ARTICOLE ŞI NOTE MATEMATICE

SOME CONSEQUENCES OF W.J.BLUNDON'S INEQUALITY

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Abstract. This paper presents some rafined geometric inequalities in triangle, based on Blundon's inequality.

Keywords: Blundon's inequality, Gerretsen's inequality.
MSC : 51M16

In any triangle ABC we shall denote $a = BC$, $b = AC$, $c = AB$, $p = \dfrac{a+b+c}{2}$, R the radius of circumcircle and r the radius of incircle.

The *Blundon*'s inequality was obtained for the first time by *E. Rouché* in the year 1851, but in the mathematical literature it is known as *Blundon*'s inequality.

We shall present several results published in many research papers, which are true in any triangle.

Theorem 1. *In any triangle ABC are valid the following inequalities*:

1) $\dfrac{27Rr}{2} \le 16Rr - 5r^2 \le p^2 \le 4R^2 + 4Rr + 3r^2$;

2) $24Rr - 12r^2 \le a^2 + b^2 + c^2 \le 8R^2 + 4r^2$;

3) $R \ge 2r$;

4) $6\sqrt{3}r \le a + b + c \le 4R + \left(6\sqrt{3}r - 8\right) r$;

5) $\dfrac{4r\left(12R^2 - 11Rr + r^2\right)}{3R - 2r} \le p^2 \le \dfrac{R\left(4R+r\right)^2}{2(2R-r)} \le 4R^2 + 4Rr + 3r^2$;

6) $\left|p^2 - \left(2R^2 + 10Rr - r^2\right)\right| \le 2(R-2r)\sqrt{R(R-2r)}$;

7) $a^2 + b^2 + c^2 \le \dfrac{72R^4}{9R^2 - 4r^2}$;

8) $a^3 + b^3 + c^3 \le 4pR(2R - r) \le 4R\left[2R + \left(3\sqrt{3} - 3\right) r\right](2R - r)$.

In the paper [1] *W. J. Blundon* proved the inequality $p \le 2R + \left(3\sqrt{3} - 4\right) r$. *Blundon*'s inequality which is represented by inequality 6) from Theorem 1 was proved in the paper [2].

Also in this paper *W. J. Blundon* proved that if $p \le kR + hr$ in any triangle ABC then $2R + \left(3\sqrt{3} - 4\right) r \le kR + hr$.

The inequality 7) from Theorem 1 was given by prof. *I. V. Maftei.* The inequality 5) from Theorem 1 was proved by *S. J. Bilčev* and *E. A. Velikova* in the paper [4] and represents an extension of the inequality established by

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A. Bager in the paper [5]. The inequality 8) from Theorem 1 was given in the book [6].

In the following we shall use the next result:

Lemma 1. *In any triangle* ABC *are valid the followings identities*:

$$a^2+b^2+c^2=2\left(p^2-r^2-4Rr\right), \tag{1}$$

$$a^3+b^3+c^3=2p\left(p^2-3r^2-6Rr\right). \tag{2}$$

In the next theorem we shall improve the *Gerretsen*'s inequality which states that in any triangle ABC we have $a^2+b^2+c^2\le 8R^2+4r^2$.

Theorem 2. *In any triangle* ABC *are true the following inequalities*:

$$a^2+b^2+c^2\le\frac{36\left(8R^4+tr^4\right)}{36R^2+(t-16)r^2},\qquad \forall\, t\in[-2,6] \tag{3}$$

$$a^2+b^2+c^2\le\frac{36\left(4R^4+3r^4\right)}{18R^2-5r^2}\le\frac{72R^4}{9R^2-4r^2}\le 8R^2+4r^2, \tag{4}$$

$$a^2+b^2+c^2\le 8R^2+\frac{p^2r^2}{2R^2}+\frac{5r^4}{2R^2}\le\frac{72R^4}{9R^2-4r^2}\le 8R^2+4r^2. \tag{5}$$

Proof. In order to prove the inequality (3) we shall consider the function $f:[-2,6]\to\mathbb{R}$,

$$f(t)=\frac{36\left(8x^4+t\right)r^2}{t+36x^2-16}\quad\text{ where }\ x=\frac{R}{r}\in[2,\infty).$$

We have $f'(t)=\dfrac{-288\left(x^2-4\right)\left(x^2-\dfrac{1}{2}\right)r^2}{\left(t+36x^2-16\right)^2}\le 0$, $\forall\, t\in[-2,6]$, because $x\in[2,\infty)$. It follows that f is a decreasing function.

The inequality

$$a^2+b^2+c^2\le\frac{36\left(4R^4+3r^4\right)}{18R^2-5r^2}, \tag{6}$$

is equivalent with the inequality $a^2+b^2+c^2\le f(6)$ which implies the inequality (3).

By identity (1) from Lemma 1 if follows that inequality (6) is equivalent with the following inequality:

$$2\left(p^2-r^2-4Rr\right)\le\frac{36\left(4R^4+3r^4\right)}{18R^2-5r^2}$$

or in another form :

$$p^2\le r^2+4Rr+\frac{18\left(4R^4+3r^4\right)}{18R^2-5r^2}. \tag{7}$$

According the *Blundon's* inequality, in order to prove inequality (7) it will be sufficient to prove that:

$$2R^2 + 10Rr - r^2 + 2\sqrt{R(R-2r)^3} \le r^2 + 4Rr + \frac{18\left(4R^4 + 3r^4\right)}{18R^2 - 5r^2}. \tag{8}$$

Using the notation $x = \dfrac{R}{r} \in [2, \infty)$ inequality (8) may be written as:

$$2x^2 + 10x - 1 + 2\sqrt{x(x-2)^3} \le 1 + 4x + \frac{18\left(4x^4 + 3\right)}{18x^2 - 5}$$

or in an equivalent form:

$$x(x-2)^3\left(36x^2 - 10\right)^2 \le (x-2)^2\left(36x^3 - 36x^2 - 26x - 22\right)^2. \tag{9}$$

The inequality (9) may be written as:

$$\left(18x^2 - 5\right)^2\left(x^2 - 2x\right) \le \left(18x^3 - 18x^2 - 13x - 11\right)^2$$

or in an equivalent form:

$$36x^2\left(x^2 - 8x + 15\right) + 336x + 121 \ge 0, \quad x \in [2, \infty). \tag{10}$$

If $x \in [2,3] \cup [5, \infty)$, the inequality (10) is true because $x^2 - 8x + 15 \ge 0$.
If $x \in (3, 5)$ we have the sequence of inequalities:

$$\begin{gathered} 36x^2\left(x^2 - 8x + 15\right) + 336x + 121 \ge \\ \ge -36x^2 + 336x + 121 \ge -900 + 336 \cdot 3 + 121 = 229 \ge 0 \end{gathered}$$

because $x^2 - 8x + 15 \ge -1$ and $x^2 < 25$.

So inequality (3) is proved.

In order to prove the inequalities (4) note that $a^2 + b^2 + c^2 \le f(6) \le \le f(0) \le f(-2)$ because f is a decreasing function.

In order to prove the inequalities (4) note that $a^2 + b^2 + c^2 \le 8R^2 + + \dfrac{p^2r^2 + 5r^4}{2R^2}$ it will be sufficient according with the Lemma 1 to prove that $2\left(p^2 - r^2 - 4Rr\right) \le 8R^2 + \dfrac{p^2r^2 + 5r^4}{2R^2}$ (or with the equivalent form:

$$p^2 \le \frac{16R^4 + 4R^2r^2 + 16R^3r + 5r^4}{4R^2 - r^2}. \tag{11}$$

In order to prove the inequality (11), it will be sufficient to prove according *Blundon's* inequality the following:

$$2R^2 + 10R - r^2 + 2(R-2r)\sqrt{R(R-2r)} \le \frac{16R^4 + 4R^2r^2 + 16R^3r + 5r^4}{4R^2 - r^2}$$

or in an equivalent form:

$$\begin{gathered} \left(4R^2 - r^2\right)\left(2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)}\right) \le \\ \le 16R^4 + 4R^2r^2 + 16R^3r + 5r^4. \end{gathered} \tag{12}$$

The inequality (12) may be written also in the following form:

$$2(R-2r)\left(4R^2-r^2\right)\sqrt{R(R-2r)}\leq \\ \leq 8R^4-24R^3r+10R^2r^2+10Rr^3+4r^4, \tag{13}$$

or with the equivalent form:

$$\left(4R^2-r^2\right)\left(R^2-2Rr\right)\leq\left(4R^3-4R^2r-3Rr^2-r^3\right)^2$$

which is equivalent with the inequality $16R^2+8Rr+r^2\geq 0$.

In order to prove the inequality $8R^2+\dfrac{p^2r^2+5r^4}{2R^2}\leq\dfrac{72R^4}{9R^2-4r^2}$ it will be sufficient according the inequality $p^2\leq 4R^2+4Rr+3r^2$ to prove that

$$8R^2+\frac{\left(4R^2+4Rr+3r^2\right)r^2+5r^4}{2R^2}\leq\frac{72R^4}{9R^2-4r^2}. \tag{14}$$

The inequality (14) may be written in an equivalent form as

$$\left(9R^2-4r^2\right)\left(8R^4+2R^2r^2+2Rr^3+4r^4\right)\leq 72R^6. \tag{15}$$

After performing some calculations, inequality (15) may be written as:

$$14R^4-18R^3r-28R^2r^2+8Rr^3+16r^4\geq 0$$

or equivalent as:

$$2(R-2r)\left(7R^3+5R^2r-4Rr^2-4r^3\right)\geq 0. \tag{16}$$

Because $R\geq 2r$ and $5R^2r-4Rr^2-4r^3=r\left[4R(R-r)+R^2-4r^2\right]\geq 0$ if follows that the inequality (16) is true.

The purpose of the following theorem is to prove that the inequality of *Gerretsen* is the best if we suppose that where $a^2+b^2+c^2\leq\alpha R^2+\beta Rr+\gamma r^2$ where $\alpha,\beta,\gamma\in\mathbb{R}$ şi $\beta\geq 0$.

This statement was proved by *L.Panaitopol* in the paper [7] with the supplementary hypotesis $\beta=0$.

Theorem 3. *If α, β and γ are real numbers with $\beta\geq 0$ and with the property that the inequality $a^2+b^2+c^2\leq\alpha R^2+\beta Rr+\gamma r^2$ is true in every triangle ABC, then we have the inequality $8R^2+4r^2\leq\alpha R^2+\beta Rr+\gamma r^2$ in every triangle ABC.*

Proof. If the triangle ABC is equilateral then from the inequality $a^2+b^2+c^2\leq\alpha R^2+\beta Rr+\gamma r^2$ it follows that:

$$4\alpha+2\beta+\gamma\geq 36. \tag{17}$$

If we consider the case of the isoscel triangle with $b=c$ and if we let a tends to zero we obtain:

$$\alpha\geq 8. \tag{18}$$

From (17), (18) and $R\geq 2r$ we shall obtain the following inequalities

$$(\alpha-8)R^2+\beta Rr+(\gamma-4)r^2\geq r^2\left[4(\alpha-8)+2\beta+\gamma-4\right]= \\ =r^2(4\alpha+2\beta+y-36)\geq 0,$$

or equivaletly $8R^2 + 4r^2 \le \alpha R^2 + \beta Rr + \gamma r^2$.

The following theorem establish an analogous inequality with the inequality of *Gerretsen*.

Theorem 4. *In every triangle ABC is true the following inequality*:

$$a^3 + b^3 + c^3 \le 16R^3 - 6Rr^2 + \left(72\sqrt{3} - 116\right) r^3. \tag{19}$$

Proof. According with Lemma 1 the inequality (19) is equivalent with the following inequality:

$$2p\left(p^2 - 3r^2 - 6Rr\right) \le 16R^3 - 6Rr^2 + \left(72\sqrt{3} - 116\right) r^3. \tag{20}$$

According *Blundon*'s inequality, in order to prove the inequality (20) it will be sufficient to prove that:

$$\begin{gathered}\sqrt{2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)}}\times \\ \times\left(2R^2 + 4Rr - 4r^2 + 2(R-2r)\sqrt{R(R-2r)}\right) \le \\ \le 8R^3 - 3Rr^2 + \left(36\sqrt{3} - 58\right) r^3.\end{gathered} \tag{21}$$

If we square the inequality (21) and we denote $x = \dfrac{R}{r}$ it follows that we have to prove that:

$$\begin{gathered}\left[2x^2 + 10x - 1 + (2x-4)\sqrt{x^2-2x}\right]\left[2x^2 + 4x - 4 + (2x-4)\sqrt{x^2-2x}\right] \le \\ \le \left(8x^3 - 3x + 36\sqrt{3} - 58\right)^2.\end{gathered} \tag{22}$$

After some calculation it follows that the inequality (22) is equivalent with:

$$\begin{gathered}\left(x^4 + 3x^3 + \frac{27}{4}x^2 - \frac{19}{2}x + \frac{3}{2}\right)^2\left(x^2 - 2x\right) \le \\ \le \left(x^5 + 2x^4 + \frac{13}{4}x^3 + \frac{576\sqrt{3} - 1528}{32}x^2 + \right. \\ \left. + \frac{1152\sqrt{3} - 1751}{32}x + \frac{2088\sqrt{3} - 3634}{32}\right)^2.\end{gathered} \tag{23}$$

In order to prove the inequality (23) it will be sufficient to prove that:

$$\begin{gathered}\left(x^4 + 3x^3 + \frac{27}{4}x^2 - 9x + 2\right)^2\left(x^2 - 2x\right) \le \\ \le \left(x^5 + 2x^4 + \frac{13}{4}x^3 - 17x^2 + 7x - 1\right)^2, \ x \in [2, \infty)\end{gathered} \tag{24}$$

After some calculations we obtain that the inequality (24) is equivalent with the following true inequality:

$$\frac{3}{2}x^7 + 6x^6 + \frac{129}{8}x^5 + \frac{7}{2}x^4 + \frac{15}{2}x^3 + 7x^2 - 6x + 1, \ x \in [2, \infty). \tag{25}$$

We shall prove in the sequel that the inequality (19) is the best of the inequalities of the type $a^3+b^3+c^3 \leq \alpha R^3+\beta R^2 r+\gamma R r^2+\delta r^3$ with $\gamma \geq -6$.

Theorem 5. *Let α, β, γ, δ real numbers with $\gamma \geq -6$ and with the property that in every triangle ABC we have*:

$$a^3+b^3+c^3 \leq \alpha R^3+\beta R^2 r+\gamma R r^2+\delta r^3.$$

Then in every triangle ABC is true the following inequality:

$$\alpha R^3+\beta R^2 r+\gamma R r^2+\delta r^3 \geq 16R^3-6Rr^2+\left(72\sqrt{3}-116\right) r^3.$$

Proof. If we consider the case of equilateral triangle then from the inequality:

$$a^3+b^3+c^3 \leq \alpha R^3+\beta R^2 r+\gamma R r^2+\delta r^3$$

we obtain that:

$$8\alpha+4\beta+2\gamma+\delta \geq 72\sqrt{3}. \tag{26}$$

In the case of the isoscel triangle with $b=c$ and with a tends zero we obtain:

$$\alpha \geq 16. \tag{27}$$

According with (26), (27) and $R \geq 2r$ it follows that:

$$\begin{aligned}
&(\alpha-16)R^3+\beta R^2 r+(\gamma+6)Rr^2+\left(\delta-72\sqrt{3}+116\right) r^3 \geq \\
&\geq \left[8(\alpha-16)+4\beta+2(\gamma+6)+\delta-72\sqrt{3}+116\right] r^3 = \\
&= \left(8\alpha+4\beta+2\gamma+\delta-72\sqrt{3}\right) r^3 \geq 0.
\end{aligned}$$

In conclusion $\alpha R^3+\beta R^2 r+\gamma R r^2+\delta r^3 \geq 16R^3-6Rr^2+\left(72\sqrt{3}-116\right) r^3$ in every triangle ABC.

In the sequel we shall prove an inequality which improves the left of inequality 5) from theorem 1.

Theorem 6. *In every triangle ABC are true the following inequalities*:

$$p^2 \geq \frac{r\left[16R^2+(16t-4)Rr-(5t+2)r^2\right]}{R+tr}, \quad t \in [-1,\infty) \tag{28}$$

$$p^2 \geq \frac{r\left(16R^2-20Rr+3r^2\right)}{R-r} \geq \frac{4r\left(12R^2-11Rr+r^2\right)}{3R-2r}. \tag{29}$$

Proof. According *Blundon*'s inequality in order to prove inequality (28) it will be sufficient to prove that:

$$\begin{aligned}
&2R^2+10Rr-r^2-2(R-2r)\sqrt{R(R-2r)} \geq \\
&\geq \frac{16R^2+(16t-4)Rr-(5t+2)r^2}{R+tr}, \quad t \in [-1,\infty).
\end{aligned} \tag{30}$$

Inequality (30) is equivalent with the inequality:

$$2R^2+(2t-2)Rr-(2t+1)r^2 \geq 2(R+tr)\sqrt{R^2-2Rr}. \tag{31}$$

If we square inequality (31) we obtain the inequality:

$$4R^4 + \left(4t^2 - 8t + 4\right) R^2r^2 + \left(4t^2 + 4t + 1\right) r^4 + (8t - 8)R^3r - (8t + 4)R^2r^2 - $$
$$- \left(8t^2 - 4t - 4\right) Rr^3 \geq 4R^4 + 8tR^3r + 4t^2R^2r^2 - 8R^3r - 16tR^2r^2 - 8t^2Rr^3,$$

which is equivalent with the following true inequality:

$$4(t + 1)Rr + (2R + r)^2 \geq 0, \qquad t \in [-1, \infty).$$

In order to prove inequality (29) we shall consider the function $f : [-1, \infty) \to \mathbb{R},\ f(t) = \dfrac{\left[\left(16Rr - 5r^2\right) t + 16R^2 - 4Rr - 2r^2\right] r}{R + tr}$.

Because $f'(t) = \dfrac{r\left(2r^2 - Rr\right)}{(R + tr))^2} \leq 0$, $\forall\, t \in [-1, \infty)$ it follows that f is a decreasing function.

Because $f(-1) \geq f(0)$ and $f(0) = \dfrac{\left(16R^2 - 4Rr - 2r^2\right) r}{R}$ and $f(-1) = = \dfrac{\left(16R^2 - 20Rr + 3r^2\right) r}{R - r}$ it follows the inequality (29).

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O REZOLVARE VECTORIALĂ A UNEI PROBLEME

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Abstract. This article illustrates how vectorial methods can sotimes provide easier solutions.

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Este binecunoscută problema următoare:

Fie două cercuri secante $\mathcal{C}_1$, $\mathcal{C}_2$ şi numărul real $k > 0$. Prin unul dintre punctele lor de intersecţie se duce o dreaptă variabilă, care intersectează a doua oară cercurile $\mathcal{C}_1$ şi $\mathcal{C}_2$ în M_1, respectiv M_2. Să se afle locul geometric al punctului $M \in (M_1M_2)$, pentru care $MM_1 = k \cdot MM_2$.

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