PROBLEMS FOR COMPETITIONS AND OLYMPIADS

Junior Level

C.O:5187. Let n be a positive integer. Show that 21 divides $10^n - 2^n - 8$ if and only if 6 divides $n^2 - 1$.

Ovidiu Ţâţan, Râmnicu Sărat

C.O:5188. Prove that

$$\frac{a\sqrt{a}}{\sqrt{b+c}} + \frac{b\sqrt{b}}{\sqrt{c+a}} + \frac{c\sqrt{c}}{\sqrt{a+b}} \geq \frac{1}{2}\left(\sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)}\right),$$

for any $a, b, c \in (0, \infty)$.

Cătălin Cristea, Craiova

C.O:5189. Suppose a and b are odd integers. Show that for any $x, y \in \mathbb{N}^*$, the number $\frac{x}{y}a^2 + \frac{y}{x}b^2$ is not a square.

Ioan Băetu, Botoşani

C.O:5190. Let a be a positive integer and let b be the number obtained by reversing the order of digits of the number a. Show that 9 do not divide $ab - 2$.

Cosmin Manea and *Dragoş Petrică*, Piteşti

Senior Level

C.O:5191. Let x, y, z be real numbers with $x + y + z = 3$. Prove that

$$2\left(x^3 + y^3 + z^3\right) \geq x^2 + y^2 + z^2 + 3.$$

Romeo Raicu, Blaj

C.O:5192. Let ABC be a triangle and consider the points M, N, P such that A and M are separated by BC, B and N are separated by CA, C and P are separated by AB. Let X, Y, Z be the intersection points of the pairs of lines AM and BC, BN and CA, CP and AB respectively. Show that triangles BMC, CNA and PAB have equal areas if and only if the centroids of the triangles ABC, MNP, XYZ are collinear.

Claudiu Mândrilă, student, Târgovişte

C.O:5193. Three circles of distinct radii r_1, r_2, r_3 are mutually external tangent. Consider the circumscribed triangle, such that each side is tangent to exactly two of the three circles. Prove that the inradius of this triangle is

$$r = \frac{\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} + \sqrt{r_1 + r_2 + r_3}}{\frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}} + \frac{1}{\sqrt{r_3}}}.$$

I. C. Drăghicescu, Bucharest

C.O:5194. a) Find an example of a polynomial $f \in \mathbb{Q}[X]$ such that $f\left(X^n\right)$ is irreducible for any positive integer n.

b) Let K be a finite field and let $f \in K[X]$, $\mathrm{grad} f \geq 1$. Show that there exists $n \in \mathbb{N}^*$ such that $f(X^n)$ is reducible.

Marian Andronache, Bucharest