$\gamma_0 \leq \frac{1}{\sqrt{2}}. \tag{12}$$

It remains to prove that $\frac{1}{\sqrt{2}} \leq f(x) = \frac{x\left(\sqrt{1+\sqrt{4x^2+1}} - 2\right)}{\sqrt{2}x - \sqrt{1+\sqrt{4x^2+1}}}$, that is $\sqrt{2}x - \sqrt{1+\sqrt{4x^2+1}} \leq x\sqrt{2}\sqrt{1+\sqrt{4x^2+1}} - 2\sqrt{2}x$, which reduces to $3\sqrt{2}x \leq \left(x\sqrt{2}+\sqrt{2}\right)\sqrt{1+\sqrt{4x^2+1}}$, or $\frac{3\sqrt{2}}{2x+\sqrt{2}} \leq \sqrt{\frac{1+\sqrt{4x^2+1}}{2x^2}}$, inequality which was proved during the proof of theorem 3.

From (10) and (12) follows that $\gamma_0 = \frac{1}{\sqrt{2}}$ and $\beta_0 = \gamma_0 + \frac{\sqrt{2}}{2} = \sqrt{2}$, which represents a contradiction. So, that the inequality (4) is the best of type (3). □

Next, we shall find the best real constants α, β and $\gamma > -\sqrt{2}$, such that the inequality

$$\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{D}{2} + \sin\frac{D}{2}\cos\frac{A}{2}\right) \leq \frac{\alpha R + \beta r}{R + \gamma r} \tag{13}$$

is true in every bicentric quadrilateral.

4. *In every bicentric quadrilateral is true the inequality*

$$\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right)\le$$
$$\le\frac{R+\left(6-4\sqrt{2}\right)r}{\sqrt{2}R+\left(4-3\sqrt{2}\right)r}. \tag{14}$$

Proof. From (1) we have that

$$\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right)\le$$
$$\le\frac{\sqrt{4R^2+r^2}+r}{2\sqrt{2}R}.$$

In order to prove (14), it will be sufficient to prove that

$$\frac{\sqrt{4R^2+r^2}+r}{2\sqrt{2}R}\le\frac{R+\left(6-4\sqrt{2}\right)r}{\sqrt{2}R+\left(4-3\sqrt{2}\right)r},$$

or

$$\frac{\sqrt{4x^2+1}+1}{2\sqrt{2}x}\le\frac{x+6-4\sqrt{2}}{\sqrt{2}x+4-3\sqrt{2}},$$

or

$$\frac{\sqrt{4x^2+1}+1}{2\sqrt{2}x}\le\frac{x+2\alpha_1}{\sqrt{2}(x-\alpha_1)},$$

or

$$(x-\alpha_1)\sqrt{4x^2+1}\le 2x^2+(4\alpha_1-1)x+\alpha_1.$$

After squaring and performing some calculations we obtain that

$$4x\left[(6\alpha_1-1)x^2+(3\alpha_1^2-\alpha)x+2\alpha_1^2\right]\ge 0$$

or

$$4(17-12\sqrt{2})(x-\sqrt{2})^2\ge 0$$

for each $x\ge\sqrt{2}$, inequality which is true. □

Next we shall prove that the inequality (14) is the best of type (13). We suppose that other constants $\alpha_0,\beta_0\in\mathbb{R}$, $\gamma_0>-\sqrt{2}$ exist such that

$$\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right)\le$$
$$\le\frac{\alpha_0R+\beta_0r}{R+\gamma_0r}$$

is a better inequality of type (13). It follows that

$$\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right)\le$$
$$\le\frac{\alpha_0R+\beta_0r}{R+\gamma_0r}\le\frac{R+(6-4\sqrt{2})r}{\sqrt{2}R+(4-3\sqrt{2})r} \tag{15}$$

is true in every bicentric quadrilateral.

If we consider the bicentric quadrilateral $A_2B_2C_2D_2$ which makes up the maximal semiperimeter $S_2 = \sqrt{4R^2 + r^2} + r$, from (15) it follows that

$$\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{D}{2} + \sin\frac{D}{2}\cos\frac{A}{2}\right) \le$$
$$\le \frac{1}{2}\left(\sin\frac{A_2}{2}\cos\frac{B_2}{2} + \sin\frac{B_2}{2}\cos\frac{C_2}{2} + \sin\frac{C_2}{2}\cos\frac{D_2}{2} + \sin\frac{D_2}{2}\cos\frac{A_2}{2}\right) =$$
$$= \frac{\sqrt{4R^2 + r^2} + r}{2\sqrt{2}R} \le \frac{\alpha_0 R + \beta_0 r}{R + \gamma_0 r} \le \frac{R + \left(6 - 4\sqrt{2}\right) r}{\sqrt{2}R + (4 - 3\sqrt{2})r} \tag{16}$$

is true in every bicentric quadrilateral since we have

$$\frac{1}{2}\left(\sin\frac{A_2}{2}\cos\frac{B_2}{2} + \sin\frac{B_2}{2}\cos\frac{C_2}{2} + \sin\frac{C_2}{2}\cos\frac{D_2}{2} + \sin\frac{D_2}{2}\cos\frac{A_2}{2}\right) =$$
$$= \frac{1}{2}\sqrt{1 + \frac{x_3}{4R^2} + \frac{x_3}{8R^2 r} S_2} \ge \frac{1}{2}\sqrt{1 + \frac{x_3}{4R^2} + \frac{x_3}{8R^2 r} S} =$$
$$= \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{D}{2} + \sin\frac{D}{2}\cos\frac{A}{2}\right).$$

In the case of the square with sides $a = b = c = d = 1$, $R = \frac{1}{\sqrt{2}}$, $r = \frac{1}{2}$, the relation (16) yields $1 \le \frac{\alpha_0/\sqrt{2} + \beta_0/2}{1/\sqrt{2} + \gamma_0/2} \le 1$, or

$$\alpha_0\sqrt{2} + \beta_0 = \gamma_0 + \sqrt{2}. \tag{17}$$

From (16) we have

$$\frac{\sqrt{4x^2 + 1}}{2\sqrt{2}x} \le \frac{\alpha_0 x + \beta_0}{x + \gamma_0} \le \frac{R + \left(6 - 4\sqrt{2}\right) r}{\sqrt{2}R + \left(4 - 3\sqrt{2}\right) r}.$$

If we take $x \to \infty$, we obtain $\frac{1}{\sqrt{2}} \le \alpha_0 \le \frac{1}{\sqrt{2}}$ or $\alpha_0 = \frac{1}{\sqrt{2}}$.
Replacing in (17) we obtain

$$\beta_0 = \gamma_0 + \sqrt{2} - 1. \tag{18}$$

From (16) we obtain that

$$\frac{\sqrt{4x^2 + 1} + 1}{2\sqrt{2}x} \le \frac{\frac{1}{\sqrt{2}}x + \gamma_0 + \sqrt{2} - 1}{x + \gamma_0} \le \frac{x + 6 - 4\sqrt{2}}{\sqrt{2}x + 4 - 3\sqrt{2}} \tag{19}$$

for each $x \ge \sqrt{2}$. The right side of inequality (19) may be written as

$$\left[5\sqrt{2} - 7 + \left(\sqrt{2} - 1\right)\gamma_0\right]\left(x - \sqrt{2}\right) \le 0$$

for each $x \geq \sqrt{2}$ or $\gamma_0 \leq \dfrac{7-5\sqrt{2}}{\sqrt{2}-1}$ for each $x \geq \sqrt{2}$. It results that

$$\gamma_0 \leq 2\sqrt{2} - 3. \tag{20}$$

The left side of inequality (19) may be written as

$$\gamma_0 \geq \frac{x\left[\sqrt{4x^2+1} - 2x - 3 + 2\sqrt{2}\right]}{2\sqrt{2}x - 1 - \sqrt{4x^2+1}}. \tag{21}$$

Consider the function $f : [\sqrt{2}, +\infty) \to \mathbb{R}$,

$$f(x) = \frac{x\left(\sqrt{4x^2+1} - 2x - 3 + 2\sqrt{2}\right)}{2\sqrt{2}x - 1 - \sqrt{4x^2+1}}.$$

From (21) we obtain that $\gamma_0 \geq f(x)$ for each $x \geq \sqrt{2}$. It follows that $\gamma_0 \geq \max\limits_{x\geq\sqrt{2}} f(x)$. We have

$$\begin{aligned}\lim_{x\to\sqrt{2}} f(x) &= \lim_{x\to\sqrt{2}} \frac{x\sqrt{4x^2+1} - 2x^2 - (3-2\sqrt{2})x}{2\sqrt{2}x - 1 - \sqrt{4x^2+1}} = \\ &= \lim_{x\to\sqrt{2}} \frac{\sqrt{4x^2+1} + \dfrac{4x^2}{\sqrt{4x^2+1}} - 4x - (3-2\sqrt{2})}{2\sqrt{2} - \dfrac{4x}{\sqrt{4x^2+1}}} = 2\sqrt{2} - 3.\end{aligned}$$

We prove that

$$f(x) \leq 2\sqrt{2} - 3 = \lim_{x\to\sqrt{2}} f(x) \tag{22}$$

for each $x \geq \sqrt{2}$ which implies that $\max\limits_{x\geq\sqrt{2}} f(x) = 2\sqrt{2} - 3$ and

$$\gamma_0 \geq 2\sqrt{2} - 3. \tag{23}$$

Inequality (22) may be written equivalently as

$$\frac{x\left(\sqrt{4x^2+1} - 2x - 3 + 2\sqrt{2}\right)}{2\sqrt{2}x - 1 - \sqrt{4x^2+1}} \leq 2\sqrt{2} - 3$$

or $(x - \alpha_1)\sqrt{4x^2+1} \leq 2x^2 + (4\alpha_1 - 1)x + \alpha_1$, inequality which it was proved in the context of theorem **4.**

From (20) and (23) it follows that $\gamma_0 = 2\sqrt{2} - 3$. From (18) we obtain $\beta_0 = \gamma_0 + \sqrt{2} - 1 = 3\sqrt{2} - 4$. So, we have $\alpha_0 = \dfrac{1}{\sqrt{2}}$, $\beta_0 = 3\sqrt{2} - 4$, $\gamma_0 = 2\sqrt{2} - 3$ which represent a contradiction.

So, the inequality (14) is the best of the type (13). □

5. [The rational refinement of Yun's inequaliy] *In every bicentric quadrilateral are true the inequalities*

$$\frac{r\sqrt{2}}{R} \leq \frac{R+2\sqrt{2}r}{2R+\sqrt{2}r} \leq$$
$$\leq \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right) \leq$$
$$\leq \frac{R+\left(6-4\sqrt{2}\right)r}{\sqrt{2}R+(4-3\sqrt{2})r} \leq 1.$$

Proof. In the following we consider the functions

$$F : \Big(-\sqrt{2}, \frac{1}{\sqrt{2}}\Big] \to \mathbb{R}, \qquad G : \Big[2\sqrt{2}-3, +\infty\Big) \to \mathbb{R},$$
$$F(\gamma_1) = \frac{R+(\sqrt{2}+2\gamma_1)r}{2R+2\gamma_1 r}, \quad G(\gamma_2) = \frac{R+\sqrt{2}(\gamma_2+\sqrt{2}-1)r}{\sqrt{2}R+\sqrt{2}\gamma_2 r}.$$

We have

$$F'(\gamma_1) = \frac{2r(R-r\sqrt{2})}{(2R+2\gamma_1 r)^2}, \quad G'(\gamma_2) = \frac{(2-\sqrt{2})r(R-\sqrt{2}r)}{(\sqrt{2}R+\sqrt{2}\gamma_2 r)^2}.$$

Since $R \geq \sqrt{2}r$, it follows that $F'(\gamma_1) \geq 0$ and $G'(\gamma_2) \geq 0$ which imply that F and G are an increasing functions. It results that

$$\lim_{x\to-\sqrt{2}} F(\gamma_1) < F(\gamma_1) \leq F\left(\frac{1}{\sqrt{2}}\right)$$

and $G(2\sqrt{2}-3) \leq G(\gamma_2) \leq G(\infty)$ or $\frac{1}{2} \leq F(\gamma_1) \leq \frac{R+2\sqrt{2}r}{2R+\sqrt{2}r}$ for each $\gamma_1 \in \Big[-\sqrt{2}, \frac{1}{\sqrt{2}}\Big]$ and

$$\frac{R+(6-4\sqrt{2})r}{\sqrt{2}R+(4-2\sqrt{3})r} \leq G(\gamma_2) \leq 1 \tag{24}$$

for each $\gamma_2 \in [2\sqrt{2}-3, +\infty)$. □

According to (24), theorem **3** and theorem **4** we obtain the following result.

6. *In every bicentric quadrilateral are true the following inequalities*

$$\frac{1}{2} < \frac{R+(\sqrt{2}+2\gamma_1)r}{2R+2\gamma_1 r} \leq \frac{R+2\sqrt{2}r}{2R+\sqrt{2}r} \leq$$
$$\leq \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right) \leq$$
$$\leq \frac{R+(6-4\sqrt{2})r}{\sqrt{2}R+(4-2\sqrt{3})r} \leq \frac{R+\sqrt{2}(\gamma_2+\sqrt{2}-1)r}{\sqrt{2}R+\sqrt{2}\gamma_2 r} \leq 1$$

for every $\gamma_1 \in \left[-\sqrt{2}, \dfrac{1}{\sqrt{2}}\right]$ *and* $\gamma_2 \in [2\sqrt{2}-3, +\infty)$.

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