PROBLEMS FOR COMPETITIONS AND OLYMPIADS

Junior Level

C.O:5115. Let a, b, c, x, y, z be real numbers such that:

$$a + b + c = x^2 + y^2 + z^2 = 1.$$

Prove that:

$$a(x + b) + b(y + c) + c(z + a) \leq 1.$$

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C.O:5116. Is there a positive integer n such that 7 divides $2^n + 1$?

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C.O:5117. Let k be an integer, $k \geq 2$. Show that there exist three distinct integers in the interval $\big(k^3, (k+1)^3\big)$ whose product is a perfect cube.

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C.O:5118. Let $ABCD$ a regular tetrahedron of unit edge and let P be a point inside it. Show that the sum of the distances from P to all the edges of the tetrahedron is greater than or equal to $\dfrac{3\sqrt{2}}{2}$.

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Senior Level

C.O:5119. Let $F : \mathbb{N}^* \to \mathbb{N}^*$ be a function satisfying the properties: $F(mn) = F(m)F(n)$ for all $m, n \in \mathbb{N}^*$ and $F(p) \geq 2$ for any prime p. Prove that there exists a function $f : \mathbb{N}^* \to \mathbb{N}^*$ such that $F(n) = \sum\limits_{d|n} f(d)$ for all $n \in \mathbb{N}^*$.

Find f when $F = 1_{\mathbb{N}^*}$.

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C.O:5120. Consider $f = aX^2 + bX + c \in \mathbb{Z}[X]$ such that $f(n)$ is a square for any $n \in \mathbb{N}^*$. Prove that there exist $g \in \mathbb{Z}[X]$ with $f = g^2$.

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C.O:5121. Let ABC be a triangle and let T_a be the area of the triangle with vertices in the tangency points of the excircle coresponding to A with the lines AB, BC, CA. Define similarly T_b and T_c. Let S be the area of the triangle with vertices in the tangency points of the incircle with the sides of the triangle. Show that

$$\frac{1}{T_a} + \frac{1}{T_b} + \frac{1}{T_c} = \frac{1}{S}.$$

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C.O:5122. Let A be a ring such that for all $x \in A$ there exist $m, n \in \mathbb{N}$, $(m, n) = 1$ with $x^m, x^n \in Z(A)$. Show that the ring is commutative.

(Denote $Z(A) = \{x \in A \mid xy = yx,\ \forall\, y \in A\}$.)

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