PROBLEMS FOR COMPETITIONS AND OLYMPIADS

Junior Level

C.O:5067. Show that there exist infinitely many positive integers n, $n \geq 4$ such that each of the numbers 2^n and 2^{n+1} has the two last digits equal. (that is $2^n = \overline{\ldots aa}$ and $2^{n+1} = \overline{\ldots bb}$

Marius Burtea, Alexandria

C.O:5068. On the side AB of the triangle ABC consider point D such that $5AD = 2DB$. Point M lies on the segment DC such that $3CM = 7DM$. Lines BM and AC meet at E, and lines AM and BC intersect at F. Find the ratio between the area$[DEF]$ and area$[ABC]$.

Vasile Şerdean, Gherla

C.O:5069. O a circle 2009 numbers are given such that the neighbors of each numbers at up to a multiple of 3. Prove that all numbers are divisible by 3.

Karina Sertov, student, Bucharest

C.O:5070. Let ABC be a triangle and D a point on BC such that triangles ABD and ACD are isosceles. Prove that one of the following statements hold:

i) $A = 90°$; ii) $A = 3B$; iii) $A = 3C$.

Dinu Şerbănescu, Bucharest

Senior Level

C.O:5071. Consider a triangle ABC with $BC = a$, $CA = b$ şi $AB = c$ and area equal to 4. Let x, y, z the distances from the orthocenter to the vertices A, B, C. Prove that if

$$a\sqrt{x} + b\sqrt{y} + cz = 4\sqrt{a+b+c},$$

then ABC is equilateral.

Lucian Dragomir, Oţelu Roşu

C.O:5072. Find all differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$ which satisfies the relation $f'(x + g(y)) = g(x) + y, \ \forall\, x, y \in \mathbb{R}$.

Marius Burtea, Alexandria

C.O:5073. Let $(G, \cdot)$ be an abelian group. For each $n \in \mathbb{N}^*$, define the set $R_n(G) = \{x \in G \mid \exists\, y \in G \text{ with } x = y^n\}$.

a) Show that $(R_n(G), \cdot)$ is a subgroup of $(G, \cdot)$.

b) If G is finite and e is the unit, prove that $\bigcap\limits_{n \in \mathbb{N}^*} R_n(G) = \{e\}$.

c) Prove that if G is a cyclic group of order $m \in \mathbb{N}^*$, then

$$R_n(G) = G \Leftrightarrow (m, n) = 1 \text{ şi } R_n(G) = \{e\} \Leftrightarrow m \mid n.$$

Şerban Olteanu, Giurgiu

C.O:5074. Compute $\lim\limits_{n\to\infty} \displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{|\cos(2n+1)x|}{x^2}\, \mathrm{d}x$.

Vasile Mircea Popa, Sibiu