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A COMPUTATIONAL SOLUTION OF QUESTION 6 FROM IMO 2011

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Abstract. The article presents a solution for the sixth question of IMO 2011 using complex coordinates.

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MSC : 51M04

Although computational solutions for geometry problems are considered less 'spectacular', they must not be disregarded. This article shows that a good organization can bring success in many occasions (it is worth mentioning that this problem was solved during the contest only by 9 students).

Problem (IMO 2011, question 6). *Let ΔABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.*

Solution. We will denote as usual $\mathrm{e}^{i\theta} = \cos\theta + \mathrm{i}\sin\theta$.

Define $V := \ell \cap \Gamma$ and assume without loss of generality that the circumcircle of ΔABC is the unit circle. Introducing polar coordinates parametrize the points A, B, C, V by their complex arguments, $A = \mathrm{e}^{\mathrm{i}\theta_A}$, $B = \mathrm{e}^{\mathrm{i}\theta_B}$, $C = \mathrm{e}^{\mathrm{i}\theta_C}$, $V = \mathrm{e}^{\mathrm{i}\theta_V}$ with $0 \le \theta_A \le \theta_V \le \theta_B \le \theta_C \le 2\pi$. The solution idea is to simply calculate and express all quantities of interest in terms of the four free parameters θ_A, θ_B, θ_C, θ_V determining uniquely the entire geometric construction. Note that the "dimensionality" of the problem can be reduced

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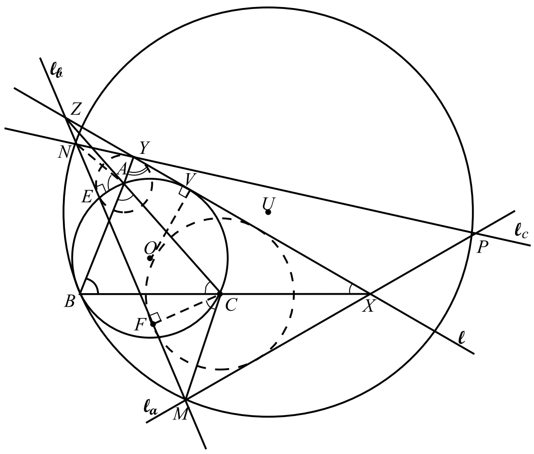
further by assuming without loss of generality that the line BC is parallel to the horizontal axis, so that

$$\theta_B + \theta_C = 3\pi, \tag{1}$$

since $\sphericalangle BAC$ is acute by assumption. The free parameters of the problem are therefore now only three: θ_A, θ_C, θ_V.



For later reference we observe now that by construction A is located at the same distance from ℓ, ℓ_B and ℓ_C. The same holds for B with respect to ℓ, ℓ_A, ℓ_C and C with respect to ℓ, ℓ_A, ℓ_B respectively, implying that NA, PB, MC are bisectors in ΔMNP. This ensures in particular $\sphericalangle NAE = \sphericalangle A$ and $\sphericalangle FCM = \sphericalangle C$.

Note now that the angles of ΔABC, ΔMNP as well as X, Y, Z (marked in the Figure above) can be successively computed in terms of the four initial parameters as follows,

$$\sphericalangle A = \frac{\theta_C - \theta_B}{2}, \quad \sphericalangle B = \pi - \frac{\theta_C - \theta_A}{2}, \quad \sphericalangle C = \frac{\theta_B - \theta_A}{2}, \tag{2}$$

$$\begin{gathered} \sphericalangle X = \pi/2 - \theta_V, \quad \sphericalangle Y = (\theta_C - \theta_A)/2 + \theta_V - \pi/2, \\ \sphericalangle Z = (\theta_B - \theta_A)/2 + \theta_V - \pi/2, \end{gathered} \tag{3}$$

and (e.g. in ΔMXZ we have $\sphericalangle M = \pi - 2(\sphericalangle X + \sphericalangle Z)$ etc.)

$$\sphericalangle M = \pi - 2\sphericalangle C, \quad \sphericalangle N = \pi - 2\sphericalangle A, \quad \sphericalangle P = \pi - 2\sphericalangle B. \tag{4}$$

The trigonometric proof we propose below consists in the following three steps:

1) Calculate the coordinates of U, the center of the circle through M, N, P.

2) Calculate R, the radius of the circle through M, N, P.

3) Prove $\|OU\|^2 = (R-1)^2$, ensuring that the circles through A, B, C and M, N, P are tangent.

1. Using complex coordinates we write, since:

$$\sphericalangle MNU = \frac{\pi}{2} - \frac{\sphericalangle MUN}{2} = \frac{\pi}{2} - \sphericalangle P,$$

$$z_U - z_N = R \cdot \frac{z_M - z_N}{\|MN\|} \cdot \mathrm{e}^{\mathrm{i}(\pi/2 - \sphericalangle P)} =$$

$$= R \cdot \mathrm{e}^{\mathrm{i}(3\pi/2 + \theta_V - 2\sphericalangle Z)} \cdot \mathrm{e}^{\mathrm{i}(\pi/2 - \sphericalangle P)} \overset{(3),(4)}{=}$$

$$\overset{(3),(4)}{=} R \cdot \mathrm{e}^{\mathrm{i}(2\theta_A - \theta_B - \theta_C - \theta_V)} \overset{(1)}{=} -R \cdot \mathrm{e}^{\mathrm{i}(2\theta_A - \theta_V)}. \tag{5}$$

We now need to calculate also z_N and to this end we write similarly

$$z_N - z_A = \|AN\| \cdot \frac{z_C - z_A}{\|AC\|} \cdot \mathrm{e}^{-\mathrm{i}(\sphericalangle A + \sphericalangle Z + \pi/2)} =$$

$$= \|AN\| \cdot \mathrm{e}^{\mathrm{i}(2\pi - \sphericalangle C)} \cdot \mathrm{e}^{-\mathrm{i}(\sphericalangle A + \sphericalangle Z + \pi/2)} \overset{(2),(3)}{=}$$

$$\overset{(2),(3)}{=} \frac{d_A}{\cos \sphericalangle A} \cdot \mathrm{e}^{\mathrm{i}(\theta_A - (\theta_B + \theta_C)/2 - \theta_V)} \overset{(1)}{=} \frac{d_A}{\cos \sphericalangle A} \cdot \mathrm{e}^{\mathrm{i}(\theta_A - \theta_V - 3\pi/2)}, \tag{6}$$

where d_A denotes the distance from A to the line MN, i.e. $d_A = \|AE\|$. From (5) and (6) we conclude that

$$z_U = -R \cdot \mathrm{e}^{\mathrm{i}(2\theta_A - \theta_V)} + \mathrm{e}^{\mathrm{i}\theta_A} + \frac{d_A}{\cos \sphericalangle A} \cdot \mathrm{e}^{\mathrm{i}(\theta_A - \theta_V - 3\pi/2)}. \tag{7}$$

Besides, from (1) and (2) we deduce

$$\cos \sphericalangle A = \cos(\theta_C - 3\pi/2) = -\sin \theta_C, \tag{8}$$

whereas

$$\begin{aligned} d_A &= AV \cdot \sin(\sphericalangle AOV/2) = 2\sin^2(\sphericalangle AOV/2) \\ &= 2\sin^2((\theta_A - \theta_V)/2) = 1 - \cos(\theta_A - \theta_V). \end{aligned} \tag{9}$$

In view of (7), (8), (9), z_U is now described only in terms of the three free parameters $\theta_A, \theta_C, \theta_V$ of the problem, and the first step of the proof is complete.

2. The calculation of R is straightforward:

$$R = \frac{MN}{2\sin(\sphericalangle P)} \overset{(4)}{=} \frac{NE + EF + FM}{2\sin(2\sphericalangle B)} =$$
$$= \frac{d_A \tan\sphericalangle A + AC\cos\sphericalangle Z + d_C\tan\sphericalangle C}{2\sin(2\sphericalangle B)} =$$
$$= \frac{-(1-\cos(\theta_A-\theta_V))\cot\theta_C + 2\sin(\sphericalangle B)\cos\sphericalangle Z}{2\sin(2\sphericalangle B)} +$$
$$+ \frac{(1-\cos(\theta_C-\theta_V))\cot((\theta_A+\theta_C)/2)}{2\sin(2\sphericalangle B)} =$$
$$= \frac{-(1-\cos(\theta_A-\theta_V))\cot\theta_C - 2\sin\dfrac{\theta_C-\theta_A}{2}\cos\left(\dfrac{\theta_C+\theta_A}{2}-\theta_V\right)}{-2\sin(\theta_C-\theta_A)} +$$
$$+ \frac{(1-\cos(\theta_C-\theta_V))\cot\dfrac{\theta_A+\theta_C}{2}}{-2\sin(\theta_C-\theta_A)}$$

which we can represent more conveniently as a linear combination of 1, $\sin\theta_V$, $\cos\theta_V$:

$$R = \alpha + \beta\sin\theta_V + \gamma\cos\theta_V, \tag{10}$$

with

$$\alpha = \frac{-\cot\theta_C + \cot\dfrac{\theta_A+\theta_C}{2}}{-2\sin(\theta_C-\theta_A)} \tag{11}$$

$$\beta = \frac{\sin\theta_A\cot\theta_C - \sin\theta_C\cot\dfrac{\theta_A+\theta_C}{2} + \cos\theta_C - \cos\theta_A}{-2\sin(\theta_C-\theta_A)} \tag{12}$$

$$\gamma = \frac{\cos\theta_A\cot\theta_C - \cos\theta_C\cot\dfrac{\theta_A+\theta_C}{2} - \sin\theta_C + \sin\theta_A}{-2\sin(\theta_C-\theta_A)}. \tag{13}$$

3. We need to show that $|z_U|^2 = (R-1)^2$. From (7) we have

$$|z_U|^2 = R^2 + 1 + \frac{d_A^2}{\cos^2\sphericalangle A} + 2\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}(-\theta_V-3\pi/2)}\frac{d_A}{\cos\sphericalangle A}\right) -$$
$$-2R\cdot\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}(-\theta_A+\theta_V)} + \mathrm{e}^{\mathrm{i}(-\theta_A-3\pi/2)}\frac{d_A}{\cos\sphericalangle A}\right) =$$
$$= R^2 + 1 + \frac{d_A^2}{\cos^2\sphericalangle A} + 2\sin\theta_V\frac{d_A}{\cos\sphericalangle A} -$$
$$-2R\left(\cos(\theta_V-\theta_A) + \sin\theta_A\frac{d_A}{\cos\sphericalangle A}\right) =$$
$$= \left[\frac{d_A^2}{\cos^2\sphericalangle A} + 2\sin\theta_V\frac{d_A}{\cos\sphericalangle A} - 2R\left(\cos(\theta_V-\theta_A) - 1 + \sin\theta_A\frac{d_A}{\cos\sphericalangle A}\right)\right] +$$

$$+R^2 - 2R + 1.$$

It follows that the proof of Step 3 is complete if we can show that the term in square brackets vanishes, i.e.

$$R = \frac{\dfrac{d_A}{\cos^2 \sphericalangle A} + \dfrac{2\sin\theta_V}{\cos \sphericalangle A}}{-2 + \dfrac{2\sin\theta_A}{\cos \sphericalangle A}}$$

which due to (1), (2), (9) is equivalent to

$$R = \frac{1 - \cos(\theta_V - \theta_A) - 2\sin\theta_V \sin\theta_C}{-2\sin\theta_A \sin\theta_C - 2\sin^2\theta_C}.$$

Expressing the r.h.s. as a linear combination of $1, \sin\theta_V, \cos\theta_V$ the equality that needs to be proved becomes

$$R = \frac{1}{-2\sin\theta_C(\sin\theta_C + \sin\theta_A)} + \frac{2 + \dfrac{\sin\theta_A}{\sin\theta_C}}{2(\sin\theta_C + \sin\theta_A)}\sin\theta_V +$$

$$+\frac{\dfrac{\cos\theta_A}{\sin\theta_C}}{2(\sin\theta_C + \sin\theta_A)}\cos\theta_V. \tag{14}$$

All we have to do now is compare (14) to (10) and identify the coefficients. For example the coefficient α of 1 in (10) can be re-written using the elementary trigonometric formulae for $\sin(2x)$ and $\sin(x-y)$ as

$$\alpha = \frac{-\cot\theta_C + \cot\dfrac{\theta_A + \theta_C}{2}}{-2\sin(\theta_C - \theta_A)} =$$

$$= \frac{-\cos\theta_C \sin\dfrac{\theta_A+\theta_C}{2} + \cos\dfrac{\theta_A+\theta_C}{2}\sin\theta_C}{-2\sin\theta_C \sin\dfrac{\theta_A+\theta_C}{2}\sin(\theta_C - \theta_A)} =$$

$$= \frac{\sin\dfrac{\theta_C-\theta_A}{2}}{-2\sin\theta_C \sin\dfrac{\theta_A+\theta_C}{2}\sin(\theta_C - \theta_A)} =$$

$$= \frac{1}{-4\sin\theta_C \sin\dfrac{\theta_A+\theta_C}{2}\cos\dfrac{\theta_C-\theta_A}{2}} = \frac{1}{-2\sin\theta_C(\sin\theta_C + \sin\theta_A)}$$

which equals the coefficient of 1 in (14). Similar arguments show that also the coefficients of $\sin\theta_V$, $\cos\theta_V$ in (10) and (14) coincide. We include below for completeness also these calculations, and we start with β:

$$\beta=\frac{\sin\theta_A\cot\theta_C-\sin\theta_C\cot\dfrac{\theta_A+\theta_C}{2}+\cos\theta_C-\cos\theta_A}{-2\sin(\theta_C-\theta_A)}=$$

$$=\frac{\sin\dfrac{\theta_A+\theta_C}{2}\sin\theta_A\cos\theta_C-\sin\theta_C\sin\theta_C\cos\dfrac{\theta_A+\theta_C}{2}}{-2\sin\theta_C\sin\dfrac{\theta_A+\theta_C}{2}\sin(\theta_C-\theta_A)}+$$

$$+\frac{(\cos\theta_C-\cos\theta_A)\sin\dfrac{\theta_A+\theta_C}{2}\sin\theta_C}{-2\sin\theta_C\sin\dfrac{\theta_A+\theta_C}{2}\sin(\theta_C-\theta_A)}. \tag{15}$$

Transforming the difference $(\cos\theta_C-\cos\theta_A)$ into a product, the numerator of the second term in (15) can be expressed as

$$2\left(1-\cos^2\frac{\theta_A+\theta_C}{2}\right)\sin\frac{\theta_A-\theta_C}{2}\sin\theta_C,$$

so that the entire numerator of (15) equals

$$2\sin\frac{\theta_A-\theta_C}{2}\sin\theta_C+\delta$$

with

$$\begin{aligned}\delta &= \sin\frac{\theta_A+\theta_C}{2}\sin\theta_A\cos\theta_C-\sin\theta_C\sin\theta_C\cos\frac{\theta_A+\theta_C}{2}+\\ &\quad+2\cos^2\frac{\theta_A+\theta_C}{2}\sin\frac{\theta_C-\theta_A}{2}\sin\theta_C=\\ &= \sin\frac{\theta_A+\theta_C}{2}\sin\theta_A\cos\theta_C-\sin\theta_C\sin\theta_C\cos\frac{\theta_A+\theta_C}{2}+\\ &\quad+\cos\frac{\theta_A+\theta_C}{2}(\sin\theta_C-\sin\theta_A)\sin\theta_C=\\ &= \sin\frac{\theta_A+\theta_C}{2}\sin\theta_A\cos\theta_C-\cos\frac{\theta_A+\theta_C}{2}\sin\theta_A\sin\theta_C=\\ &= \sin\frac{\theta_A-\theta_C}{2}\sin\theta_A.\end{aligned} \tag{16}$$

Using this back into (15) we obtain

$$\beta=\frac{2\sin\dfrac{\theta_A-\theta_C}{2}\sin\theta_C+\sin\dfrac{\theta_A-\theta_C}{2}\sin\theta_A}{-2\sin\theta_C\sin\dfrac{\theta_A+\theta_C}{2}\sin(\theta_C-\theta_A)}=$$

$$= \frac{2 + \dfrac{\sin\theta_A}{\sin\theta_C}}{4\sin\theta_C \sin\dfrac{\theta_A+\theta_C}{2}\cos\dfrac{\theta_A-\theta_C}{2}} = \frac{2 + \dfrac{\sin\theta_A}{\sin\theta_C}}{2\sin\theta_C(\sin\theta_A + \sin\theta_C)}$$

which equals indeed the coefficient of $\sin\theta_V$ in (14), as required.

As for γ, we write

$$\gamma = \frac{\cos\theta_A\cot\theta_C - \cos\theta_C\cot\dfrac{\theta_A+\theta_C}{2} - \sin\theta_C + \sin\theta_A}{-2\sin(\theta_C-\theta_A)} =$$

$$= \frac{\sin\dfrac{\theta_A+\theta_C}{2}\cos\theta_A\cos\theta_C - \sin\theta_C\cos\theta_C\cos\dfrac{\theta_A+\theta_C}{2}}{-2\sin\theta_C\sin\dfrac{\theta_A+\theta_C}{2}\sin(\theta_C-\theta_A)} +$$

$$+ \frac{(-\sin\theta_C + \sin\theta_A)\sin\dfrac{\theta_A+\theta_C}{2}\sin\theta_C}{-2\sin\theta_C\sin\dfrac{\theta_A+\theta_C}{2}\sin(\theta_C-\theta_A)}. \tag{17}$$

Using now that

$$\begin{aligned}(\sin\theta_A - \sin\theta_C)\sin\frac{\theta_A+\theta_C}{2} &= 2\sin\frac{\theta_A-\theta_C}{2}\cos\frac{\theta_A+\theta_C}{2}\sin\frac{\theta_A+\theta_C}{2}\\ &= (\cos\theta_C - \cos\theta_A)\cos\frac{\theta_A+\theta_C}{2}\end{aligned}$$

back into (17) we obtain

$$\gamma = \frac{\sin\dfrac{\theta_A+\theta_C}{2}\cos\theta_A\cos\theta_C - \sin\theta_C\cos\theta_A\cos\dfrac{\theta_A+\theta_C}{2}}{-2\sin\theta_C\sin\dfrac{\theta_A+\theta_C}{2}\sin(\theta_C-\theta_A)} =$$

$$= \frac{\sin\dfrac{\theta_A-\theta_C}{2}\cos\theta_A}{-2\sin\theta_C\sin\dfrac{\theta_A+\theta_C}{2}\sin(\theta_C-\theta_A)} =$$

$$= \frac{\cos\theta_A}{4\sin\theta_C\sin\dfrac{\theta_A+\theta_C}{2}\cos\dfrac{\theta_C-\theta_A}{2}} = \frac{\cos\theta_A}{2\sin\theta_C(\sin(\theta_C) + \sin(\theta_A))}$$

which equals, as required, the coefficient of $\cos\theta_V$ in (14). The proof is complete.