PROBLEMS FOR COMPETITIONS AND OLYMPIADS

Junior Level

C.O:5043. Show that the number $1001^2 + 1002^2 + \ldots + 2009^2$ is divisible by 35.

Gh. Achim, Mizil, Prahova

C.O:5044. Consider the triangle ABC with $A = 90°$ and $C = 30°$. Let $D \in (BC)$, $P \in (AB)$, such that $\frac{BD}{DC} = \frac{1}{3}$, $\frac{AP}{PB} = \frac{3}{2}$ and let E be the foot of the bisector of the angle B. Show that the lines CP, AD, BE are concurrent.

Marian Teler, Costeşti, Argeş

C.O:5045. Find all non-negative integers m, n so that:

$$5n^2 + 7n + 8 = \left(m^2 + m\right)(2n + 1).$$

Petre Stângescu, Bucharest

C.O:5046. Show that in any triangle ABC the following inequality holds:

$$\frac{b^2 + c^2}{m_a} + \frac{c^2 + a^2}{m_b} + \frac{a^2 + b^2}{m_c} \leq 12R.$$

Gh. Szöllösy, Sighetu Marmaţiei

C.O:5047. Consider $a \in \mathbb{N}^*$. Prove that the fraction $\frac{a^{6n+2} + a^{3n+1} + 1}{a^{6n+4} + a^{3n+2} + 1}$ reduces by $a^2 + a + 1$.

Ioan Ucu Crişan, Arad

C.O:5048. Let $a, b \in \mathbb{R}^*$ and $x, y \geq 0$. Prove that:

$$\frac{2\left(a^2 y + b^2 x\right)}{ab(a+b)^2} \leq \frac{1}{2}\left(\frac{x}{a^2} + \frac{y}{b^2}\right) \leq \frac{x + y}{(a+b)^2} + \frac{a^4 y + b^4 x}{a^2 b^2 (a+b)^2}.$$

Ovidiu Pop, Satu Mare

C.O:5049. Find all non-negative integers n such that $2n + 1$ divides

$$n^{n+3} \cdot 2^n + 4n^2 + 5n + 8.$$

Petre Stângescu, Bucharest

C.O:5050. Consider $a, b, c > 0$ with $a + b + c = 1$. Show that:

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + 3(ab + bc + ca) \geq \frac{11}{2}.$$

Gh. Ghiţă, Buzău

Senior Level

C.O:5051. Prove that there are infinitely many real numbers x such that $\left\{\frac{1}{x^n}\right\} = \{x\}^n$ for all $n \in \mathbb{N}^*$.

Francisc Bozgan, student, Bucharest

C.O:5052. A scalene triangle ABC has the internal bisectors AD, BE and CF, where $D \in (BC)$, $E \in (CA)$, $F \in (AB)$. The bisector lines of the segments $[AD]$, $[BE]$ and $[CF]$ intersect the lines BC, AC and AB at points A', B' and C' respectively. Prove that the points A', B' and C' are collinear.

Dan Nedeianu, Drobeta Tr. Severin

C.O:5053. Let $m, n \in \mathbb{N}^*$ and consider the complex numbers $z_1, z_2, \ldots, z_n$. Prove that:

$$2^n \left(|z_1|^m + |z_2|^m + \ldots + |z_n|^m\right) \leq \sum |\pm z_1 \pm z_2 \pm \ldots \pm z_n|^m,$$

for all choises of the signs $+$ and $-$.

Dan Ştefan Marinescu and *Viorel Cornea*, Hunedoara

C.O:5054. a) Let p be an integer, $p \geq 2$. Show that for any $x \in (0, \infty)$ and $n \in \mathbb{N}^*$, one has:

$$\left[\frac{\sqrt[p]{[x]}}{n}\right] = \left[\frac{\sqrt[p]{x}}{n}\right].$$

b) Let $q \in (1, \infty)$. Show that there exists $n \in \mathbb{N}^*$ and $x \in (0, \infty)$ such that:

$$\left[\frac{[x]^q}{n}\right] \neq \left[\frac{x^q}{n}\right].$$

Paul Georgescu and *Gabriel Popa*, Iaşi

C.O:5055. Show that for any $n \geq 3$ there exists a permutation σ of the set $\{1, 2, \ldots, n\}$ such that $\sigma(i) + \sigma(k) \neq 2\sigma(j)$, for all i, j, k with $1 \leq i < j < k \leq n$.

Vasile Pop, Cluj-Napoca

C.O:5056. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a continuous function. Prove that the following statements are equivalent:

a) $\forall n \in \mathbb{N}^*, \forall x, y \in I, |x - y| = \dfrac{1}{n} \Rightarrow |f(x) - f(y)| \leq |x - y|$;

b) $|f(x) \cdot f(y)| \leq |x - y|, \forall x, y \in I$.

Dan Ştefan Marinescu and *Viorel Cornea*, Hunedoara

C.O:5057. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function and consider $s : [0, 1] \to [0, 1]$ defined by $s(x) = \sup \left\{ c \in [0, x] \,\middle|\, \displaystyle\int_0^x f(t)\mathrm{d}t = xf(c) \right\}, \forall x \in [0, 1]$.

a) Show that $\displaystyle\int_0^x f(t)\mathrm{d}t = xf(s(x)), \forall x \in [0, 1]$.

b) Prove that s is a non-decreasing function.

Dan Ştefan Marinescu and *Viorel Cornea*, Hunedoara

C.O:5058. For an integer $n \geq 2$, denote by $\mathcal{M}_{2,n}(\{0,1\})$ the set of $2 \times n$ matrices whose elements belong to $\{0,1\}$. For $A \in \mathcal{M}_{2,n}(\{0,1\})$ let m_A be the number of nonzero 2×2 minors of A. Find

$$m = \max_{A \in \mathcal{M}_{2,n}(\{0,1\})} m_A.$$

Laura Năstăsescu, Bucharest

Editor's note. The problems CO:5051-C.O:5057 were on the long list of the Romanian Mathematical Olympiad, 2009.