Fie $X' \in (B'N')$ şi $Y \in (AP)$. Notăm $\frac{X'B'}{X'N'} = x$, $\frac{YP}{YA} = y$, $\frac{ZA}{ZB} = z$. Să se demonstreze că planele $(AA'X')$, $BB'Y)$, $(CC'Z)$ se intersectează după o dreaptă dacă şi numai dacă $x \cdot y \cdot z = 1$.

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SOME APPLICATIONS OF A LIMIT PROBLEM

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Abstract. We solve some problems using a limit problem which we proposed in Gazeta Matematică in 2007.

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L. Pârşan [3] proposed in Gazeta Matematică, in 1978, the following problem.

Problem 1. *Let $a, r \in (0, +\infty)$ and let $(a_n)_{n\in\mathbb{N}}$ be the sequence defined by $a_n = a + (n-1)r$, for each $n \in \mathbb{N}$. Evaluate* $\lim\limits_{n\to\infty} \frac{a_{2n+1}a_{2n+3}\cdots a_{4n+1}}{a_{2n}a_{2n+2}\cdots a_{4n}}$.
(*The limit is* $\sqrt{2}$.)

We mention that $\mathbb{N} = \{1, 2, \ldots, n, \ldots\}$.

We have given a generalization of Problem .

Problem 2. (*A. Sîntămărian* [4], [5]). *Let $k \in \mathbb{N} \cup \{0\}$, $p \in \mathbb{N} \setminus \{1\}$, $q, s \in \mathbb{N}$ and $a, r \in (0, +\infty)$. We consider the sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n = a + (n-1)r$, for each $n \in \mathbb{N}$.*

Evaluate $\lim\limits_{n\to\infty} \frac{a_{qn+k+1}a_{qn+k+1+p}\cdots a_{qn+k+1+s(n-1)p}}{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}}$.

$\left(\textit{The limit is } \sqrt[p]{\frac{ps+q}{q}}.\right)$

Using the above-mentioned problem, we have found the following limits (we specify that the notations are those from Problem 2):

$$\lim_{n\to\infty} \sqrt[n]{\frac{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}}{(n!)^s}} = \left(\sqrt[p]{\frac{ps+q}{q}}\right)^q [(ps+q)r]^s \text{ ([4], [5]);}$$

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$$\lim_{n\to\infty} \frac{\sqrt[n]{a_{qn+k}a_{qn+k+p}\cdots a_{qn+k+s(n-1)p}}}{n^s} = \left(\sqrt[p]{\frac{ps+q}{q}}\right)^q \left[\frac{(ps+q)r}{\mathrm{e}}\right]^s \quad ([4],\ [5]);$$

$$\lim_{n\to\infty} \frac{\binom{4n}{2n}}{4^n\binom{2n}{n}} = \frac{\sqrt{2}}{2} \quad ([1],[5]); \qquad \lim_{n\to\infty} \frac{4^n\binom{4n}{2n}}{\binom{2n}{n}\binom{8n}{4n}} = 1, \quad ([1]);$$

$$\lim_{n\to\infty} \frac{3^{3n}\binom{2n}{n}^2}{\binom{3n}{n}\binom{6n}{3n}} = 2 \qquad ([5]).$$

Further on we shall give other problems which can be solved using the limit from Problem 2.

Problem 3. *Evaluate* $\lim\limits_{n\to\infty} \dfrac{5^{5n}\binom{2n}{n}^3}{\binom{10n}{5n}\binom{5n}{n}\binom{4n}{2n}}$.

Solution. We have:

$$x_n := \frac{5^{5n}\binom{2n}{n}^3}{\binom{10n}{5n}\binom{5n}{n}\binom{4n}{2n}} = \frac{[(5n+5)(5n+10)\cdots(10n)]^5}{(5n+1)(5n+2)\cdots(10n)} =$$

$$= \left[\frac{(5n+5)(5n+10)\cdots(10n)}{(5n+4)(5n+9)\cdots(10n-1)}\right]^4 \cdot \left[\frac{(5n+4)(5n+9)\cdots(10n-1)}{(5n+3)(5n+8)\cdots(10n-2)}\right]^3 \times$$

$$\times \left[\frac{(5n+3)(5n+8)\cdots(10n-2)}{(5n+2)(5n+7)\cdots(10n-3)}\right]^2 \cdot \frac{(5n+2)(5n+7)\cdots(10n-3)}{(5n+1)(5n+6)\cdots(10n-4)},$$

for each $n \in \mathbb{N}$.

Choosing $a = r = 1$, $p = 5$, $q = 5$ and $s = 1$ in Problem 2, we obtain:

$$\lim_{n\to\infty} \frac{(5n+k+1)(5n+k+6)\cdots(10n+k-4)}{(5n+k)(5n+k+5)\cdots(10n+k-5)} = \sqrt[5]{2},$$

for each $k \in \{1,2,3,4\}$. So:

$$\lim_{n\to\infty} x_n = (\sqrt[5]{2})^4(\sqrt[5]{2})^3(\sqrt[5]{2})^2\sqrt[5]{2} = 4. \qquad \square$$

Problem 4. *Let* $p \in \mathbb{N}\setminus\{1\}$ *and* $\alpha, \beta \in \mathbb{N}$, *with* $\alpha < \beta$. *We consider the sequences* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$, *with* $x_1 \neq 0$ *and* $y_1 \neq 0$, *defined by the recurrence relations :*

$$nx_{n+1} = \left(n + \frac{1}{p}\right)x_n \quad \text{and} \quad ny_{n+1} = (n+1)y_n,$$

for each $n \in \mathbb{N}$.

Evaluate $\lim\limits_{n\to\infty} \dfrac{y_{\alpha n}(x_{\alpha n}+x_{\alpha n+1}+\cdots+x_{\beta n})}{x_{\alpha n}(y_{\alpha n}+y_{\alpha n+1}+\cdots+y_{\beta n})}$.

Solution. Having in view the recurrence relation $nx_{n+1} = \left(n+\dfrac{1}{p}\right)x_n$, for $n \in \mathbb{N}$, we can write that:

$$\prod_{j=1}^{n} pjx_{j+1} = \prod_{j=1}^{n}(pj+1)x_j,$$

$$\sum_{j=1}^{n} pjx_{j+1} = \sum_{j=1}^{n}[p(j-1)x_j + (p+1)x_j],$$

for each $n \in \mathbb{N}$. We get:

$$x_{n+1} = \frac{(p+1)(2p+1)\cdots(np+1)}{p(2p)\cdots(np)}x_1,$$

$$x_1 + x_2 + \cdots + x_n = \frac{pn}{p+1}x_{n+1},$$

for any $n \in \mathbb{N}$. Therefore:

$$\begin{aligned} x_{\alpha n}+x_{\alpha n+1}+\cdots+x_{\beta n} &= \frac{p\beta n}{p+1}x_{\beta n+1} - \frac{p(\alpha n-1)}{p+1}x_{\alpha n} = \\ &= \frac{p\beta n}{p+1}\cdot\frac{(p+1)(2p+1)\cdots(\beta np+1)}{p(2p)\cdots(\beta np)}x_1 - \frac{p(\alpha n-1)}{p+1}x_{\alpha n} = \\ &= \frac{p\beta n}{p+1}\cdot\frac{(\alpha np+1)((\alpha n+1)p+1)\cdots(\beta np+1)}{(\alpha np)((\alpha n+1)p)\cdots(\beta np)}x_{\alpha n} - \frac{p(\alpha n-1)}{p+1}x_{\alpha n}, \end{aligned}$$

for each $n \in \mathbb{N}$.

The above relations, obtained for the sequence $(x_n)_{n\in\mathbb{N}}$, hold for $p=1$ too. Hence:

$$y_{n+1} = (n+1)y_1,$$

$$y_{\alpha n} + y_{\alpha n+1} + \cdots + y_{\beta n} = \frac{(\beta^2-\alpha^2)n+\alpha+\beta}{2\alpha}y_{\alpha n},$$

for each $n \in \mathbb{N}$.

It follows that:

$$\begin{aligned} \frac{y_{\alpha n}(x_{\alpha n}+x_{\alpha n+1}+\cdots+x_{\beta n})}{x_{\alpha n}(y_{\alpha n}+y_{\alpha n+1}+\cdots+y_{\beta n})} &= \frac{2p\alpha\beta n}{(p+1)[(\beta^2-\alpha^2)n+\alpha+\beta]}\times \\ &\times\frac{(\alpha np+1)((\alpha n+1)p+1)\cdots(\beta np-(\beta-\alpha)p+1)}{(\alpha np)((\alpha n+1)p)\cdots(\beta np-(\beta-\alpha)p)}\times \\ &\times\frac{(\beta np-(\beta-\alpha)p+p+1)\cdots(\beta np+1)}{(\beta np-(\beta-\alpha)p+p)\cdots(\beta np)} - \frac{2p\alpha(\alpha n-1)}{(p+1)[(\beta^2-\alpha^2)n+\alpha+\beta]}, \end{aligned}$$

for each $n \in \mathbb{N}$.

Choosing $a = r = 1$, $k = 0$, $q = \alpha p$ and $s = \beta - \alpha$ in Problem , we obtain:

$$\lim_{n\to\infty} \frac{(\alpha np+1)((\alpha n+1)p+1)\cdots(\beta np-(\beta-\alpha)p+1)}{(\alpha np)((\alpha n+1)p)\cdots(\beta np-(\beta-\alpha)p)} = \sqrt[p]{\frac{\beta}{\alpha}}.$$

So:

$$\lim_{n\to\infty} \frac{y_{\alpha n}(x_{\alpha n}+x_{\alpha n+1}+\cdots+x_{\beta n})}{x_{\alpha n}(y_{\alpha n}+y_{\alpha n+1}+\cdots+y_{\beta n})} = \frac{2p}{p+1}\cdot\frac{\frac{\beta}{\alpha}\sqrt[p]{\frac{\beta}{\alpha}}-1}{\left(\frac{\beta}{\alpha}\right)^2-1}. \qquad \square$$

Regarding Problem see [7]. Some particular cases of Problem we have given in [5, problems 66, 67] (see as well [6], [2]).

Problem 5. *Let $\lambda,\mu,\alpha,\beta\in\mathbb{N}$, with $\lambda\neq\mu$ and $\alpha<\beta$. We consider the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, with $x_1\neq 0$ and $y_1\neq 0$, defined by the recurrence relations*:

$$nx_{n+1} = \left(n+\frac{1}{\lambda}\right)x_n \quad\text{and}\quad ny_{n+1} = \left(n+\frac{1}{\mu}\right)y_n,$$

for each $n\in\mathbb{N}$. Evaluate $\lim\limits_{n\to\infty} \dfrac{y_{\alpha n}(x_{\alpha n}+x_{\alpha n+1}+\cdots+x_{\beta n})}{x_{\alpha n}(y_{\alpha n}+y_{\alpha n+1}+\cdots+y_{\beta n})}$.

Solution. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence, with $z_1\neq 0$, defined by the recurrence relation $nz_{n+1} = (n+1)z_n$, for each $n\in\mathbb{N}$. Using Problem , we can write that:

$$\lim_{n\to\infty} \frac{y_{\alpha n}(x_{\alpha n}+x_{\alpha n+1}+\cdots+x_{\beta n})}{x_{\alpha n}(y_{\alpha n}+y_{\alpha n+1}+\cdots+y_{\beta n})} =$$

$$= \lim_{n\to\infty}\left[\frac{z_{\alpha n}(x_{\alpha n}+x_{\alpha n+1}+\cdots+x_{\beta n})}{x_{\alpha n}(z_{\alpha n}+z_{\alpha n+1}+\cdots+z_{\beta n})}\cdot\frac{y_{\alpha n}(z_{\alpha n}+z_{\alpha n+1}+\cdots+z_{\beta n})}{z_{\alpha n}(y_{\alpha n}+y_{\alpha n+1}+\cdots+y_{\beta n})}\right] =$$

$$= \frac{2\lambda}{\lambda+1}\cdot\frac{\left(\frac{\beta}{\alpha}\right)^{\frac{\lambda+1}{\lambda}}-1}{\left(\frac{\beta}{\alpha}\right)^2-1}\cdot\frac{\mu+1}{2\mu}\cdot\frac{\left(\frac{\beta}{\alpha}\right)^2-1}{\left(\frac{\beta}{\alpha}\right)^{\frac{\mu+1}{\mu}}-1} = \frac{\lambda(\mu+1)}{\mu(\lambda+1)}\cdot\frac{\left(\frac{\beta}{\alpha}\right)^{\frac{\lambda+1}{\lambda}}-1}{\left(\frac{\beta}{\alpha}\right)^{\frac{\mu+1}{\mu}}-1}. \qquad \square$$

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