PROBLEMS FOR COMPETITIONS AND OLYMPIADS

Junior Level

C.O:5035. Find all integers x such that:

$$9^x + 4^x + 2^x = 8^x + 6^x + 1.$$

Nicolae Papacu, Slobozia

C.O:5036. Find all non-negative integers n for which the number $n^2 + 9n + 8$ can be expressed as a product of 4 consecutive integers.

Mihai Opincariu, Brad, Hunedoara

C.O:5037. Exhibit an example of a finite set M of points in space, not all in the same plane, with the property that for any pair of points $A, B \in M$, $A \neq B$, there exist points $C, D \in M$, $C \neq D$, such that AB and CD are parallel lines.

* * *

C.O:5038. Let n be a positive integer. Show that for any positive rational x there exist $k, p \in \mathbb{N}$ with $k > n$, such that:

$$x = \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{k+1}\right) + \ldots + \left(\frac{1}{k+p}\right).$$

* * *

Senior Level

C.O:5039. Prove that for any $n \in \mathbb{N}$, $n \geq 3$, there exist primes $p_1, p_2, \ldots, p_n$ such that $p_1 < p_2 < \ldots < p_n$ and $p_1 + p_2 > p_n$.

* * *

C.O:5040. A set M of 42 points in a plane has the property that the number of triangles with all three vertices in the set M is not greater than 2009. Prove that at least 33 points of M are collinear.

George Cazacu, Bucharest

C.O:5041. Find all arithmetic progressions of real numbers $(x_n)_{n\geq 1}$ such that the sequence $(\sin x_n)_{n\geq 1}$ is convergent.

Marian Cucoaneş, Mărăşeşti, Vrancea

C.O:5042. Consider the polynomial $f = X^n + pX + ap^2$, where $n \geq 2$, p is a prime and $a \in \mathbb{Z}$, $(a, p) = 1$. Prove that the following statements are equivalent:

a) f is irreducible in $\mathbb{Q}[X]$;

b) f has no roots in $\mathbb{Q}$.

* * *