GAZETA MATEMATICĂ

SERIA A

ANUL XLIII (CXXII) **Nr. 3 – 4/ 2025**

ARTICLES

A famous formula of Euler revisited

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Abstract. In this paper we give a new proof of the following famous formula of Euler

$$2\sum_{n=1}^{\infty}\frac{H_n}{n^k}=(k+2)\zeta(k+1)-\sum_{i=2}^{k-1}\zeta(i)\zeta(k+1-i),$$

where $k\geq 2$ is an integer and H_n denotes the nth harmonic number. The new proof is based on calculating a logarithmic integral by two different methods.

Keywords: Abel's summation formula, Euler series, Goldbach series, logarithmic integrals, Riemann zeta function values.

MSC: 40A05, 40C10

1. Introduction and the main results

If $n\geq 1$ is an integer, then the nth harmonic number is defined by the formula $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$. A famous formula of Euler in which the nth harmonic number is involved reads

$$2\sum_{n=1}^{\infty}\frac{H_n}{n^k}=(k+2)\zeta(k+1)-\sum_{i=2}^{k-1}\zeta(i)\zeta(k+1-i), \tag{1}$$

where $k\geq 2$ is an integer and the second sum is nil when $k=2$. This formula was discovered and proved by Euler, see the information given in [2, p. 278]. Several proofs of (1) have appeared in the literature [2], [4], [6, pp.172–175] to mention a few.

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Two special cases of formula (1), when $k = 2$ and $k = 3$, are worth mentioning:

(a) $\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$ due to Leonhard Euler,

(b) $\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4}\zeta(4)$ due to Christian Goldbach.

In this paper we give a *new proof* of formula (1) which is based on calculating the logarithmic integral

$$\int_0^1 \ln^{k-1} x \frac{\ln(1-x)}{1-x} \mathrm{d}x$$

by two different methods. In the last part of the paper we give new proofs of formulæ (a) and (b) above by series manipulations which may be of independent interest.

Before we prove formula (1) we collect some results we need in our analysis. Recall that *Abel's summation formula* ([1, p. 55], [3, Lemma A.1, p. 258], [5, p. 38]) states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences of real numbers and $A_n = \sum_{k=1}^{n} a_k$, then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}).$$

We will be using the *infinite version* of this formula

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} (A_n b_{n+1}) + \sum_{k=1}^{\infty} A_k (b_k - b_{k+1}). \tag{2}$$

The nth generalized harmonic number of order k is defined by $H_n^{(k)} = 1 + \frac{1}{2^k} + \cdots + \frac{1}{n^k}$, where $n, k \geq 1$ are integers.

If $p, q \geq 2$ are integers, then the following symmetric formula holds true

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} + \sum_{n=1}^{\infty} \frac{H_n^{(q)}}{n^p} = \zeta(p+q) + \zeta(p)\zeta(q). \tag{3}$$

We prove formula (3) by Abel's summation formula (2) with $a_n = \frac{1}{n^q}$ and $b_n = H_n^{(p)}$, and we have

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} = \lim_{n\to\infty} H_n^{(q)} H_{n+1}^{(p)} - \sum_{n=1}^{\infty} \frac{H_n^{(q)}}{(n+1)^p}$$

$$
\begin{aligned}
&= \zeta(p)\zeta(q) - \sum_{n=1}^{\infty} \frac{H_{n+1}^{(q)} - \frac{1}{(n+1)^q}}{(n+1)^p} \\
&= \zeta(p)\zeta(q) + \sum_{n=1}^{\infty} \left(\frac{1}{(n+1)^{p+q}} - \frac{H_{n+1}^{(q)}}{(n+1)^p} \right) \\
&= \zeta(p)\zeta(q) + \sum_{n=1}^{\infty} \left(\frac{1}{n^{p+q}} - \frac{H_n^{(q)}}{n^p} \right) \\
&= \zeta(p)\zeta(q) + \zeta(p+q) - \sum_{n=1}^{\infty} \frac{H_n^{(q)}}{n^p},
\end{aligned}
$$

and formula (3) is proved.

We also need the following formula which can be proved by integration by parts, when $m \in \mathbb{N}$, or by using the Gamma function of Euler Γ, when $m \notin \mathbb{N}$:

$$
\int_0^1 x^k \ln^m x \mathrm{d}x = \begin{cases} (-1)^m \frac{\Gamma(m+1)}{(k+1)^{m+1}}, & \text{if} \quad k > -1,\ m \notin \mathbb{N}; \\ (-1)^m \frac{m!}{(k+1)^{m+1}}, & \text{if} \quad k > -1,\ m \in \mathbb{N}. \end{cases} \tag{4}
$$

The proof of formula (1). To prove formula (1), we consider $k \geq 2$ an integer and we calculate the logarithmic integral $I_k = \int_0^1 \ln^{k-1} x \frac{\ln(1-x)}{1-x} \mathrm{d}x$ by two different methods.

First, we use the generating function for the nth harmonic number $\sum\limits_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x}$, $x \in (-1,1)$. and we obtain that

$$
\begin{aligned}
I_k &= -\int_0^1 \ln^{k-1} x \sum_{n=1}^{\infty} H_n x^n \mathrm{d}x \\
&= -\sum_{n=1}^{\infty} H_n \int_0^1 x^n \ln^{k-1} x \mathrm{d}x \\
&\overset{(4)}{=} (-1)^k (k-1)! \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^k} \\
&= (-1)^k (k-1)! \sum_{n=1}^{\infty} \frac{H_{n+1} - \frac{1}{n+1}}{(n+1)^k} \\
&= (-1)^k (k-1)! \left(\sum_{n=1}^{\infty} \frac{H_n}{n^k} - \zeta(k+1) \right).
\end{aligned} \tag{5}
$$

Second, using the geometric series and the power series expansion of the logarithmic function $\ln(1-x) = -\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$, $x \in [-1, 1)$, we have that

$$\begin{aligned}
I_k &= \int_0^1 \ln^{k-1} x \ln(1-x) \sum_{n=0}^{\infty} x^n \mathrm{d}x \\
&= \sum_{n=0}^{\infty} \int_0^1 x^n \ln^{k-1} x \ln(1-x) \mathrm{d}x \\
&= -\sum_{n=0}^{\infty} \int_0^1 x^n \ln^{k-1} x \sum_{m=1}^{\infty} \frac{x^m}{m} \mathrm{d}x \\
&= -\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \int_0^1 x^{n+m} \ln^{k-1} x \mathrm{d}x \\
&\overset{(4)}{=} (-1)^k (k-1)! \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m(n+m+1)^k} \\
&\overset{n+1 \to n}{=} (-1)^k (k-1)! \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m(n+m)^k}.
\end{aligned} \tag{6}$$

Now we calculate the preceding double series. We have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m(n+m)^k} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{\infty} \frac{1}{(n+m)^{k-1}} \left(\frac{1}{m} - \frac{1}{n+m} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{1}{m(n+m)^{k-1}} - \sum_{m=1}^{\infty} \frac{1}{(n+m)^k} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{\infty} \frac{1}{m(n+m)^{k-1}} - \sum_{n=1}^{\infty} \frac{\zeta(k) - H_n^{(k)}}{n} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{\infty} \frac{1}{(n+m)^{k-2}} \left(\frac{1}{m} - \frac{1}{n+m} \right) \frac{1}{n} - \sum_{n=1}^{\infty} \frac{\zeta(k) - H_n^{(k)}}{n} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sum_{m=1}^{\infty} \frac{1}{m(n+m)^{k-2}} - \left(\zeta(k-1) - H_n^{(k-1)} \right) \right) - \sum_{n=1}^{\infty} \frac{\zeta(k) - H_n^{(k)}}{n} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=1}^{\infty} \frac{1}{m(n+m)^{k-2}} - \sum_{n=1}^{\infty} \frac{\zeta(k-1) - H_n^{(k-1)}}{n^2} - \sum_{n=1}^{\infty} \frac{\zeta(k) - H_n^{(k)}}{n}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{m=1}^{\infty}\frac{1}{(n+m)^{k-3}}\left(\frac{1}{m}-\frac{1}{n+m}\right)\frac{1}{n} \\
&\qquad -\sum_{n=1}^{\infty}\frac{\zeta(k-1)-H_n^{(k-1)}}{n^2}-\sum_{n=1}^{\infty}\frac{\zeta(k)-H_n^{(k)}}{n} \\
&= \sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\left(\frac{1}{m(n+m)^{k-3}}-\left(\zeta(k-2)-H_n^{(k-2)}\right)\right) \\
&\qquad -\sum_{n=1}^{\infty}\frac{\zeta(k-1)-H_n^{(k-1)}}{n^2}-\sum_{n=1}^{\infty}\frac{\zeta(k)-H_n^{(k)}}{n} \\
&= \sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{1}{m(n+m)^{k-3}}-\sum_{n=1}^{\infty}\frac{\zeta(k-2)-H_n^{(k-2)}}{n^3} \\
&\qquad -\sum_{n=1}^{\infty}\frac{\zeta(k-1)-H_n^{(k-1)}}{n^2}-\sum_{n=1}^{\infty}\frac{\zeta(k)-H_n^{(k)}}{n}.
\end{aligned}
\tag{7}
$$

We continue this iterative procedure and we obtain that the double sum is

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n^{k-1}}\sum_{m=1}^{\infty}\frac{1}{m(n+m)}-\sum_{n=1}^{\infty}\frac{\zeta(2)-H_n^{(2)}}{n^{k-1}}-\cdots \\
&\qquad -\sum_{n=1}^{\infty}\frac{\zeta(k-1)-H_n^{(k-1)}}{n^2}-\sum_{n=1}^{\infty}\frac{\zeta(k)-H_n^{(k)}}{n} \\
&= \sum_{n=1}^{\infty}\frac{H_n}{n^k}-(\zeta(2)\zeta(k-1)+\cdots+\zeta(k-1)\zeta(2))+\sum_{n=1}^{\infty}\sum_{i=2}^{k-1}\frac{H_n^{(i)}}{n^{k+1-i}} \\
&\qquad -\sum_{n=1}^{\infty}\frac{\zeta(k)-H_n^{(k)}}{n},
\end{aligned}
\tag{8}
$$

since

$$
\sum_{m=1}^{\infty}\frac{1}{m(n+m)}=\sum_{m=1}^{\infty}\frac{1}{n}\left(\frac{1}{m}-\frac{1}{n+m}\right)=\frac{H_n}{n}.
$$

We calculate the series $\sum\limits_{n=1}^{\infty}\frac{\zeta(k)-H_n^{(k)}}{n}$ by using formula (2) with $a_n=\frac{1}{n}$ and $b_n=\zeta(k)-H_n^{(k)}$ and we have, since $A_n=H_n$ and $b_n-b_{n+1}=\frac{1}{(n+1)^k}$,

that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\zeta(k)-H_n^{(k)}}{n} &= \lim_{n\to\infty} H_n\left(\zeta(k)-H_{n+1}^{(k)}\right)+\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^k}\\ &= \sum_{n=1}^{\infty}\frac{H_{n+1}-\frac{1}{n+1}}{(n+1)^k}\\ &= \sum_{n=1}^{\infty}\left(\frac{H_n}{n^k}-\frac{1}{n^{k+1}}\right)\\ &= \sum_{n=1}^{\infty}\frac{H_n}{n^k}-\zeta(k+1).\end{aligned} \tag{9}$$

We used that $\lim_{n\to\infty} H_n\left(\zeta(k)-H_{n+1}^{(k)}\right)=0$, for $k\geq 2$.

Combining (8) and (9) we have that

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m(n+m)^k} &= \zeta(k+1)-\sum_{i=2}^{k-1}\zeta(i)\zeta(k+1-i)\\ &\quad+\sum_{n=1}^{\infty}\sum_{i=2}^{k-1}\frac{H_n^{(i)}}{n^{k+1-i}}.\end{aligned} \tag{10}$$

We observe that

$$\sum_{i=2}^{k-1}\frac{H_n^{(i)}}{n^{k+1-i}} \overset{k+1-i=j}{=} \sum_{j=2}^{k-1}\frac{H_n^{(k+1-j)}}{n^j},$$

and this implies that

$$s=\sum_{n=1}^{\infty}\sum_{i=2}^{k-1}\frac{H_n^{(i)}}{n^{k+1-i}}=\sum_{n=1}^{\infty}\sum_{j=2}^{k-1}\frac{H_n^{(k+1-j)}}{n^j}.$$

It follows that

$$\begin{aligned}2s &= \sum_{n=1}^{\infty}\sum_{i=2}^{k-1}\left(\frac{H_n^{(i)}}{n^{k+1-i}}+\frac{H_n^{(k+1-i)}}{n^i}\right)\\ &= \sum_{i=2}^{k-1}\sum_{n=1}^{\infty}\left(\frac{H_n^{(i)}}{n^{k+1-i}}+\frac{H_n^{(k+1-i)}}{n^i}\right)\\ &\overset{(3)}{=} \sum_{i=2}^{k-1}\left(\zeta(k+1)+\zeta(i)\zeta(k+1-i)\right)\\ &= (k-2)\zeta(k+1)+\sum_{i=2}^{k-1}\zeta(i)\zeta(k+1-i).\end{aligned}$$

This implies that

$$s=\frac{k-2}{2}\zeta(k+1)+\frac{1}{2}\sum_{i=2}^{k-1}\zeta(i)\zeta(k+1-i). \tag{11}$$

Replacing (11) in (10) one has that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m(n+m)^k}=\frac{k}{2}\zeta(k+1)-\frac{1}{2}\sum_{i=2}^{k-1}\zeta(i)\zeta(k+1-i). \tag{12}$$

Combining formulæ (5), (6) and (12) it follows that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n^k}&=\zeta(k+1)+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m(n+m)^k}\\&=\frac{k+2}{2}\zeta(k+1)-\frac{1}{2}\sum_{i=2}^{k-1}\zeta(i)\zeta(k+1-i),\end{aligned}$$

and formula (1) is proved.

A gem with a proof of Euler's series. Now we give a new proof, a gem in the theory of series, of Euler's formula $\sum\limits_{n=1}^{\infty}\frac{H_n}{n^2}=2\zeta(3)$ by calculating the series $\sum\limits_{n=1}^{\infty}\frac{1}{n}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)$ by two different ways.

An analytic proof, based on computing a polylogarithm integral by two different ways, can be found in [5, pp. 238–239]. We calculate the series $\sum\limits_{n=1}^{\infty}\frac{1}{n}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)$ by two different ways.

We apply formula (2), with $a_n=\frac{1}{n}$ and $b_n=\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}$, and we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{1}{n}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)\\&=\lim_{n\to\infty}H_n\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}\right)+\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}\\&=\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}=\sum_{n=1}^{\infty}\frac{H_{n+1}-\frac{1}{n+1}}{(n+1)^2}=\sum_{n=1}^{\infty}\left(\frac{H_n}{n^2}-\frac{1}{n^3}\right)\\&=\sum_{n=1}^{\infty}\frac{H_n}{n^2}-\zeta(3).\end{aligned} \tag{13}$$

On the other hand,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right) &= \sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=1}^{\infty}\frac{1}{(n+m)^2}\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{m}+\frac{1}{n}\right)\frac{1}{(n+m)^2}\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm(n+m)} \qquad (14)\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=1}^{\infty}\frac{1}{m(n+m)}\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^2},\end{aligned}$$

since $\sum_{n=1}^{\infty}\frac{1}{m(n+m)}=\frac{H_n}{n}$. Combining (13) and (14) the result follows.

A gem with a proof of Goldbach's series. We give a new proof of Goldbach's formula $\sum_{n=1}^{\infty}\frac{H_n}{n^3}=\frac{5}{4}\zeta(4)$ by series manipulations. An analytic proof, based on computing a polylogarithm integral by two different ways, can be found in [5, pp. 239–240].

We calculate

$$S:=\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)$$

as follows:

$$\begin{aligned}S &= \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{m=1}^{\infty}\frac{1}{(n+m)^2}\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{n^2}+\frac{1}{m^2}\right)\frac{1}{(n+m)^2}\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n^2+m^2}{n^2m^2(n+m)^2}\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(n+m)^2-2nm}{n^2m^2(n+m)^2}\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^2m^2}-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm(n+m)^2}\end{aligned}$$

$$= \frac{1}{2}\zeta^2(2) - \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{m=1}^{\infty}\frac{1}{n+m}\left(\frac{1}{m}-\frac{1}{n+m}\right)$$

$$= \frac{5}{4}\zeta(4) - \sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sum_{m=1}^{\infty}\frac{1}{m(n+m)} - \sum_{m=1}^{\infty}\frac{1}{(n+m)^2}\right)$$

$$= \frac{5}{4}\zeta(4) - \sum_{n=1}^{\infty}\frac{H_n}{n^3} + \sum_{n=1}^{\infty}\frac{1}{n^2}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right),$$

and the result follows.

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A generalization of Iran 1995 inequality

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Abstract. Let a_1, a_2, a_3 be nonnegative real numbers such that

$$a_1a_2 + a_2a_3 + a_3a_1 = 3.$$

By Iran 1995 inequality, the expression

$$E_3 = \frac{1}{(a_2+a_3)^2} + \frac{1}{(a_3+a_1)^2} + \frac{1}{(a_1+a_2)^2}$$

has the minimum value (equal to $\frac{3}{4}$) for $a_1 = a_2 = a_3$, as well as for $a_1 = 0$ and $a_2 = a_3$ (or any permutation). In this paper, we establish the minimum value of the expression

$$E_n = \frac{1}{(\sum_{i\neq 1} a_i)^2} + \frac{1}{(\sum_{i\neq 2} a_i)^2} + \cdots + \frac{1}{(\sum_{i\neq n} a_i)^2},$$

where $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that

$$\sum_{1\leq i<j\leq n} a_ia_j = \frac{n(n-1)}{2}.$$

We will show that if $4 \leq n \leq 18$, the minimum occurs for $a_1 = \cdots = a_{n-2} = 0$ and $a_{n-1} = a_n$ (or any permutation), and if $n \geq 18$, the minimum occurs for $a_1 = \cdots = a_{n-2} = 0$ and $\frac{a_{n-1}}{a_n} + \frac{a_n}{a_{n-1}} = \sqrt{n-2} - 2$ (or any permutation).

Keywords: Nonnegative variables, symmetric constraint and inequality, minimum value, Iran 1995 inequality.

MSC: 26D10, 26D15

1. Preliminary results

For $a_1 = \cdots = a_{n-2} = 0$ and $a_{n-1} = a_n = \sqrt{\frac{n(n-1)}{2}}$, the constraint

$$\sum_{1\leq i<j\leq n} a_ia_j = \frac{n(n-1)}{2}$$

is satisfied, and the expression

$$E_n = \frac{1}{(\sum_{i\neq 1} a_i)^2} + \frac{1}{(\sum_{i\neq 2} a_i)^2} + \cdots + \frac{1}{(\sum_{i\neq n} a_i)^2}$$

becomes

$$E_n = \frac{n+6}{2n(n-1)}.$$

We will show that

$$m_1 = \frac{n+6}{2n(n-1)}$$

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is the minimum value of E_n for $3 \le n \le 18$.

For $a_1 = \cdots = a_{n-2} = 0$, $2a_{n-1}a_n = n(n-1)$, and $\dfrac{a_{n-1}}{a_n} + \dfrac{a_n}{a_{n-1}} = \sqrt{n-2} - 2$ (with $n \ge 18$), the constraint

$$\sum_{1 \le i < j \le n} a_i a_j = \frac{n(n-1)}{2}$$

is satisfied, and the expression

$$E_n = \frac{1}{(\sum_{i \ne 1} a_i)^2} + \frac{1}{(\sum_{i \ne 2} a_i)^2} + \cdots + \frac{1}{(\sum_{i \ne n} a_i)^2}$$

becomes

$$E_n = \frac{4(\sqrt{n-2} - 1)}{n(n-1)}.$$

We claim that

$$m_2 = \frac{4(\sqrt{n-2} - 1)}{n(n-1)}$$

is the minimum value of E_n for $n \ge 18$.

To prove these claims, we will use the following proposition.

Proposition 1. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 \le a_2 \le \cdots \le a_n$ and $a_n > 0$, and let*

$$p = \sum_{i=1}^{n} a_i$$

and

$$q = \sum_{1 \le i < j \le n} a_i a_j.$$

For $n \ge 3$ and fixed p and q (with $p > 0$ and $q > 0$), the expression

$$E_n = \frac{1}{(p - a_1)^2} + \frac{1}{(p - a_2)^2} + \cdots + \frac{1}{(p - a_n)^2}$$

has the minimum value for one of the cases

$$0 = a_1 = \cdots = a_{n-k-1} \le a_{n-k} \le a_{n-k+1} = \cdots = a_n,$$

where $k \in \{1, 2, \ldots, n-1\}$.

Proof. Let

$$p_1 = a_1^2 + a_2^2 + \cdots + a_n^2.$$

Since $p_1 = p^2 - 2q$, the sum p_1 is fixed. We will apply the Equal Variables Theorem (see [1], [2], and [3]) to the function $f(x) = \dfrac{1}{(p-x)^2}$ defined on $[0, p)$. Let

$$g(x) = f'(x),$$

with

$$g(x)=\frac{2}{(p-x)^3},\qquad g''(x)=\frac{24}{(p-x)^5}.$$

Since the function $g(x)$ is strictly convex on $[0,p)$ and the sums p and p_1 are fixed, from EV-Theorem it follows that E_n has the minimum value when $a_1=0$ or $0<a_1\le a_2=a_3=\cdots=a_n$. These cases are equivalent to

$$0=a_1=\cdots=a_{n-k-1}\le a_{n-k}\le a_{n-k+1}=\cdots=a_n,$$

where $k\in\{1,2,\ldots,n-1\}$. □

2. Mains results

In this section we present a generalization of Iran 1995 inequality to n variables.

Theorem 2. *Let $a_1,a_2,\ldots,a_n$ be nonnegative real numbers such that*

$$\sum_{1\le i<j\le n}a_ia_j=\frac{n(n-1)}{2}.$$

If $3\le n\le 18$, then

$$\frac{1}{(\sum_{i\ne 1}a_i)^2}+\frac{1}{(\sum_{i\ne 2}a_i)^2}+\cdots+\frac{1}{(\sum_{i\ne n}a_i)^2}\ge\frac{n+6}{2n(n-1)},$$

with equality for $a_1=\cdots=a_{n-2}=0$ and $a_{n-1}=a_n=\sqrt{\dfrac{n(n-1)}{2}}$ (or any permutation).

Proof. By Proposition 1, it suffices to consider $a_1\le a_2\le\cdots\le a_n$ and

$$0=a_1=\cdots=a_{n-k-1}\le a_{n-k}\le a_{n-k+1}=\cdots=a_n,\qquad k\in\{1,2,\ldots,n-1\}.$$

Denoting $a_{n-k}:=x$ and $a_{n-k+1}=\cdots=a_n:=y$, we need to prove that

$$\frac{n-k-1}{(x+ky)^2}+\frac{1}{k^2y^2}+\frac{k}{[x+(k-1)y]^2}\ge\frac{n+6}{2n(n-1)}$$

for $0\le x\le y$ and

$$2kxy+k(k-1)y^2=n(n-1).$$

Write this inequality in the homogeneous form

$$\frac{n-k-1}{(x+ky)^2}+\frac{1}{k^2y^2}+\frac{k}{[x+(k-1)y]^2}\ge\frac{n+6}{2k[2xy+(k-1)y^2]}.$$

Due to homogeneity, we may set $y=1$ and $x\le 1$. Then the inequality becomes

$$\frac{n-k-1}{(x+k)^2}+\frac{1}{k^2}+\frac{k}{(x+k-1)^2}\ge\frac{n+6}{2k(2x+k-1)}.\qquad(1)$$

For $k = 1$, the inequality (1) turns into

$$\frac{n-2}{(x+1)^2} + 1 + \frac{1}{x^2} \geq \frac{n+6}{4x},$$

i.e.,

$$(x-1)^2[4x^2 + (10-n)x + 4] \geq 0.$$

It is true since

$$4x^2 + (10-n)x + 4 = 4\left(x + \frac{10-n}{8}\right)^2 + \frac{(n-2)(18-n)}{16} \geq 0.$$

Consider further $k \in \{2, \ldots, n-1\}$ and $n \leq 18$. For fixed x and k, we may write the inequality (1) as $E(n) \geq 0$, where $k+1 \leq n \leq 18$ and

$$E(n) = An - \frac{k+1}{(x+k)^2} + \frac{1}{k^2} + \frac{k}{(x+k-1)^2} - \frac{6}{2k(2x+k-1)}$$

with

$$A = \frac{1}{(x+k)^2} - \frac{1}{2k(2x+k-1)} = \frac{x(2k-x) + k(k-2)}{2k(x+k)^2(2x+k-1)}.$$

Since $A \geq 0$, we have $E(n) \geq E(k+1)$. So, it suffices to prove the inequality (1) for $n = k+1$, i.e.,

$$\frac{1}{k^2} + \frac{k}{(x+k-1)^2} \geq \frac{k+7}{2k(2x+k-1)},$$

or

$$\frac{1}{k} + \frac{k^2}{(x+k-1)^2} \geq \frac{k+7}{2(2x+k-1)}. \tag{2}$$

For $k = 2$, the inequality (2) becomes

$$\frac{1}{2} + \frac{4}{(x+1)^2} \geq \frac{9}{2(2x+1)},$$

i.e.,

$$x(x-1)^2 \geq 0.$$

For $k = 3$, the inequality (2) becomes

$$\frac{1}{3} + \frac{9}{(x+2)^2} \geq \frac{5}{2(x+1)},$$

i.e.,

$$2x^3 - 5x^2 + 10x + 2 \geq 0.$$

Indeed, we have

$$2x^3 - 5x^2 + 10x + 2 > 5x(2-x) \geq 0.$$

For $k \geq 4$, since

$$\frac{k^2}{(x+k-1)^2} \geq \frac{2k}{x+k-1} - 1,$$

the inequality (2) is true if

$$\frac{1}{k}+\frac{2k}{x+k-1}-1\geq\frac{k+7}{2(2x+k-1)},$$

which is equivalent to

$$\frac{1-k}{k}+\frac{(k-1)(7x+3k-7)}{2(x+k-1)(2x+k-1)}\geq 0,$$
$$\frac{-1}{k}+\frac{7x+3k-7}{2(x+k-1)(2x+k-1)}\geq 0,$$
$$x(k+6-4x)+k(k-4)+k-2\geq 0.$$

Since $x\leq 1$ and $k\geq 4$, the last inequality is clearly true. □

Theorem 3. *Let $a_1,a_2,\ldots,a_n$ be nonnegative real numbers such that*

$$\sum_{1\leq i<j\leq n}a_ia_j=\frac{n(n-1)}{2}.$$

If $n\geq 18$, then

$$\frac{1}{(\sum_{i\neq 1}a_i)^2}+\frac{1}{(\sum_{i\neq 2}a_i)^2}+\cdots+\frac{1}{(\sum_{i\neq n}a_i)^2}\geq\frac{4(\sqrt{n-2}-1)}{n(n-1)},$$

with equality for $a_1=\cdots=a_{n-2}=0$, $2a_{n-1}a_n=n(n-1)$, and $\dfrac{a_{n-1}}{a_n}+\dfrac{a_n}{a_{n-1}}=\sqrt{n-2}-2$ (or any permutation).

Proof. Following the same method as in the proof of Theorem 1, we need to show that

$$\frac{n-k-1}{(x+k)^2}+\frac{1}{k^2}+\frac{k}{(x+k-1)^2}\geq\frac{4(\sqrt{n-2}-1)}{k(2x+k-1)} \tag{3}$$

for $0\leq x\leq 1$.

For $k=1$, the inequality (3) becomes

$$\frac{n-2}{(x+1)^2}+1+\frac{1}{x^2}\geq\frac{2(\sqrt{n-2}-1)}{x},$$

i.e.,

$$\frac{n-2}{(x+1)^2}+\frac{(x+1)^2}{x^2}\geq\frac{2\sqrt{n-2}}{x},$$
$$\left(\frac{\sqrt{n-2}}{x+1}-\frac{x+1}{x}\right)^2\geq 0.$$

Consider further $k\in\{2,\ldots,n-1\}$ and $n\geq 18$. For fixed x and k, we may write the inequality (3) as $F(n)\geq 0$, where

$$F(n)=\frac{n}{(x+k)^2}-\frac{4\sqrt{n-2}}{k(2x+k-1)}+A(x,k).$$

We will prove that

$$F(n) \geq F(18) \geq 0.$$

We have

$$\begin{aligned} F(n) - F(18) &= \frac{n-18}{(x+k)^2} - \frac{4(\sqrt{n-2}-4)}{k(2x+k-1)} \\ &= (\sqrt{n-2}-4)\left[\frac{\sqrt{n-2}+4}{(x+k)^2} - \frac{4}{k(2x+k-1)}\right] \\ &\geq 4(\sqrt{n-2}-4)\left[\frac{2}{(x+k)^2} - \frac{1}{k(2x+k-1)}\right] \\ &= \frac{4(\sqrt{n-2}-4)B}{k(x+k)^2(2x+k-1)}, \end{aligned}$$

where

$$B = k(k-2) + 2kx - x^2.$$

Since $\sqrt{n-2}-4 \geq 0$ and $B \geq x(2k-x) \geq 0$, we have $F(n) \geq F(18)$. Next, to show that $F(18) \geq 0$, we need to prove the inequality (3) for $n = 18$, i.e.,

$$\frac{k(17-k)}{(x+k)^2} + \frac{1}{k} + \frac{k^2}{(x+k-1)^2} \geq \frac{12}{2x+k-1}. \tag{4}$$

For $k = 2$, the inequality (4) turns into

$$\frac{30}{(x+2)^2} + \frac{1}{2} + \frac{4}{(x+1)^2} \geq \frac{12}{2x+1}.$$

It is true if

$$\frac{30}{(x+2)^2} + \frac{1}{2(x+1)^2} + \frac{4}{(x+1)^2} \geq \frac{12}{2x+1},$$

which is equivalent to

$$\frac{20}{(x+2)^2} + \frac{3}{(x+1)^2} \geq \frac{8}{2x+1},$$

$$x(20 + 23x - 2x^2 - 8x^3) \geq 0.$$

Since $0 \leq x \leq 1$, the last inequality is clearly true.

For $k \geq 3$, since

$$\frac{k^2}{(x+k-1)^2} \geq \frac{2k}{x+k-1} - 1 \geq \frac{2k}{2x+k-1} - 1,$$

the inequality (4) holds if

$$\frac{k(17-k)}{(x+k)^2} + \frac{1}{k} + \frac{2k}{2x+k-1} - 1 \geq \frac{12}{2x+k-1},$$

i.e.,

$$\frac{k(17-k)}{(x+k)^2} - \frac{k-1}{k} + \frac{2(k-6)}{2x+k-1} \geq 0,$$

$$-2(k-1)x^3-(3k^2+6k+1)x^2-2k(k^2-8k+1)x+2k^2(4k-9)\geq 0.$$

Since $0\leq x\leq 1$, it suffices to show that

$$-2(k-1)x-(3k^2+6k+1)x-2k(k^2-8k+1)x+2k^2(4k-9)\geq 0,$$

i.e.,

$$Cx+2k^2(4k-9)\geq 0,$$

where

$$C=-2k^3+13k^2-10k+1.$$

This inequality is obviously true for $C\geq 0$ while for $C<0$ it holds

$$Cx+2k^2(4k-9)\geq C+2k^2(4k-9)=6k^3-5k^2-10k+1$$
$$>5k^3-5k^2-10k=5k(k^2-k-2)=5k(k-2)(k+1)>0.$$

□

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A new proof of Sandham–Yyeung equality

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Abstract. More than a lustrum ago, Ovidiu Furdui and Alina Sîntămărian [5] gave a beautiful heuristic proof for the famous equality of Sandham–Yeung. The purpose of this note is to make their proof rigorous by using a logarithmic integral formula in [10] and an obscure infinite series in [8].

Keywords: Classical harmonic numbers, nonlinear harmonic sums, linear harmonic sums, Riemann Zeta function, Logarithmic integrals, polylogarithm function.

MSC: Primary 40A25; Secondary 11M06

1. Introduction

The Sandham–Yeung equality

$$\left(\frac{H_1}{1}\right)^2+\left(\frac{H_2}{2}\right)^2+\left(\frac{H_3}{3}\right)^2+\cdots+\left(\frac{H_n}{n}\right)^2=\frac{17}{4}\zeta\left(4\right) \tag{1}$$

is one of the most proved results in nonlinear harmonic sums. Nowadays, there is a lot of accessible literature on Sandham-Yeung quadratic series; see e.g. [7, p. 267], [3], [4], [12], [5], [6], [9, Equation (4.99)], [11], [2] and the references therein; as a matter of fact, any curios reader can easily find an abundance of approaches and references by simply making a search.

In this short note we propose a new proof based on the following polylogarithmic and logarithmic integrals

$$\int_0^1\frac{\mathrm{Li}_2\left(x\right)\ln\left(x\right)}{1-x}\,\mathrm{d}x\quad\text{and}\quad\int_0^1\frac{\ln\left(x\right)\ln^2\left(1-x\right)}{x}\,\mathrm{d}x.$$

Let $H_n:=\sum_{k=1}^n k^{-1}$ be the classical harmonic number for $n=1,2,\ldots,$ by $H_n^{(r)}$ denote the nth generalized harmonic number of order r defined by $H_n^{(r)}=1+1/2^r+1/3^r+\cdots+1/n^r$, let $\zeta\left(s\right)$ denote the Riemann zeta function, which is defined by $\zeta\left(s\right)=\sum_{n=1}^{\infty}\frac{1}{n^s},\ \Re(s)>1$, and $\mathrm{Li}_n(x)$ denote the polylogarithm function, which is defined for $|x|\leq 1$ by $\mathrm{Li}_{\mathrm{n}}\left(x\right)=\sum_{k=1}^{\infty}\frac{x^k}{k^n},\ n\in\mathbb{N},\ n\geq 2$.

To evaluate (1), we shall establish some lemmas.

Lemma 1. *The following identity holds:*

$$\sum_{n=1}^{\infty}H_n^2x^{n-1}=\frac{\ln^2\left(1-x\right)+\mathrm{Li}_2\left(x\right)}{x\left(1-x\right)}.$$

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Proof. In the article [8, p. 700], Mező had evaluated the following ordinary generating function for the sequence $\{H_n^2\}_{n\geq 1}$:

$$\sum_{n=1}^{\infty} H_n^2 x^n = \frac{\ln^2(1-x) + \operatorname{Li}_2(x)}{1-x} \quad \text{for} \quad |x| < 1. \tag{2}$$

Dividing by x on both sides of (2), gives us the desired result. □

Lemma 2. *The following equality holds:*

$$\int_0^1 \frac{\ln(x)\ln^2(1-x)}{1-x}\,\mathrm{d}x = -2\zeta(4).$$

Proof. Due to symmetry, we have

$$\int_0^1 \frac{\ln(x)\ln^2(1-x)}{1-x}\,\mathrm{d}x = \int_0^1 \frac{\ln(1-x)\ln^2(x)}{x}\,\mathrm{d}x.$$

In the recent article [10, p. 85], Stewart had obtained the following general logarithmic integral formula:

$$\int_0^z \frac{\ln^2(x)\ln(1-x)}{x}\,\mathrm{d}x = -2\operatorname{Li}_4(z) - \operatorname{Li}_2(z)\ln^2(z) + 2\operatorname{Li}_3(z)\ln(z). \tag{3}$$

Plugging in $z = 1$ on both sides of (3) gives

$$\int_0^1 \frac{\ln(1-x)\ln^2(x)}{x}\,\mathrm{d}x = -2\operatorname{Li}_4(1) = -2\zeta(4). \qquad \square$$

Remark 3. In [10, p. 85], the expression for $\int_0^z \frac{\ln^2(x)\ln(1-x)}{x}\,\mathrm{d}x$ has a typo, the correct formula being that stated in (3) above.

Lemma 4. *The following equality holds:*

$$\int_0^1 \frac{\ln(x)\ln^2(1-x)}{x}\,\mathrm{d}x = -\frac{1}{2}\zeta(4).$$

Proof. Due to symmetry, we have

$$\int_0^1 \frac{\ln(x)\ln^2(1-x)}{x}\,\mathrm{d}x = \int_0^1 \frac{\ln(1-x)\ln^2(x)}{1-x}\,\mathrm{d}x.$$

Olaikhan [9, Equation (3.87)] had given the following general logarithmic integral formula:

$$\int_0^1 \frac{\ln^b(x)\ln(1-x)}{1-x}\,\mathrm{d}x = (-1)^b\, b!\left[\zeta(b+2) - \sum_{n=1}^{\infty}\frac{H_n}{n^{b+1}}\right] \tag{4}$$

for $b = 1, 2, 3, \ldots$.

Plugging in $b = 2$ on both sides of (4) gives

$$\int_0^1 \frac{\ln^2(x)\ln(1-x)}{1-x}\,\mathrm{d}x = 2\left[\zeta(4) - \sum_{n=1}^{\infty}\frac{H_n}{n^3}\right].$$

In [13, pp. 129–130], Whittaker and Klamkin had evaluated the following linear harmonic sum

$$\sum_{n=1}^{\infty}\frac{H_n}{n^3} = \frac{5}{4}\zeta(4),$$

giving us that the following equality holds:

$$\int_0^1 \frac{\ln^2(x)\ln(1-x)}{1-x}\,\mathrm{d}x = 2\left(\zeta(4) - \frac{5}{4}\zeta(4)\right) = -\frac{1}{2}\zeta(4). \quad \square$$

Lemma 5. *The following equality holds:*

$$\int_0^1 \frac{\ln(x)\,\mathrm{Li}_2(x)}{1-x}\,\mathrm{d}x = -\frac{3}{4}\zeta(4).$$

Proof. Olaikhan [9, Equation (2.2)] had given the following identity involving the generalized harmonic number of order r:

$$\sum_{n=1}^{\infty} H_n^{(r)} x^n = \frac{\mathrm{Li}_r(x)}{1-x} \quad \text{for} \quad |x| < 1. \tag{5}$$

Plugging in $r = 2$ on both sides of (5) gives

$$\sum_{n=1}^{\infty} H_n^{(2)} x^n = \frac{\mathrm{Li}_2(x)}{1-x}.$$

It follows that

$$\begin{aligned}\int_0^1 \frac{\ln(x)\,\mathrm{Li}_2(x)}{1-x}\,\mathrm{d}x &= \int_0^1 \ln(x)\left(\sum_{n=1}^{\infty} H_n^{(2)} x^n\right)\mathrm{d}x \\ &= \sum_{n=1}^{\infty} H_n^{(2)}\left(\int_0^1 x^n \ln(x)\,\mathrm{d}x\right).\end{aligned}$$

Invoking the well-known integral formula $\displaystyle\int_0^1 x^{\alpha}\ln(x)\,\mathrm{d}x = -\frac{1}{(\alpha+1)^2}$, for $\alpha > -1$, one obtains

$$-\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^2} = -\sum_{n=1}^{\infty}\frac{H_{n+1}^{(2)} - \dfrac{1}{(n+1)^2}}{(n+1)^2}$$

$$= -\sum_{n=1}^{\infty} \frac{H_{n+1}^{(2)}}{(n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^4}$$
$$= -\sum_{i=1}^{\infty} \frac{H_i^{(2)}}{i^2} + \sum_{i=1}^{\infty} \frac{1}{i^4}.$$

Olaikhan [9, Equation (2.98)] had evaluated the following linear harmonic sum

$$\sum_{i=1}^{\infty} \frac{H_i^{(2)}}{i^2} = \frac{7}{4}\zeta(4),$$

giving us that the following equality holds:

$$\int_0^1 \frac{\ln(x)\operatorname{Li}_2(x)}{1-x}\,\mathrm{d}x = -\frac{7}{4}\zeta(4) + \zeta(4) = -\frac{3}{4}\zeta(4). \qquad \square$$

Lemma 6. *The following equality holds:*

$$\int_0^1 \frac{\ln(x)\operatorname{Li}_2(x)}{x}\,\mathrm{d}x = -\zeta(4).$$

Proof. $\displaystyle\int_0^1 \frac{\ln(x)\operatorname{Li}_2(x)}{x}\,\mathrm{d}x$
$$= \int_0^1 \frac{\ln(x)}{x}\left(\sum_{n=1}^{\infty}\frac{x^n}{n^2}\right)\mathrm{d}x$$
$$= \sum_{n=1}^{\infty}\frac{1}{n^2}\left(\int_0^1 x^{n-1}\ln(x)\;\mathrm{d}x\right)$$
$$= -\sum_{n=1}^{\infty}\frac{1}{n^4}$$
$$= -\zeta(4). \qquad \square$$

Now we are ready to state the main result of this note.

Theorem 7. *The following identity holds:*

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2 = \frac{17}{4}\zeta(4).$$

Proof. Indeed we have
$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2 = \sum_{n=1}^{\infty} H_n^2\left(-\int_0^1 x^{n-1}\ln(x)\right)\mathrm{d}x$$

$$
\begin{aligned}
&= -\int_0^1 \ln(x)\left(\sum_{n=1}^{\infty} H_n^2 x^{n-1}\right) \mathrm{d}x \\
&= -\int_0^1 \frac{\ln(x)\ln^2(1-x)}{1-x}\,\mathrm{d}x - \int_0^1 \frac{\ln(x)\ln^2(1-x)}{x}\,\mathrm{d}x \\
&\quad -\int_0^1 \frac{\ln(x)\,\mathrm{Li}_2(x)}{1-x}\,\mathrm{d}x - \int_0^1 \frac{\ln(x)\,\mathrm{Li}_2(x)}{x}\,\mathrm{d}x \\
&= 2\zeta(4) + \frac{1}{2}\zeta(4) + \frac{3}{4}\zeta(4) + \zeta(4) \\
&= \frac{17}{4}\zeta(4),
\end{aligned}
$$

and the theorem is proved. □

Here, in the proofs of Lemma 5, Lemma 6, and Theorem 7, Bernstein's theorem [1, Thm. 9.30, p. 243] justifies interchanging the order of integration and summation because of the positivity of the coefficients.

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Traian Lalescu national mathematics contest for university students, 2025 edition

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Abstract. We present the problems from the Sections A and B of the 2025 edition of Traian Lalescu National Mathematics Contest for University Students, hosted by the Constanţa Maritime University between May 8 and May 10, 2025. We also include the solutions provided by the authors.

Keywords: Real functions, series, convexity, Taylor formula, rank, minimal polynomial, linear dependence.

MSC: 15A03, 15A18, 15A24, 15A36, 26A06, 26A24, 40A05, 40A10

The 2025 edition of the Traian Lalescu National Mathematics Contest for University Students and of the National Session of Scientific Communications for University Students was organized between May 8 and May 10 by the Constanţa Maritime University, with the support of the Traian Lalescu Foundation and the Ministry of Education.

The participating students represented 13 universities from Braşov, Bucharest, Cluj-Napoca, Constanţa, Craiova, Iaşi, and Timişoara. They competed in one of the following sections:

- Section A — first- and second-year students from faculties of Mathematics: 16 students;
- Section B — first-year students that follow some specialization in Electrical Engineering or in Computer Science: 20 students;
- Section C — first-year students from technical faculties with a specialization outside the field of Electrical Engineering: 25 students;
- Section D — second-year students that follow some specialization in Electrical Engineering: 14 students;
- Section E — second-year students from technical faculties with a specialization outside the field of Electrical Engineering: 8 students;
- Scientific Communications, Mathematics Section: 5 students;
- Scientific Communications, Computer Science Section: 6 students.

The interested reader may find additional details on the competition's website `https://cmu-edu.eu/traian-lalescu`

We present the statements of the problems given in Sections A and B of the contest, together with the authors' solutions. The solutions given by the contestants mainly followed the official ones.

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1. SECTION A

Problem 1. Find all C^2 functions $f : \mathbb{R} \to (0, \infty)$ such that there exists $\alpha > 1$ for which we have

$$f''(x)f(x) \geq \alpha\big(f'(x)\big)^2, \ \forall x \in \mathbb{R}.$$

Ionel Rovenţa, University of Craiova

Author's solution. The first-order derivative of the function $\frac{1}{f^{\alpha-1}}$ is

$$\Big(\frac{1}{f^{\alpha-1}(x)}\Big)' = -\frac{(\alpha-1)f'(x)}{f^{\alpha}(x)}.$$

Hence, we observe that the second-order derivative of the same function, $\frac{1}{f^{\alpha-1}}$, satisfies

$$\Big(\frac{1}{f^{\alpha-1}(x)}\Big)'' = -(\alpha-1)\frac{f''(x)f(x) - \alpha\big(f'(x)\big)^2}{f^{2\alpha-1}(x)} \leq 0.$$

This implies that the function $\frac{1}{f^{\alpha-1}}$ is positive and concave. However, all positive and concave functions defined on $\mathbb{R}$ are constant, so the set of solutions is represented by the family of constant functions defined on $\mathbb{R}$.

Problem 2. Find all matrices $A \in \mathcal{M}_n(\mathbb{Z})$ for which we have

$$2025A^{2025} = A^{2024} + A^{2023} + \cdots + A.$$

Marian Daniel Vasile, West University of Timişoara

Author's solution. Let λ be an eigenvalue of A. We will first prove that $|\lambda| < 1$.

Let us assume that $|\lambda| \geq 1$. Then

$$|\lambda|^{2025} \geq |\lambda|^k, \ \forall k \in \{1, 2, 3, \ldots, 2024\},$$

hence $2025|\lambda|^{2025} > |\lambda|^{2024} + |\lambda|^{2023} + \cdots + |\lambda|^2 + |\lambda|$, since on the left-hand side we have more positive terms. At the same time, using the given equation for A, we also get

$$\begin{aligned} 2025|\lambda|^{2025} &= |\lambda^{2024} + \lambda^{2023} + \cdots + \lambda^2 + \lambda| \\ &\leq |\lambda|^{2024} + |\lambda|^{2023} + \cdots + |\lambda|^2 + |\lambda|, \end{aligned}$$

from which we get a contradiction.

We therefore have $|\lambda| < 1$. This implies that there exists $n_0 \in \mathbb{N}$ such that

$$|\lambda|^k \leq \frac{1}{2n}, \ \forall k \geq n_0.$$

Then for any $k \geq n_0$, we have

$$\left|\operatorname{tr}(A^k)\right| = \left|\sum_{i=1}^{n} \lambda_i^k\right| \leq \frac{n}{2n} = \frac{1}{2}$$

and, since $\operatorname{tr}(A^k) \in \mathbb{Z}$, we get $\operatorname{tr}(A^k) = 0$.

Using Newton sums, we obtain that all eigenvalues of A are 0, hence $A^n = O_n$. Let $m_A \in \mathbb{Z}[X]$ be the minimal polynomial of A. Then

$$m_A \mid \left(X^n,\ 2025X^{2025} - X^{2024} - X^{2023} - \cdots - X^2 - X\right) \Rightarrow m_A \mid X,$$

so $A = O_n$.

Problem 3. Let $f : [0, \infty) \to \mathbb{R}$ be a function that admits the first-order derivative in 0 and such that $f(0) = 0$. Compute the value of

$$\lim_{n\to\infty} \frac{1}{n} \int_{2^n}^{2^{n+1}} f\left(\frac{\ln x}{x}\right) \mathrm{d}x.$$

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Author's solution. Let $\varepsilon > 0$. Since f admits first-order derivative in 0, there exists $\delta_\varepsilon > 0$ such that, for all $0 < t < \delta_\varepsilon$, we have

$$\left|\frac{f(t) - f(0)}{t} - f'(0)\right| < \varepsilon \Rightarrow \left|f(t) - f'(0)t\right| < \varepsilon t.$$

Since

$$\lim_{n\to\infty} \frac{(n+1)\ln 2}{2^n} = 0,$$

there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$ we have $\dfrac{(n+1)\ln 2}{2^n} < \delta_\varepsilon$.

Let $n \geq n_\varepsilon$. For all $x \in [2^n, 2^{n+1}]$, we have

$$0 < \frac{\ln x}{x} \leq \frac{\ln 2^{n+1}}{2^n} = \frac{(n+1)\ln 2}{2^n} < \delta_\varepsilon$$

and

$$\left|f\left(\frac{\ln x}{x}\right) - f'(0)\frac{\ln x}{x}\right| < \varepsilon \frac{\ln x}{x}.$$

This implies

$$\int_{2^n}^{2^{n+1}} \left|f\left(\frac{\ln x}{x}\right) - f'(0)\frac{\ln x}{x}\right| \mathrm{d}x \leq \varepsilon \int_{2^n}^{2^{n+1}} \frac{\ln x}{x} \mathrm{d}x.$$

Since

$$\left|\int_{2^n}^{2^{n+1}} \left[f\left(\frac{\ln x}{x}\right) - f'(0)\frac{\ln x}{x}\right] \mathrm{d}x\right| \leq \int_{2^n}^{2^{n+1}} \left|f\left(\frac{\ln x}{x}\right) - f'(0)\frac{\ln x}{x}\right| \mathrm{d}x,$$

we get

$$\left|\int_{2^n}^{2^{n+1}}\left[f\left(\frac{\ln x}{x}\right)-f'(0)\frac{\ln x}{x}\right]\mathrm{d}x\right|\leq\varepsilon\int_{2^n}^{2^{n+1}}\frac{\ln x}{x}\mathrm{d}x,$$

that is,

$$\left|\frac{\displaystyle\int_{2^n}^{2^{n+1}}f\left(\frac{\ln x}{x}\right)\mathrm{d}x}{\displaystyle\int_{2^n}^{2^{n+1}}\frac{\ln x}{x}\mathrm{d}x}-f'(0)\right|\leq\varepsilon.$$

We therefore have

$$\lim_{n\to\infty}\frac{\displaystyle\int_{2^n}^{2^{n+1}}f\left(\frac{\ln x}{x}\right)\mathrm{d}x}{\displaystyle\int_{2^n}^{2^{n+1}}\frac{\ln x}{x}\mathrm{d}x}=f'(0). \tag{1}$$

At the same time,

$$\int_{2^n}^{2^{n+1}}\frac{\ln x}{x}\mathrm{d}x=\frac{\ln^2 x}{2}\bigg|_{2^n}^{2^{n+1}}=\frac{(2n+1)\ln^2 2}{2},$$

which implies

$$\lim_{n\to\infty}\frac{1}{n}\int_{2^n}^{2^{n+1}}\frac{\ln x}{x}\mathrm{d}x=\ln^2 2. \tag{2}$$

Combining (1) and (2), we finally get

$$\lim_{n\to\infty}\frac{1}{n}\int_{2^n}^{2^{n+1}}f\left(\frac{\ln x}{x}\right)\mathrm{d}x=f'(0)\ln^2 2.$$

Problem 4. Let $A,B\in\mathcal{M}_n(\mathbb{C})$. Prove that the following statements are equivalent:

1) For all matrices $C\in\mathcal{M}_n(\mathbb{C})$, there exist $X,Y\in\mathcal{M}_n(\mathbb{C})$ such that $AX+YB=C$;

2) For all matrices $C\in\mathcal{M}_n(\mathbb{C})$, there exist $U,V\in\mathcal{M}_n(\mathbb{C})$ such that $A^2U+VB^2=C$.

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Author's solution. 2) $\Rightarrow$ 1) If $A^2U+VB^2=C$, we consider $X=AU$ and $Y=VB$, and get $AX+YB=C$.

For 1) $\Rightarrow$ 2), we first introduce another statement:

3) $\det(A)\neq 0$ or $\det(B)\neq 0$.

We first prove that 1) $\Leftrightarrow$ 3). We start with 3) $\Rightarrow$ 1). If $\det(A) \neq 0$, we consider $Y = O_n$ and $X = A^{-1}C$, so $AX + YB = C$. If $\det(B) \neq 0$, we consider $X = O_n$ and $Y = CB^{-1}$, so once again $AX + YB = C$.

We now show that 1) $\Rightarrow$ 3). Let us suppose by contradiction that $\det(A) = \det(B) = 0$ and we prove that there exists $C \in \mathcal{M}_n(\mathbb{C})$ such that the equation $AX + YB = C$ has no solutions.

Since $\det(A) = 0$, all the rows of A are linearly dependent. This means that there exist elementary transformations such that all the elements in the first row of A are 0, hence, there exists an invertible matrix P such that $PA = A_1$, where all the elements in the first row of A_1 are 0. In a similar way, there exists an invertible matrix Q such that $BQ = B_1$, where all the elements in the first column of B_1 are 0. This implies that

$$AX + YB = C \Leftrightarrow P^{-1}A_1X + YB_1Q^{-1} = C \Leftrightarrow A_1XQ + PYB_1 = PCQ. \tag{3}$$

By choosing $C = P^{-1}Q^{-1}$ in (3), we obtain $A_1X_1 + Y_1B_1 = I_n$, where $X_1 = XQ$ and $Y_1 = PY$, which has no solutions, since the $(1,1)-$entry in the left-hand side matrix is 0 and in the right-hand side matrix is 1.

Therefore, we have established that 1) $\Leftrightarrow$ 3), from which the implication 1) $\Rightarrow$ 2) follows immediately.

2. Section B

Problem 1. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix whose entries are either a or b, where a and b are distinct positive real numbers. Moreover, we have $\operatorname{rank}(A) = 2$ and each row and each column of A contains at least one of a and at least one of b.

a) Prove that any two columns of A that start with the same element are equal.

b) Let U and V be two square matrices, whose entries are only 0 or 1, such that $A = aU + bV$. Prove that $\operatorname{rank}(U) = \operatorname{rank}(V) = 2$.

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Author's solution. a) Let k be a column index for which $a_{1k} \neq a_{11}$ and let p be a row index such that $a_{p1} \neq a_{11}$. These indices exists, according to our initial hypothesis. It is clear that $a_{p1} = a_{1k}$. Then it is easy to see that

$$\Delta := \begin{vmatrix} a_{11} & a_{1k} \\ a_{p1} & a_{pk} \end{vmatrix} \neq 0.$$

We shall prove that all columns A_j of A for which $a_{1j} = a_{11}$ are equal and, clearly, all columns A_j of A for which $a_{1j} = a_{1k}$ are equal.

Let A_j, with $j \neq 1$, be a column of A such that $a_{1j} = a_{11}$. Since $\operatorname{rank}(A) = 2$ and $\Delta \neq 0$, A_j is a unique linear combination of the columns

A_1 and A_k:

$$A_j = \alpha A_1 + \beta A_k,\ \alpha, \beta \in \mathbb{R}.$$

Then

$$\begin{cases} \alpha a_{11} + \beta a_{1k} = a_{1j} = a_{11} \\ \alpha a_{p1} + \beta a_{pk} = a_{pj} \end{cases} \Leftrightarrow \begin{cases} \alpha a_{11} + \beta a_{1k} = a_{1j} = a_{11} \\ \alpha a_{1k} + \beta a_{pk} = a_{pj}. \end{cases}$$

Let us assume that $a_{pj} = a_{11}$. Then, by taking the minor formed by using the qth row of A (containing $a_{qj} = a_{1k}$ — or, equivalently, $a_{qj} \neq a_{11}$), we get

$$\delta := \begin{vmatrix} a_{11} & a_{1k} & a_{1j} \\ a_{p1} & a_{pk} & a_{pj} \\ a_{q1} & a_{qk} & a_{qj} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{1k} & a_{11} \\ a_{1k} & a_{pk} & a_{11} \\ a_{q1} & a_{qk} & a_{1k} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{1k} & a_{11} \\ a_{1k} - a_{11} & a_{pk} - a_{1k} & 0 \\ a_{q1} & a_{qk} & a_{1k} \end{vmatrix}.$$

If $a_{pk} = a_{1k}$, then $\delta = (a_{1k} - a_{11})(a_{1k}^2 - a_{11}a_{qk}) \neq 0$.

If $a_{pk} = a_{11}$, then

$$\delta = (a_{1k} - a_{11}) \begin{vmatrix} a_{11} & a_{1k} & a_{11} \\ 1 & -1 & 0 \\ a_{q1} & a_{qk} & a_{1k} \end{vmatrix} = (a_{11} - a_{1k}) \begin{vmatrix} a_{1k} + a_{11} & a_{11} \\ a_{qk} + a_{q1} & a_{1k} \end{vmatrix} \neq 0,$$

since $a_{qk} + a_{q1} \in \{a_{11} + a_{11}, a_{11} + a_{1k}, a_{1k} + a_{1k}\}$.

In both situations, we would get that $\operatorname{rank}(A) \geq 3$, which is false. We obtain $a_{pj} = a_{1k}$ and, from the uniqueness of the solution to the system, we get $\alpha = 1$ and $\beta = 0$, that is, $A_j = A_1$.

b) If $a_{ij} = a$, then

$$a = au_{ij} + bv_{ij} \Rightarrow a(1 - u_{ij}) = bv_{ij}.$$

Since $u_{ij}, v_{ij} \in \{0, 1\}$ and $a \neq b$, we have $1 - u_{ij} = v_{ij} = 0$.

In the same way, if $a_{ij} = b$, then $u_{ij} = 1 - v_{ij} = 0$.

We have obtained in this manner that U and V are uniquely determined:

$$u_{ij} = \begin{cases} 1, & \text{if } a_{ij} = a, \\ 0, & \text{if } a_{ij} = b, \end{cases} \quad \text{and} \quad v_{ij} = \begin{cases} 0, & \text{if } a_{ij} = a, \\ 1, & \text{if } b_{ij} = b. \end{cases}$$

Since in A there exists a minor of the form

$$\begin{vmatrix} a & b \\ b & \star \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} b & a \\ a & \star \end{vmatrix},$$

there is also a minor in V of the form

$$\begin{vmatrix} 0 & 1 \\ 1 & \star \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 1 & 0 \\ 0 & \star \end{vmatrix},$$

which implies that $\operatorname{rank}(V) \geq 2$. In a similar way, $\operatorname{rank}(U) \geq 2$.

At the same time, according to a), the columns of U that start with the same element are equal, so $\operatorname{rank}(U) \leq 2$. Similarly, $\operatorname{rank}(V) \leq 2$, so the conclusion follows.

Problem 2. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers convergent to $l \in \mathbb{R}$ such that $a_n \neq l$, for all $n \in \mathbb{N}$.

a) Let us suppose that the sequence is monotone. Study the limit of the sequence $\big([a_n]\big)_{n\in\mathbb{N}}$ and prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n a_n}{n}$ converges and that the series $\displaystyle\sum_{n=1}^{\infty} \frac{\big[(-1)^n a_n\big]}{n}$ diverges.

b) Study the limit of the sequence $\big([a_n]\big)_{n\in\mathbb{N}}$ and the nature of the series from point a) in the general case (in the case in which the sequence is not monotone).

Note. By $[x]$ we understand floor of x (the integer part of the real number x).

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Author's solution. a) In the case in which $l \notin \mathbb{Z}$, the sequence $\big([a_n]\big)_{n\in\mathbb{N}}$ converges to $[l]$, without using monotonicity. Indeed, we have that there exists $\varepsilon > 0$ such that

$$[l] < l - \varepsilon < l < l + \varepsilon < [l] + 1. \tag{4}$$

Since $a_n \to l$, one has, for all n large enough, $a_n \in (l-\varepsilon, l+\varepsilon)$, so there exists $n_0 \in \mathbb{N}$ such that $[a_n] = [l]$, for all $n \geq n_0$. In this way, we get $[a_n] \to [l]$.

Let us now assume that $l \in \mathbb{Z}$. If the sequence is decreasing, then there exists $n_0 \in \mathbb{N}$ such that

$$l < a_n < l + 1,\ \forall n \geq n_0, \tag{5}$$

hence $[a_n] = l$, for all $n \geq n_0$. This implies $[a_n] \to l$. If the sequence is increasing, then, since $a_n \neq l$, $\forall n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that

$$l - 1 < a_n < l,\ \forall n \geq n_0, \tag{6}$$

so $[a_n] = l - 1$, $\forall n \geq n_0$. In this case, $[a_n] \to l - 1$.

In the case in which the sequence is monotone, then, since it is also bounded because it is convergent, the first series is convergent by Abel's test.

For the second series, we consider first the case $l \notin \mathbb{Z}$. In this case, we do not need monotonicity in order to show that the series is divergent. Indeed, for all $\varepsilon > 0$ such that (4) holds, since $a_n \to l$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_1 = 2n_0$ we have

$$[l] < l - \varepsilon < (-1)^{2n} a_{2n} < l + \varepsilon < [l] + 1,\ \ -[l] - 1 < (-1)^{2n+1} a_{2n+1} < -[l].$$

Then, for all $n \geq n_1$ it holds

$$\big[(-1)^n a_n\big] = \begin{cases} [l], & n = \text{even}, \\ -[l] - 1, & n = \text{odd}. \end{cases}$$

Of course, the nature of the series does not change when we eliminate the first terms, hence

$$\sum_{n=1}^{\infty}\frac{\left[(-1)^n a_n\right]}{n}\sim\sum_{n=n_1}^{\infty}\frac{\left[(-1)^n a_n\right]}{n}=\sum_{n=n_1}^{\infty}\frac{(-1)^n[l]}{n}-\sum_{n=n_0}^{\infty}\frac{1}{2n+1},$$

which is divergent.

Let now $l\in\mathbb{Z}$. If $(a_n)_{n\in\mathbb{N}}$ is decreasing, then, from (5), there exists $n_0\in\mathbb{N}$ such that, for all $n\geq n_0$, we have

$$[l]=l<(-1)^{2n}a_{2n}<[l]+1\quad\text{and}\quad -[l]-1<(-1)^{2n+1}a_{2n+1}<-[l].$$

If the sequence is increasing, from (6), there exists $n_0\in\mathbb{N}$ such that, for all $n\geq n_0$, we have

$$[l]-1<(-1)^{2n}a_{2n}<[l]\quad\text{and}\quad -[l]<(-1)^{2n+1}a_{2n+1}<-[l]+1.$$

In both cases, the same argument as before works in order to prove that the second series is divergent.

b) If $l\notin\mathbb{Z}$, we have shown previously that $[a_n]\to[l]$.

If $l\in\mathbb{Z}$, let

$$a_n:=\begin{cases}l-\dfrac{1}{n}, & n=\text{even},\\[2mm] l+\dfrac{1}{n}, & n=\text{odd}.\end{cases}$$

This sequence has the properties $a_n\to l$ and

$$[a_n]=\begin{cases}l-1, & n=\text{even},\\ l, & n=\text{odd},\end{cases}$$

which clearly imply that the sequence $\big([a_n]\big)_{n\in\mathbb{N}}$ is divergent (it has no limit).

We shall prove that the first series is not necessarily convergent. For

$$a_n:=\frac{(-1)^n}{\ln(n+1)},$$

the series becomes

$$\sum_{n=1}^{\infty}\frac{1}{n\ \ln(n+1)}\sim\sum_{n=1}^{\infty}\frac{1}{n\ \ln(n)},$$

hence, it is divergent.

The second series is not necessarily divergent. For

$$a_n:=l+\frac{(-1)^n}{n}\to l,$$

we get

$$(-1)^n a_n = \begin{cases} l + \dfrac{1}{n}, & n = \text{even}, \\ -l + \dfrac{1}{n}, & n = \text{odd}. \end{cases}$$

Then,

$$\left[(-1)^n a_n\right] = \begin{cases} l, & n = \text{even}, \\ -l, & n = \text{odd}, \end{cases}$$

so the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, l}{n},$$

which is convergent.

Problem 3. Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be such that $A^2 = B^2 = I_n$ and $AB + BA = -2(A + B + I_n)$. Prove that:

a) $AB = BA$;

b) $\operatorname{rank}(A + I_n) + \operatorname{rank}(B + I_n) \leq n$.

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Author's solution. a) Let $C = A + I_n$ and $D = B + I_n$. Then $CD = AB + A + B + I_n$ and $DC = BA + A + B + I_n$. This implies $CD + DC = AB + BA + 2(A + B + I_n) = O_n$, according to our hypothesis. At the same time, $I_n = A^2 = (C - I_n)^2 = C^2 - 2C + I_n$, which implies $C^2 = 2C$, and, in a similar way, $D^2 = 2D$.

We also have that

$$CD = -DC = -\frac{1}{2}D^2C = \frac{1}{2}D(-DC) = \frac{1}{2}(DC)D =$$
$$= \frac{1}{2}(-CD)D = -C\left(\frac{1}{2}D^2\right) = -CD,$$

so $CD = O_n$ and this also implies $DC = O_n$. Therefore, $AB - BA = CD - DC = O_n$, so $AB = BA$.

b) We have $(C+D)^2 = C^2 + D^2 + CD + DC = 2C + 2D$. Together with $C^2 = 2C$ and $D^2 = 2D$, we see that C, D, and $C + D$ all verify the equation $X^2 = 2X$, so all three matrices are diagonalisable and their eigenvalues are from the set $\{0, 2\}$.

This implies that $\operatorname{rank}(C) = \frac{1}{2}\operatorname{tr}(C)$, $\operatorname{rank}(D) = \frac{1}{2}\operatorname{tr}(D)$, and

$$\operatorname{rank}(C + D) = \frac{1}{2}\operatorname{tr}(C + D) = \frac{1}{2}\operatorname{tr}(C) + \frac{1}{2}\operatorname{tr}(D) = \operatorname{rank}(C) + \operatorname{rank}(D).$$

Hence $\operatorname{rank}(A + I_n) + \operatorname{rank}(B + I_n) = \operatorname{rank}(C + D) \leq n$.

The equality holds if and only if $C+D$ is invertible and, since we also have $(C+D)^2 = 2(C+D)$, this leads to $C+D = 2I_n$, so $A+B = O_n$.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function, of class C^{2025}, for which $f^{(2k+1)}(0) = 0$, for all $k \in \overline{1, 1011}$, and $f^{(2025)}(0) > 0$.

a) Prove that there exists $\varepsilon > 0$ such that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1 + \frac{1}{n^{f(x)}}\right)$$

is convergent for all $x \in (0, \varepsilon)$.

b) For all $x \in (0, \varepsilon)$, we denote the sum of the previous series by $S(x)$ and let $g(x) = \dfrac{1}{S(x)}$. Study the convergence of the series

$$\sum_{n=1}^{\infty} g\Big(\frac{1}{n^{\alpha}}\Big),$$

where $\alpha > 0$.

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Author's solution. Since f is odd, we have f' even, f'' odd, f''' even, and so on. Then $f(0) = f''(0) = \ldots = f^{(2024)}(0) = 0$. Using our hypothesis and Taylor's formula, we get that for any $x > 0$ there exists $c_x \in (0, x)$ such that:

$$f(x) = \frac{f^{(2025)}(c_x)}{2025!} \cdot x^{2025}. \tag{7}$$

Since f is of class C^{2025} and $f^{(2025)}(0) > 0$, there exists $\varepsilon > 0$ such that $f^{(2025)} > 0$ in $(0, \varepsilon)$, so, by using (7), we have that $f(x) > 0$ for any $x \in (0, \varepsilon)$.

Using the limit comparison test for series, we obtain that for any fixed $x \in (0, \varepsilon)$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1 + \frac{1}{n^{f(x)}}\right) \sim \sum_{n=1}^{\infty} \frac{1}{n^{f(x)+1}},$$

which is convergent.

b) We start by observing that $S(x) \geq \ln 2$ for all $x \in (0, \varepsilon)$, so g is well-defined.

For any fixed $x \in (0, \varepsilon)$, we consider the function $h_x : (0, \infty) \to \mathbb{R}$ defined by

$$h_x(t) = t \ln\left(1 + t^{f(x)}\right), \ \forall t \in (0, \infty).$$

This is a strictly increasing function, so $h_x\left(\frac{1}{t}\right)$ is strictly decreasing. Then, for any $n \geq 1$ and any $x \in (0, \varepsilon)$, we have

$$h_x\Big(\frac{1}{n}\Big) \geq h_x\Big(\frac{1}{t}\Big) \geq h_x\Big(\frac{1}{n+1}\Big), \ \forall t \in [n, n+1],$$

so

$$h_x\Big(\frac{1}{n}\Big) \geq \int_n^{n+1} h_x\Big(\frac{1}{t}\Big)\,\mathrm{d}t \geq h_x\Big(\frac{1}{n+1}\Big).$$

By summing from 1 to N and letting $N \to \infty$, we obtain

$$S(x) = \sum_{n=1}^{\infty} h_x\Big(\frac{1}{n}\Big) \geq \int_1^{\infty} h_x\Big(\frac{1}{t}\Big)\,\mathrm{d}t \geq \sum_{n=2}^{\infty} h_x\Big(\frac{1}{n}\Big) = S(x) - \ln(2). \tag{8}$$

Using a), we get that the integral $\displaystyle\int_1^{\infty} h_x\left(\frac{1}{t}\right)\mathrm{d}t$ converges. We now compute the limit:

$$\ell = \lim_{x\to 0_+} x^{2025} S(x) \overset{(8)}{=} \lim_{x\to 0_+} x^{2025} \int_1^{\infty} \frac{1}{t}\,\ln\Big(1 + \frac{1}{t^{f(x)}}\Big)\mathrm{d}t.$$

We also compute the following integral

$$I_x = \int_1^{\infty} \frac{1}{t}\,\ln\Big(1 + \frac{1}{t^{f(x)}}\Big)\mathrm{d}t$$

using the following change of variables

$$\frac{1}{t^{f(x)}} = y, \quad \mathrm{d}t = -\frac{1}{f(x)}\, y^{-\frac{1}{f(x)}-1}\mathrm{d}y,$$

so the integral becomes

$$I_x = \frac{1}{f(x)} \int_0^1 \frac{\ln(1+y)}{y}\mathrm{d}y = \frac{1}{f(x)} \cdot \frac{\pi^2}{12}.$$

One obtains that $\displaystyle\int_0^1 \frac{\ln(1+y)}{y}\mathrm{d}y = \frac{\pi^2}{12}$ by integrating term-by-term the power series expansion, but, at the same time, we only need the integral to converge to a strictly positive real number.

Furthermore, using (7) and that f is of class C^{2025}, we get the following

$$\lim_{x\to 0_+} \frac{f(x)}{x^{2025}} = \lim_{x\to 0_+} \frac{f^{2025}(c_x)}{2025!} = \frac{f^{2025}(0)}{2025!}.$$

Then,

$$\ell = \lim_{x\to 0_+} x^{2025} S(x) = \lim_{x\to 0_+} \frac{x^{2025}}{f(x)} \cdot \frac{\pi^2}{12} = \frac{2025!}{f^{2025}(0)} \cdot \frac{\pi^2}{12} > 0.$$

This implies that, for any $\alpha > 0$, we have

$$\lim_{n\to\infty} n^{2025\alpha} g\left(\frac{1}{n^\alpha}\right) = \lim_{x\to 0_+} \frac{1}{x^{2025} S(x)} = \frac{1}{\ell} > 0,$$

from which we get, by using the limit comparison test, that

$$\sum_{n=1}^{\infty} g\left(\frac{1}{n^\alpha}\right) \sim \sum_{n=1}^{\infty} \frac{1}{n^{2025\alpha}},$$

so the series is convergent if $\alpha > \dfrac{1}{2025}$ and divergent otherwise.

MATHEMATICAL NOTES

On the weak hazard rate ordering obtained using geometric transformations and multivariate Bernoulli distribution

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Abstract. In this paper we present results obtained on how one of the translation, rotation or dilatation applied to a d-dimensional random vector Bernoulli distributed gives us a dominant vector in the weak hazard rate sense.

Keywords: Weak hazard rate order, geometric transformation.
MSC: 60E15

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, P)$ be a probabilty space and $X : \Omega \to \mathbb{R}^d$ a random vector, $d \geq 2$, $X \sim B(1,p)$ (X is multivariate Bernoulli distributed), that is, $P(X = \alpha) = p_\alpha \in (0;1)$, $\alpha \in \mathbb{R}^d$, $\alpha_i \in \{0;1\}$, $i = \overline{1,d}$.

For $x, y \in \mathbb{R}^d$ we say that $x \leq y$ if $x_i \leq y_i$ for $i = \overline{1,d}$, and $x < y$ if $x \leq y$ and $x \neq y$.

We denote $F_X^*(x) = P(X > x)$ and $U(x) = \left\{y \in \mathbb{R}^d : y \geq x\right\}$.

For a random vector X with F_X^* differentiable, we consider the hazard rate function $r_X : \mathrm{Supp}(F_X^*) \to [0;\infty)$, $r_X(x) = \nabla(-\ln F_X^*(x))$, where $\mathrm{Supp}(F_X^*) = \{x \in \mathbb{R}^d : F_X^*(x) > 0\}$.

Definition 1. *Let X and Y be two d-dimensional random vectors with r_X, r_Y hazard rate functions. We say that X is* smaller *than Y in weak hazard rate sense and we denote $X \prec_{\mathrm{whr}} Y$ if $r_X(x) \geq r_Y(x)$ $\forall x \in \mathrm{Supp}(F_X^*) \cap \mathrm{Supp}(F_Y^*)$.*

Although Definition 1 is the basic one, it cannot be applied in the case of discrete distributions, therefore we will work with the weak hazard rate order in the sense of the following

Theorem 2. (see [2]) *Let X and Y be two d-dimensional random vectors. Then $X \prec_{\mathrm{whr}} Y$ if and only if $\frac{F_Y^*}{F_X^*}$ is nondecreasing on $\mathrm{Supp}(F_Y^*)$.*

We want to find a geometric transform φ such that $X \prec_{\mathrm{whr}} \varphi(X)$. Proposal for φ: translation, rotation or dilatation. In these cases it holds $\varphi(X) \sim B(1,q)$, $q_\alpha = p_{\varphi^{-1}(\alpha)}$.

In [1] the same problem was studied for dilatation and translation in the case of uniform distribution on $\mathcal{B}\left(\mathbb{R}^2\right)$.

The following two results are useful in the proofs of the main results.

[1)]Ion Barbu High School, Bucharest, Romania, luigi_catana@yahoo.com

Theorem 3. (see [2, pg. 290]) *If* $X \prec_{\text{whr}} Y$, *then* $\text{Supp}(F_X^*) \subset \text{Supp}(F_Y^*)$.

Theorem 4. *If* $X \prec_{\text{whr}} Y$, *then* $F_X^*(x) \leq F_Y^*(x)\ \forall x \in \mathbb{R}^d$.

Proof. $X \prec_{\text{whr}} Y \Longrightarrow \frac{F_Y^*(y)}{F_X^*(y)} \leq \frac{F_Y^*(x)}{F_X^*(x)}\ \forall x, y \in \text{Supp}(F_Y^*)$, $y \leq x$. Letting $y \to -\infty$, we obtain the conclusion. □

2. Main results

Theorem 5. *Let* $X \sim B(1,p)$, $p \in (0;1)^d$, and $t \in \mathbb{R}^d$. *We have that* $X \prec_{\text{whr}} X + t \Longleftrightarrow t \geq (1;1;\ldots;1)^T$.

Proof. "$\Longrightarrow$" We suppose that there exists some entry of t with $t_k < 1$. Without loss of generality, we consider $t_1 < 1$.

For $x = (\varepsilon;1;1;\ldots;1)$, $y = (1-\varepsilon;1;1;\ldots;1)$, $\varepsilon \in \left(0;\min\left(t_1;\frac{1}{2}\right)\right)$, we have that $x < y$ and

$$\frac{F_Y^*(x)}{F_X^*(x)} = \frac{p_{(0;1;1;\ldots;1)} + p_{(1;1;1;\ldots;1)}}{p_{(1;1;1;\ldots;1)}} > \frac{p_{(1;1;1;\ldots;1)}}{p_{(1;1;1;\ldots;1)}} = \frac{F_Y^*(y)}{F_X^*(y)},$$

a contradiction.

Therefore $t \geq (1;1;\ldots;1)^T$.

"$\Longleftarrow$" Let $x \in \text{Supp}(F_Y^*)$. Let $y \in \mathbb{R}^d$, $y \geq x$. We distinguish two cases.

Case 1: $x < (1;1;\ldots;1)^T$

If $y < (1;1;\ldots;1)^T$, then $\frac{F_Y^*(x)}{F_X^*(x)} = \frac{F_Y^*(y)}{F_X^*(y)}$.

If $y \not< (1;1;\ldots;1)^T$, then $y \notin \text{Supp}(F_X^*)$, thus $\frac{F_Y^*(x)}{F_X^*(x)} < \infty = \frac{F_Y^*(y)}{F_X^*(y)}$.

Case 2: $x \not< (1;1;\ldots;1)^T$

Then $x, y \notin \text{Supp}(F_X^*)$, thus $\frac{F_Y^*(x)}{F_X^*(x)} = \infty = \frac{F_Y^*(y)}{F_X^*(y)}$. □

In the following we denote by B_d the hyperoctahedral group.

Theorem 6. *Let* $X \sim B(1,p)$, $p \in (0;1)^d$. *There is no rotation matrix* $R \notin B_d$ *such that* $X \prec_{\text{whr}} RX$.

Proof. Let us suppose that there exists a rotation matrix $R \notin B_d$ such that $X \prec_{\text{whr}} RX$. We have that $\|Rx\| = \|x\| \leq \left\|(1;1;\ldots;1)^T\right\| = \sqrt{d}$ $\forall x \in \text{Supp}(X)$. Thus, $U\left((1;1;\ldots;1)^T\right) \cap \text{Supp}(RX) = \emptyset$. Then there exists $\varepsilon > 0$ such that

$$F_X^*(1-\varepsilon;1-\varepsilon;\ldots;1-\varepsilon) = p_{(1;1;\ldots;1)} > 0 = F_Y^*(1-\varepsilon;1-\varepsilon;\ldots;1-\varepsilon),$$

a contradiction.

Therefore there is no rotation matrix $R \notin B_d$ such that $X \prec_{\text{whr}} RX$. □

Remark 7. In the case $R \in B_d$, the problem $X \prec_{\text{whr}} RX$ is reduced to computationally studying a finite number of inequalities.

Theorem 8. *Let* $X \sim B(1,p)$, $p \in (0;1)^d$, *and* $\lambda \in (0;\infty)$. *We have that* $X \prec_{\text{whr}} \lambda X$ *if and only if* $\lambda \geq 1$.

Proof. $X \prec_{\text{whr}} \lambda X \Longrightarrow \text{Supp}(F_X^*) \subset \text{Supp}(F_{\lambda X}^*) \Longrightarrow \lambda \geq 1$.

For $\lambda \geq 1$ it is proved similarly as implication "$\Longleftarrow$" from the proof of Theorem 5. □

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[1] L. I. Catana, Multivariate weak hazard rate order according to hazard rate function, *Gaz. Mat. Ser. A* **37** (2019), no. 1–2, 27–30.

[2] M. Shaked, J. G. Shantikumar, *Stochastic Orders*, Springer Series in Statistics, Springer, 2007.

PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of May 2026**.

PROPOSED PROBLEMS

578. Let $A \in \mathcal{M}_{n,m}(\mathbb{C})$, $B \in \mathcal{M}_{m,n}(\mathbb{C})$ and $C \in \mathcal{M}_m(\mathbb{C})$ such that $\operatorname{rank} A = \operatorname{rank} B = m$. Show that there exists a unique matrix $D \in \mathcal{M}_n(\mathbb{C})$ such that $BDA = C$ and determine an explicit formula for D.

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania

579. Let $a \in (0,1)$. Find all continuously differentiable functions $f\colon [0,\infty) \to \mathbb{R}$ with $f(0) = 0$ such that

$$(x-y)\left(f'(x-y)\right)^2 \leq f(x-y)\cdot f'(ax) \quad \text{for all } x, y \text{ with } x > y \geq 0.$$

Proposed by Stănescu Florin, Șerban Cioculescu Secondary School, Găești, Dâmbovița, Romania.

580. Prove that $3/4$ is the largest positive value of the power exponent k such that the inequality

$$(ab)^k + (ac)^k + (ad)^k + (bc)^k + (bd)^k + (cd)^k \leq 6$$

holds for any $a \geq b \geq c \geq d \geq 0$ with $ab + bc + cd + da = 4$.

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania.

581. Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_1 \in (0,1)$ and

$$x_{n+1} = \frac{x_1(1-x_1^2) + x_2\sqrt{1-x_2^2} + \cdots + x_n \sqrt[n]{1-x_n^2}}{n}$$

for all $n \geq 1$.

Prove that the sequence $(x_n)_{n\geq 1}$ is convergent to a limit $C > 0$ and, moreover, $\lim\limits_{n\to\infty} n(x_n - C) = -C\ln(1-C^2)$.

Proposed by Dumitru Popa, Department of Mathematics, Ovidius University of Constanța, Romania.

582. a) Let $(V, \langle\cdot,\cdot\rangle)$ be an Euclidean space over $\mathbb{C}$ and $T : V \to V$ an endomorphism with the property that $\langle T(z), z\rangle = 0$ for all $z \in V$. Prove that $T = 0$.

b) Give an example of a real Euclidean space W and an endomorphism $S : W \to W$, $S \neq 0$, for which $\langle S(x), x\rangle = 0$ for all $x \in W$.

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania

583. Let $n \geq 3$ be an integer. Find the largest real constant C with the property that

$$\frac{a_1 + a_2 + \cdots + a_n}{\sqrt[n]{a_1 a_2 \cdots a_n}} - n \geq C \cdot \frac{\left(\sqrt[n]{a_1} - \sqrt[n]{a_n}\right)^2}{\sqrt[n]{a_1 a_n}}$$

holds for all positive real numbers $a_1 \geq a_2 \geq \cdots \geq a_n$.

Proposed by Leonard Giugiuc, Greci Secondary School, Mehedinți, Romania.

584. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function with $\displaystyle\int_0^1 f(x)\,\mathrm{d}x = 0$ and let $\alpha \in \mathbb{R}$, $\alpha > 0$. Prove that there is $c \in (0,1)$ such that

$$\int_0^c x^{\alpha+2} f(x)\,\mathrm{d}x = -\alpha c^{\alpha+1} \int_0^c x f(x)\,\mathrm{d}x.$$

Proposed by Stănescu Florin, Șerban Cioculescu Secondary School, Găești, Dâmbovița, Romania.

SOLUTIONS

563. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative such that $f(0) = f(1)$ and $\displaystyle\int_0^1 x f(x)\mathrm{d}x = 0$. Prove that

$$\int_0^1 (f'(x))^2 \mathrm{d}x \geq 180 \left(\int_0^1 f(x)\mathrm{d}x\right)^2.$$

Proposed by Cezar Lupu, Beijing Institute of Mathematical Sciences and Applications (BIMSA), Tsinghua University, Beijing, P. R. China, and Tudorel Lupu, Decebal High School, Constanța, Romania.

Solution by the authors. We will use a very tricky Cauchy–Schwarz! Indeed, by the Cauchy–Schwarz inequality we have

$$\int_0^1 \left[\left(x - \frac{1}{2}\right)^2 - \frac{1}{12}\right]^2 \mathrm{d}x \int_0^1 (f'(x))^2\,\mathrm{d}x \geq \left(\int_0^1 \left[\left(x - \frac{1}{2}\right)^2 - \frac{1}{12}\right] f'(x)\,\mathrm{d}x\right)^2.$$

Now, a short computation gives us

$$\int_0^1 \left[\left(x-\frac{1}{2}\right)^2-\frac{1}{12}\right]^2 \mathrm{d}x = \int_0^1 \left(x^2-x+\frac{1}{6}\right)^2 \mathrm{d}x$$
$$= \frac{1}{36}\int_0^1 (6x^2-6x+1)^2\,\mathrm{d}x = \frac{1}{36\cdot 5}.$$

On the other hand, integrating by parts, we have

$$\int_0^1 \left[\left(x-\frac{1}{2}\right)^2-\frac{1}{12}\right] f'(x)\,\mathrm{d}x = \int_0^1 \left(x^2-x+\frac{1}{6}\right) f'(x)\,\mathrm{d}x$$
$$= \frac{1}{6}(f(1)-f(0)) - \int_0^1 \left(x^2-x+\frac{1}{6}\right)' f(x)\,\mathrm{d}x$$
$$= \frac{1}{6}(f(1)-f(0)) - \int_0^1 (2x-1)f(x)\,\mathrm{d}x$$
$$= \int_0^1 f(x)\,\mathrm{d}x.$$

Here we used the hypothesis that $f(0)=f(1)$ and $\displaystyle\int_0^1 xf(x)\,\mathrm{d}x=0$. It follows that

$$\frac{1}{36\cdot 5}\int_0^1 (f'(x))^2\,\mathrm{d}x \geq \left(\int_0^1 f(x)\,\mathrm{d}x\right)^2,$$

and thus the conclusion follows immediately. The equality case holds for functions such as $f(x)=\lambda(60x^3-90x^2+30x+1)$, where $\lambda\in\mathbb{R}$.

Solution by Ulrich Abel and Vitaliy Kushnirevych, Technische Hochschule Mittelhessen, Germany. By the assumptions $\int_0^1 xf(x)\,\mathrm{d}x = 0$ and $f(0)=f(1)$, we have, for arbitrary reals c,

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 (1-2x)f(x)\,\mathrm{d}x = -\int_0^1 \left(x-x^2+c\right) f'(x)\,\mathrm{d}x,$$

since $\left(x-x^2+c\right) f(x)\Big|_0^1 = 0$. Application of the Cauchy–Schwarz inequality yields

$$\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2 = \left(\int_0^1 \left(x-x^2+c\right) f'(x)\,\mathrm{d}x\right)^2 \leq M(c)\int_0^1 \left(f'(x)\right)^2\,\mathrm{d}x,$$

where

$$M(c) = \int_0^1 \left(x - x^2 + c\right)^2 \mathrm{d}x = c^2 + \frac{c}{3} + \frac{1}{30} = \frac{1}{180} + \left(c + \frac{1}{6}\right)^2.$$

Since $M(-1/6) = 1/180$, we conclude that

$$\int_0^1 \left(f'(x)\right)^2 \mathrm{d}x \geq \frac{1}{M(-1/6)} \left(\int_0^1 f(x)\,\mathrm{d}x\right)^2 = 180 \left(\int_0^1 f(x)\,\mathrm{d}x\right)^2.$$

We received a very similar solution from Henry Ricardo, from Westchester Area Math Circle, New York, USA. He arrived to essentially the same formula, written as $\int\limits_0^1 f(x)\,\mathrm{d}x = \int\limits_0^1 (x^2 - x + a) f(x)\,\mathrm{d}x$, and then he uses Cauchy–Schwarz inequality. And he shows that $\int\limits_0^1 (x^2 - x + a) f(x)\,\mathrm{d}x = 1/180$ is obtained for $a = 1/6$.

The same approach appears in a solution by Brian Bradie, from Christopher Newport University, Newport News, Virginia, USA, and one by Marian Cucoaneș, from Valeriu D. Cotea Technical College, Focșani, Romania.

We also received an incomplete solution from Mihalcea Nicolae. His idea is to regard the problem as an optimization one, of determining a function f that minimizes the integral $\int_0^1 (f'(x))^2\,\mathrm{d}x$. He proves that the inequality holds for polynomials of degree 3 and he determines the cases of equality. Then he claims that one may reduce to the polynomial case of degree 3 by using the calculus of variations and the Euler–Lagrange equations. However, he gives no details on how this is done.

564. Let $a_1, \ldots, a_7$ be nonnegative real numbers such that $a_1a_2 + a_2a_3 + \cdots + a_7a_1 = 7$. Prove that

$$\text{(a)} \qquad \frac{1}{5a_1 + 3} + \frac{1}{5a_2 + 3} + \cdots + \frac{1}{5a_7 + 3} \geq \frac{7}{8}.$$

$$\text{(b)}^* \qquad \frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \cdots + \frac{1}{a_7 + 1} \geq \frac{7}{2}.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania.

Solution by the author. (a) We first show that

$$64\left(\frac{1}{5a_1 + 3} + \frac{1}{5a_2 + 3}\right) + 5a_1a_2 \geq 21.$$

Denoting $s = \dfrac{a_1 + a_2}{2}$ and $p = \sqrt{a_1a_2}$, the inequality writes as follows:

$$\frac{64(10s + 6)}{25p^2 + 30s + 9} + 5p^2 \geq 21,$$

$$2(15p^2+1)s + 25p^4 - 96p^2 + 39 \geq 0.$$

Since $s \geq p$, we have

$$2(15p^2+1)s + 25p^4 - 96p^2 + 39 \geq 2(15p^2+1)p + 25p^4 - 96p^2 + 39$$
$$= 25p^4 + 30p^3 - 96p^2 + 2p + 39 = (p-1)^2(25p^2+80p+39) \geq 0.$$

Thus, from

$$\sum_{\text{cyc}} \left[64\left(\frac{1}{5a_1+3} + \frac{1}{5a_2+3} \right) + 5a_1a_2 - 21 \right] \geq 0$$

and the hypothesis we get the desired inequality. The equality occurs for $a_1 = \cdots = a_7 = 1$.

(b)* This is an open problem. We prove that this is the best result possible in the sense that for $k > 1$ the inequality

$$\frac{1}{a_1+k} + \frac{1}{a_2+k} + \cdots + \frac{1}{a_7+k} \geq \frac{7}{1+k}$$

doesn't always hold. Choosing $a_1 = a_4 = 3$, $a_2 = a_3 = 1$, and $a_5 = a_7 = 0$, the equality constraint is satisfied and the inequality becomes

$$\frac{1}{a_6+k} + \frac{2}{3+k} + \frac{2}{k} \geq \frac{5}{1+k}.$$

Moreover, for $a_6 \to \infty$ we get the necessary condition

$$\frac{2}{3+k} + \frac{2}{k} \geq \frac{5}{1+k},$$

which is equivalent to $-(k-1)(k+6) \geq 0$, so it implies $k \leq 1$.

Remark. From the proof of (a), it follows that the inequality

$$\frac{1}{5a_1+3} + \cdots + \frac{1}{5a_n+3} \geq \frac{n}{8}$$

holds for $n \geq 2$ and any nonnegative real numbers $a_1, a_2, \ldots, a_n$ such that $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$.

Note from the editor. Given $n \geq 2$ and $k > 0$, the author is interested in the following statement which we denote by $P(n,k)$: For any $a_1, \ldots, a_n > 0$ such that $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$ we have $\dfrac{1}{a_1+k} + \cdots + \dfrac{1}{a_n+k} \geq \dfrac{n}{1+k}$. The (a) part of the problem states that $P(7, 3/5)$ holds, but, as seen in the Remark, the proof applies to all $P(n, 3/5)$ for $n \geq 2$. The open problem (b)* is just $P(7,1)$ and the author proves that $P(7,k)$ fails for $k > 1$. This leaves the problem, assuming that $P(7,1)$ holds, then does $P(7,k)$ also hold for $k < 1$? The answer is yes and it follows from the following general result.

Proposition 1. If $P(n,k_0)$ holds for some $k_0 > 0$, then $P(n,k)$ holds for all $k \in (0, k_0]$.

Proof. It is sufficient to prove that, for every $a_1, \ldots, a_n > 0$ such that $a_1a_2 + \cdots + a_na_1 = n$, if $\dfrac{1}{a_1+k_0} + \cdots + \dfrac{1}{a_n+k_0} \geq \dfrac{n}{1+k_0}$, then $\dfrac{1}{a_1+k} + \cdots + \dfrac{1}{a_n+k} \geq \dfrac{n}{1+k}$ for $k \in (0, k_0]$. We fix $a_1, \ldots, a_n$ and we denote

$$f(k) = \frac{1}{a_1+k} + \cdots + \frac{1}{a_n+k} - \frac{n}{1+k}.$$

We must prove that if $f(k_0) \geq 0$, then $f(k) \geq 0$ for $k \in (0, k_0]$.

Suppose first that $a_1 = \cdots = a_n$, i.e., that $(a_1, \ldots, a_n) = (1, \ldots, 1)$. Then $f \equiv 0$, so our statement holds trivially. Suppose now that $a_1, \ldots, a_n$ are not all equal. Suppose that our statement is false and put $k_1 = \sup\{k \mid 0 < k \leq k_0,\ f(k) < 0\}$. We have $0 < k_1 \leq k_0$ and, since $f(k_0) \geq 0$, by the continuity of f, $f(k_1) = 0$ and $f(k) \geq 0$ for $k \in [k_1, k_0]$. It follows that $\dfrac{1}{a_1+k_1} + \cdots + \dfrac{1}{a_n+k_1} = \dfrac{n}{1+k_1}$. Since not all $\dfrac{1}{a_1+k_1}, \ldots, \dfrac{1}{a_n+k_1}$ are equal, we have the strict inequality $\dfrac{1}{(a_1+k_1)^2} + \cdots + \dfrac{1}{(a_n+k_1)^2} > \dfrac{1}{n}\left(\dfrac{1}{a_1+k_1} + \cdots + \dfrac{1}{a_n+k_1}\right)^2 = \dfrac{n}{(1+k_1)^2}$. Hence,

$$f'(k_1) = -\frac{1}{(a_1+k_1)^2} - \cdots - \frac{1}{(a_n+k_1)^2} + \frac{n}{(1+k_1)^2} < 0.$$

It follows that there is $\varepsilon > 0$ such that $f(k) \geq f(k_1) = 0$ for $k \in [k_1 - \varepsilon, k_1]$. Since also $f(k) \geq 0$ for $k \in [k_1, k_0]$, we get $f(k) \geq 0$ for $k \in [k_1 - \varepsilon, k_0]$. It follows that $k_1 = \sup\{k \mid 0 < k \leq k_0,\ f(k) < 0\} \leq k_1 - \varepsilon$. Contradiction. □

As a consequence, since $P(n, 3/5)$ holds for all $n \geq 2$, the same happens for $P(n, k)$, with $k \in (0, 3/5]$. The method used by the author to obtain this result can be adapted to slightly increase the bound from $3/5 = 0.6$ to $(\sqrt{5}-1)/2 = 0.618\ldots$, but not further than that.

The author's solution depends on finding an inequality

$$\frac{1}{a_1+k} + \frac{1}{a_2+k} \geq \alpha a_1a_2 + \beta$$

for all $a_1, a_2 \geq 0$. By taking cyclic summation and dividing by 2, this implies $\dfrac{1}{a_1+k} + \cdots + \dfrac{1}{a_n+k} \geq \dfrac{1}{2}\left(\alpha(a_1a_2 + \cdots + a_na_1) + n\beta\right) = \dfrac{n}{2}(\alpha+\beta)$. So, in order to obtain the desired inequality, we need to have $\alpha + \beta = \dfrac{2}{1+k}$. (We cannot have $\alpha + \beta > \dfrac{2}{1+k}$ because then we will have a strict inequality, which is impossible: when $a_1 = \cdots = a_n = 1$, we have equality.)

If we fix $a_1a_2 = p^2$ for some $p > 0$, then $(a_1, a_2) = (x, p^2/x)$ for some $x > 0$, so our condition writes as $\dfrac{1}{x+k} + \dfrac{1}{p^2/x+k} \geq \alpha p^2 + \beta$ for all $p, x > 0$, i.e., as $f(p) \geq \alpha p^2 + \beta$ for all $p > 0$, where $f(p) = \inf\limits_{x>0} \left(\dfrac{1}{x+k} + \dfrac{1}{p^2/x+k} \right)$.

Lemma 1. $f(p) = \min\left\{ \dfrac{2}{p+k}, \dfrac{1}{k} \right\}$ for all $p > 0$.

Proof. Put

$$g(x) = \frac{1}{x+k} + \frac{1}{p^2/x+k} = \frac{1}{x+k} + \frac{x}{kx+p^2}.$$

Then we have

$$g'(x) = -\frac{1}{(x+k)^2} + \frac{p^2}{(kx+p^2)^2} = \frac{(p^2-k^2)(x^2-p^2)}{(x+k)^2(kx+p^2)^2}.$$

If $k < p$, then $g'(x) < 0$ for $x \in (0, p)$ and $g'(x) > 0$ for $x \in (p, \infty)$. Thus $\inf\limits_{x>0} g(x) = g(p) = \dfrac{2}{p+k}$. If $k > p$, then $g'(x) > 0$ for $x \in (0, p)$ and $g'(x) < 0$ for $x \in (p, \infty)$. Thus $\inf\limits_{x>0} g(x) = \min\{ \lim\limits_{x\to 0^+} g(x), \lim\limits_{x\to\infty} g(x) \} = \dfrac{1}{k}$. (Both limits when $x \to 0^+$ and when $x \to \infty$ of $g(x) = \dfrac{1}{x+k} + \dfrac{1}{p^2/x+k}$ are $\dfrac{1}{k}$.) And if $k = p$, then $g' \equiv 0$, so g is a constant. In particular, $g \equiv g(p) = \dfrac{2}{p+k} = \dfrac{1}{k}$.

If k is $>$, $=$, or $< p$, then $\dfrac{2}{p+k}$ is $>$, $=$, or $< \dfrac{1}{k}$, respectively. Hence, in all cases we have $\inf\limits_{x>0} g(x) = \min\left\{ \dfrac{2}{p+k}, \dfrac{1}{k} \right\}$. □

Lemma 2. If $k > 0$ and $\alpha, \beta \in \mathbb{R}$, with $\alpha + \beta = \dfrac{2}{1+k}$, then $\dfrac{2}{p+k} \geq \alpha p^2 + \beta$ for all $p > 0$ iff $\alpha = -\dfrac{1}{(1+k)^2}$, $\beta = \dfrac{3+2k}{(1+k)^2}$.

Proof. Let $g(p) = \dfrac{2}{p+k} - \alpha p^2 - \beta$. We want necessary and sufficient conditions such that $g(p) \geq 0$ for all $p > 0$. Note that $g(1) = \dfrac{2}{1+k} - \alpha - \beta = 0$. If $g(p) \geq 0$ for all $p \geq 0$, then $p = 1$ is a point of minimum, so we have $g'(1) = 0$, i.e., $-\dfrac{-2}{(1+k)^2} - 2\alpha = 0$, so $\alpha = -\dfrac{1}{(1+k)^2}$. It follows that $\beta = \dfrac{2}{1+k} - \alpha = \dfrac{3+2k}{(1+k)^2}$.

Conversely, if $\alpha = -\dfrac{1}{(1+k)^2}$ and $\beta = \dfrac{3+2k}{(1+k)^2}$, then

$$g(p) = \frac{2}{p+k} + \frac{p^2}{(1+k)^2} - \frac{3+2k}{(1+k)^2},$$

so that

$$(p+k)(1+k)^2 g(p) = 2(1+k)^2 + (p+k)(p^2-3-2k) = (p-1)^2(p+2+k) \geq 0.$$

(Note that the factor $(p-1)^2$ was expected to appear. We have $g(1) = g'(1) = 0$, so $p = 1$ is a double zero for $g(p)$. Therefore, it is a double zero for $(p+k)(1+k)^2 g(p)$ as well.) Hence $g(p) \geq 0$ for all $p > 0$, which concludes the proof. □

Proposition 2. If $k > 0$, then there are $\alpha, \beta \in \mathbb{R}$, with $\alpha + \beta = \dfrac{2}{1+k}$, such that $\dfrac{1}{a_1+k} + \dfrac{1}{a_2+k} \geq \alpha a_1 a_2 + \beta$ for all $a_1, a_2 \geq 0$, iff $k \leq (\sqrt{5}-1)/2$. Moreover, if these α, β exist, then they are unique, namely $\alpha = -\dfrac{1}{(1+k)^2}$ and $\beta = \dfrac{3+2k}{(1+k)^2}$.

Proof. Recall that $\dfrac{1}{a_1+k} + \dfrac{1}{a_2+k} \geq \alpha a_1 a_2 + \beta$ for all $a_1, a_2 \geq 0$ iff $f(p) \geq \alpha p^2 + \beta$ for all $p > 0$. By Lemma 1, this is equivalent to $\dfrac{2}{p+k} \geq \alpha p^2 + \beta$ and $\dfrac{1}{k} \geq \alpha p^2 + \beta$ for all $p > 0$. By Lemma 2, the first condition is equivalent to $\alpha = -\dfrac{1}{(1+k)^2}$ and $\beta = \dfrac{3+2k}{(1+k)^2}$. Since $\alpha < 0$, the second condition writes as $\dfrac{1}{k} \geq \sup\limits_{p>0}(\alpha p^2 + \beta) = \beta = \dfrac{3+2k}{(1+k)^2}$, which is equivalent to $(1+k)^2 \geq k(3+2k)$, i.e., $0 \geq k^2 + k - 1$. Since the roots of $X^2 + X - 1$ are $(-\sqrt{5}-1)/2 < 0$ and $(\sqrt{5}-1)/2 > 0$, this is equivalent to $k - (\sqrt{5}-1)/2 \leq 0$. □

In the particular case when $k = 3/5$, we get $\alpha = 25/64$ and $\beta = 21 \cdot 5/64$, so we have the inequality $\dfrac{1}{a_1+3/5} + \dfrac{1}{a_2+3/5} \geq -\dfrac{25}{64} a_1 a_2 + \dfrac{21 \cdot 5}{64}$. After multiplying by $\dfrac{64}{5}$, we get $64\left(\dfrac{1}{5a_1+3} + \dfrac{1}{5a_2+3}\right) \geq -5a_1 a_2 + 21$, as in the author's solution.

565. Let $f \in C^{2n+1}([0,1],\mathbb{R})$ for some $n \geq 0$ such that $f(1/2) = f'(1/2) = f''(1/2) = \cdots = f^{(2n)}(1/2) = 0$. Prove that

$$\int_0^1 \left(f^{(2n+1)}(x)\right)^2 \mathrm{d}x - \left(\int_0^1 f^{(2n+1)}(x)\,\mathrm{d}x\right)^2$$

$$\geq 2^{4n+2}(4n+3)((2n+1)!)^2 \left(\int_0^1 f(x)\,\mathrm{d}x\right)^2.$$

Proposed by Florin Stănescu, Șerban Cioculescu School, Găești, Romania.

Solution by the author. Define $g : [0,1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} x^{2n+1} & \text{if } 0 \leq x \leq 1/2, \\ -(1-x)^{2n+1} & \text{if } 1/2 < x \leq 1. \end{cases}$$

We note that $\int_0^1 g(x)\,\mathrm{d}x = 0$, so for every $\lambda \in \mathbb{R}$ we have

$$\lambda \int_0^1 f^{(2n+1)}(x)g(x)\,\mathrm{d}x = \int_0^1 g(x)\left(1 + \lambda f^{(2n+1)}(x)\right)\,\mathrm{d}x.$$

Then, by the integral Cauchy–Bunyakovsky–Schwarz inequality, we have

$$\lambda^2 \left(\int_0^1 f^{(2n+1)}(x)g(x)\,\mathrm{d}x\right)^2 = \left(\int_0^1 g(x)\left(1 + \lambda f^{(2n+1)}(x)\right)\,\mathrm{d}x\right)^2$$
$$\leq \int_0^1 g(x)^2\,\mathrm{d}x \int_0^1 \left(1 + \lambda f^{(2n+1)}(x)\right)^2 \mathrm{d}x.$$

If $I = \int_0^1 g^2(x)\,\mathrm{d}x$, $I_1 = \int_0^1 f^{(2n+1)}(x)g(x)\,\mathrm{d}x$, $I_2 = \int_0^1 f^{(2n+1)}(x)\,\mathrm{d}x$, and $I_3 = \int_0^1 \left(f^{(2n+1)}(x)\right)^2 \mathrm{d}x$, then the inequality above writes as $\lambda^2 I_1^2 \leq I(1 + 2\lambda I_2 + \lambda^2 I_3)$, which is equivalent to $F(\lambda) \geq 0$, where $F(X) = (I_3 - I^{-1}I_1^2)X^2 + 2I_2X + 1$. Since $F(X)$ takes only nonnegative values, its discriminant is ≤ 0. Hence, $4I_2^2 - 4(I_3 - I^{-1}I_1^2) \leq 0$, which is equivalent to $I_3 - I_2^2 \geq I^{-1}I_1^2$. But $I_3 - I_2^2$ is the left side of the inequality we want to prove. So it is enough to prove that $I^{-1}I_1$ is the right side of the inequality, i.e.,

$$2^{4n+2}(4n+3)((2n+1)!)^2 \left(\int_0^1 f(x)\,\mathrm{d}x\right)^2.$$

Define $g_1 : [0, 1/2] \to \mathbb{R}$ by $g_1(x) = x^{2n+1}$ and $g_2 : [1/2, 1] \to \mathbb{R}$ by $g_2(x) = -(1-x)^{2n+1} = (x-1)^{2n+1}$. Then we obtain

$$I = \int_0^{1/2} g_1^2(x)\,\mathrm{d}x + \int_{1/2}^1 g_2^2(x)\,\mathrm{d}x = \frac{x^{4n+3}}{4n+3}\bigg|_0^{1/2} + \frac{(x-1)^{4n+3}}{4n+3}\bigg|_{1/2}^1$$
$$= 2 \cdot \frac{(1/2)^{4n+3}}{4n+3} = \frac{1}{2^{4n+2}(4n+3)}.$$

We also have $I_1 = \int\limits_0^{1/2} f^{2n+1}(x)g_1(x)\,\mathrm{d}x + \int\limits_{1/2}^1 f^{2n+1}(x)g_2(x)\,\mathrm{d}x$. To evaluate this sum, note that for every $a < b$, $m \geq 1$, and $f, g \in C^m([a,b], \mathbb{R})$, by applying successively integration by parts, we have

$$\int_a^b f^{(m)}(x)g(x)\,\mathrm{d}x = f^{(m-1)}(x)g(x)\Big|_a^b - \int_a^b f^{(m-1)}(x)g'(x)\,\mathrm{d}x$$
$$= f^{(m-1)}(x)g(x)\Big|_a^b - f^{(m-2)}(x)g'(x)\Big|_a^b + \int_a^b f^{(m-2)}(x)g''(x)\,\mathrm{d}x = \cdots$$
$$= \sum_{i=0}^{m-1}(-1)^i f^{(m-1-i)}(x)g^{(i)}(x)\Big|_a^b + (-1)^m \int_a^b f(x)g^{(m)}(x)\,\mathrm{d}x.$$

In our case, since $g_1(x) = x^{2n+1}$, we have $g_1(0) = g_1'(0) = \cdots = g_1^{(2n)}(0) = 0$ and $g_1^{(2n+1)} \equiv (2n+1)!$. Then for $0 \leq i \leq 2n$ we have $f^{(2n-i)}(1/2) = g_1^{(i)}(0) = 0$, so

$$f^{(2n-i)}(x)g_1^{(i)}(x)\Big|_0^{1/2} = f^{(2n-i)}(1/2)g_1^{(i)}(1/2) - f^{(2n-i)}(0)g_1^{(i)}(0) = 0 - 0 = 0.$$

It follows that

$$\int_0^{1/2} f^{(2n+1)}(x)g_1(x)\,\mathrm{d}x = (-1)^{2n+1}\int_0^{1/2} f(x)g_1^{(2n+1)}(x)\,\mathrm{d}x$$
$$= -(2n+1)!\int_0^{1/2} f(x)\,\mathrm{d}x.$$

Similarly, since $g_2(x) = (x-1)^{2n+1}$, we have $g_2(1) = g_2'(1) = \cdots = g_2^{(2n)}(1) = 0$ and $g_2^{(2n+1)} \equiv (2n+1)!$. Then for $0 \leq i \leq 2n$ we have

$f^{(2n-i)}(1/2) = g_2^{(i)}(1) = 0$, so

$$f^{(2n-i)}(x)g_1^{(i)}(x)\Big|_{1/2}^{1} = f^{(2n-i)}(1)g_1^{(i)}(1) - f^{(2n-i)}(1/2)g_1^{(i)}(1/2) = 0 - 0 = 0.$$

It follows that

$$\begin{aligned}\int_{1/2}^{1} f^{(2n+1)}(x)g_2(x)\,\mathrm{d}x &= (-1)^{2n+1}\int_{1/2}^{1} f(x)g_2^{(2n+1)}(x)\,\mathrm{d}x \\ &= -(2n+1)!\int_{1/2}^{1} f(x)\,\mathrm{d}x.\end{aligned}$$

Hence,

$$I_1 = -(2n+1)!\int_0^{1/2} f(x)\,\mathrm{d}x - (2n+1)!\int_{1/2}^{1} f(x)\,\mathrm{d}x = -(2n+1)!\int_0^1 f(x)\,\mathrm{d}x.$$

In conclusion,

$$\begin{aligned}I^{-1}I_1^2 &= \left(\frac{1}{2^{4n+2}(4n+3)}\right)^{-1}\left(-(2n+1)!\int_0^1 f(x)\,\mathrm{d}x\right)^2 \\ &= 2^{4n+2}(4n+3)((2n+1)!)^2\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2,\end{aligned}$$

as claimed.

566. Prove that for all natural numbers $n \geq 4$ the equation

$$\sum_{k=1}^{n}\frac{1}{k^2+2k+x} = \frac{3}{4}$$

has a unique solution in the interval $(-1, 0)$, denoted by x_n, and find the value of the limit $\lim\limits_{n\to\infty} nx_n$.

Proposed by Dumitru Popa, Department of Mathematics, Ovidius University of Constanța, Romania.

Solution by the author. From $\dfrac{1}{k^2+2k} = \dfrac{1}{2}\left(\dfrac{1}{k} - \dfrac{1}{k+2}\right)$ we deduce that

$$\sum_{k=1}^{n}\frac{1}{k^2+2k} = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}.$$

For $n \geq 4$ let $f_n : [-1, 0] \to \mathbb{R}$ be defined by $f_n(x) = \sum\limits_{k=1}^{n}\dfrac{1}{k^2+2k+x} - \dfrac{3}{4}$ and note that f_n is continuous strictly decreasing. We have $f_n(0) =$

$\sum_{k=1}^{n} \frac{1}{k^2+2k} - \frac{3}{4} = -\frac{2n+3}{2(n+1)(n+2)} < 0$. Since $n \geq 4$,

$$f_n(-1) = \sum_{k=1}^{n} \frac{1}{k^2+2k-1} - \frac{3}{4} \geq \sum_{k=1}^{4} \frac{1}{k^2+2k-1} - \frac{3}{4} = \frac{5}{23 \cdot 28} > 0.$$

Hence, the equation $f_n(x) = 0$ has an unique solution $x_n \in (-1, 0)$. Thus $f_n(x_n) = 0$, i.e., $\sum_{k=1}^{n} \frac{1}{k^2+2k+x_n} = \frac{3}{4}$. Then, for all $n \geq 4$, we have

$$\sum_{k=1}^{n} \frac{1}{k^2+2k} - \sum_{k=1}^{n} \frac{1}{k^2+2k+x_n} = \sum_{k=1}^{n} \frac{1}{k^2+2k} - \frac{3}{4}$$

or

$$x_n \sum_{k=1}^{n} \frac{1}{(k^2+2k)(k^2+2k+x_n)} = -\frac{2n+3}{2(n+1)(n+2)}.$$

Therefore, since $-1 < x_n < 0$, it holds

$$\begin{aligned} |x_n| \sum_{k=1}^{n} \frac{1}{(k^2+2k)^2} &< |x_n| \sum_{k=1}^{n} \frac{1}{(k^2+2k)(k^2+2k+x_n)} \\ &= \frac{2n+3}{2(n+1)(n+2)}, \end{aligned}$$

whence $|x_n| \leq \frac{\frac{2n+3}{2(n+1)(n+2)}}{\sum_{k=1}^{n} \frac{1}{(k^2+2k)^2}}$ and hence $\lim_{n\to\infty} x_n = 0$. (Notice that the series $\sum_{n=1}^{\infty} \frac{1}{(n^2+2n)^2}$ is convergent and its sum is > 0). Now note that

$$nx_n = \frac{-\frac{2n^2+3n}{2(n+1)(n+2)}}{\sum_{k=1}^{n} \frac{1}{(k^2+2k)(k^2+2k+x_n)}}.$$

From $x_n > -1$ we deduce that

$$\left| \sum_{k=1}^{n} \frac{1}{(k^2+2k)(k^2+2k+x_n)} - \sum_{k=1}^{n} \frac{1}{(k^2+2k)^2} \right|$$

$$= |x_n| \sum_{k=1}^{n} \frac{1}{(k^2+2k)^2(k^2+2k+x_n)} < |x_n| \sum_{k=1}^{n} \frac{1}{(k^2+2k)^2(k^2+2k-1)}.$$

Since $\lim\limits_{n\to\infty} x_n = 0$ and the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{(n^2+2n)^2(n^2+2n-1)}$ is convergent, we get

$$\lim_{n\to\infty}\left(|x_n|\sum_{k=1}^{n}\frac{1}{(k^2+2k)^2(k^2+2k-1)}\right)=0$$

and, by the squeeze theorem,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(k^2+2k)(k^2+2k+x_n)}=\sum_{k=1}^{\infty}\frac{1}{(k^2+2k)^2}.$$

Hence, $\lim\limits_{n\to\infty} nx_n = -\dfrac{1}{\sum\limits_{k=1}^{\infty}\frac{1}{(k^2+2k)^2}}$. From

$$\frac{1}{(k^2+2k)^2}=\frac{1}{4}\left(\frac{1}{k^2}+\frac{1}{(k+2)^2}-\frac{2}{k(k+2)}\right)$$

we get

$$\sum_{k=1}^{\infty}\frac{1}{(k^2+2k)^2}=\frac{1}{4}\left(\frac{\pi^2}{3}-1-\frac{1}{4}-\frac{3}{2}\right)=\frac{4\pi^2-33}{48}$$

and so $\lim\limits_{n\to\infty} nx_n = -\dfrac{48}{4\pi^2-33}$.

Editor's note. We received a solution from Mattia van Reijn, Bruges, Belgium, but one which is too long to include here. He proved the following more general result.

Let $(a_k)_{k\geq 1}$ be a sequence of positive real numbers such that $\sum\limits_{k\geq 1}\dfrac{1}{a_k} = A < \infty$ and let $\min\limits_{k\geq 1} a_k = \mu > 0$. (The existence of μ is assured by the fact that $\lim\limits_{k\to\infty} a_k = \infty$.)

The convergence of $\sum\limits_{k\geq 1}\dfrac{1}{a_k}$ implies that the series $\sum\limits_{k\geq 1}\dfrac{1}{a_k^2}$ too is convergent and we denote by B its sum. We also assume that $\lim\limits_{n\to\infty} n\sum\limits_{k=n+1}^{\infty}\dfrac{1}{a_k}$ exists and it is equal to L. Suppose that $-\mu < u < 0$.

Then there is an integer $n_0 > 0$ such that for every $n \geq n_0$ the equation

$$\sum_{k=1}^{n}\frac{1}{a_n+x}=A$$

has a unique solution $x_n \in (u, 0)$ and we have

$$\lim_{n\to\infty} nx_n = -L/B.$$

In our case $a_k = k^2 + 2k$, $u = -1$, $n_0 = 4$, and, as in author's solution, we have $A = \dfrac{3}{4}$, $B = \dfrac{\pi^2}{12} - \dfrac{11}{16} = \dfrac{4\pi^2 - 33}{48}$, and $L = 1$. This concludes the solution of the original problem.

If $\sum\limits_{k=1}^{\infty} \dfrac{1}{a_k^3} = C$ and we have a finite limit $M = \lim\limits_{n\to\infty} n\left(n \sum\limits_{k=n+1}^{\infty} \dfrac{1}{a_k} - L\right)$, then

$$\lim_{n\to\infty} n\left(nx_n + \frac{L}{B}\right) = \frac{L^2 C}{B^3} - \frac{M}{B}.$$

When $a_k = k^2 + 2k$, we have $B = \dfrac{4\pi^2 - 33}{48}$, $L = 1$, $C = \dfrac{21 - 2\pi^2}{32}$, and $M = -\dfrac{3}{2}$, so

$$\lim_{n\to\infty} n\left(nx_n + \frac{48}{4\pi^2 - 33}\right) = \frac{C}{B^3} + \frac{3}{2B} = \frac{3456(21 - 2\pi^2)}{(4\pi^2 - 33)^3} + \frac{72}{4\pi^2 - 33}.$$

567. Let $n \geq 4$ and let $a_1, \ldots, a_n \geq 0$ such that

$$a_1^2 + \cdots + a_n^2 + (n^2 - 3n + 1)\, a_1 \cdots a_n \geq (n-1)^2.$$

Prove that

$$a_1 + \cdots + a_n + \frac{1}{n-1}\sqrt{\frac{1}{n-1} \sum_{1\leq i\leq j\leq n} (a_i - a_j)^2} \geq n.$$

Proposed by Leonard Giugiuc, Greci School, Mehedinți, Romania.

Solution by the author. Our result will follow from the following

Lemma. *Assume that $n \geq 4$ and $a_1, \ldots, a_n \geq 0$ such that*

$$a_1 + \cdots + a_n + \frac{1}{n-1}\sqrt{\frac{1}{n-1} \sum_{1\leq i\leq j\leq n} (a_i - a_j)^2} \leq n. \tag{1}$$

Then

$$a_1^2 + \cdots + a_n^2 + (n^2 - 3n + 1)\, a_1 \cdots a_n \leq (n-1)^2, \tag{2}$$

with equality if and only if $(a_1, \ldots, a_n)$ is $(1, \ldots, 1)$ or a permutation of $(n - 1, 0, \ldots, 0)$.

Proof. If $a_1 + \cdots + a_n = 0$, then $(a_1, \ldots, a_n) = (0, \ldots, 0)$ and the inequality holds strictly.

We denote $a_1 + \cdots + a_n = p$ and $a_1^2 + \cdots + a_n^2 = q$. Then $p^2/n \leq q \leq p^2$. (The left equality holds iff $a_1 = \ldots = a_n = p/n$. The right one iff $(a_1, \ldots, a_n) = (p, 0, \ldots, 0)$ and permutations.) Then there are unique $0 \leq x \leq y$ such that $(n-1)x + y = p$ and $(n-1)x^2 + y^2 = q$ and, by EV Theorem, $a_1 \cdots a_n \leq x^{n-1} y$, with equality iff $(a_1, \ldots, a_n)$ is a permutation of $(x, \ldots, x, y)$. Also, since $(n-1)x + y = p \leq n$ and $x \leq y$, we have $x \leq 1$.

We have

$$\sum_{1\le i<j\le n}(a_i-a_j)^2 = n(a_1^2+\cdots+a_n^2)-(a_1+\cdots+a_n)^2 = nq-p^2$$
$$= n((n-1)x^2+y^2)-((n-1)x+y)^2 = (n-1)(y-x)^2.$$

Hence, the condition from the hypothesis writes as

$$(n-1)x+y+\frac{1}{n-1}(y-x) = \frac{n(n-2)}{n-1}x+\frac{n}{n-1}y \le n,$$

i.e., $(n-2)x+y\le n-1$.

We have

$$a_1^2+\cdots+a_n^2+(n^2-3n+1)\,a_1\cdots a_n \le (n-1)x^2+y^2+(n^2-3n+1)x^{n-1}y,$$

so it is enough to prove that $(n-1)x^2+y^2+(n^2-3n+1)x^{n-1}y\le(n-1)^2$, i.e., $f(y)\le 0$, where $f(y)=y^2+(n^2-3n+1)x^{n-1}y+(n-1)x^2-(n-1)^2$.

Since f is a quadratic polynomial with a positive leading coefficient, the set $I=\{y\in\mathbb{R}\mid f(y)\le 0\}$ is an interval. (If y_1,y_2 are the roots of f, then $I=[y_1,y_2]$ if $y_1,y_2\in\mathbb{R}$ and $I=\emptyset$ otherwise.) Since y is bounded by the constrains $x\le y$ and $(n-2)x+y\le n-1$, i.e., $y\in[x,n-1-(n-2)x]$, we must prove that $[x,n-1-(n-2)x]\subseteq I$. But I is an interval, so this is equivalent to $x,n-1-(n-2)x\in I$, i.e., $f(x)\le 0$ and $f(n-1-(n-2)x)\le 0$. Moreover, the equality can only hold at the endpoints, i.e., only at $y=x$ or $y=n-1-(n-2)x$.

We have $f(x)=nx^2+(n^2-3n+1)x^n-(n-1)^2$. Since $x\in[0,1]$ and $n\ge 4$, we get $f(x)\le n\cdot 1^2+(n^2-3n+1)\cdot 1^n-(n-1)^2=0$, with equality iff $x=1$.

We have $f(n-1-(n-2)x)=g(x)$, with $g:[0,1]\to\mathbb{R}$ given by $g(x)=(n-1-(n-2)x)^2+(n^2-3n+1)x^{n-1}(n-1-(n-2)x)+(n-1)x^2-(n-1)^2$. Then $g(x)=xh(x)$, where $h(x)=-2(n-1)(n-2)+(n^2-3n+3)x+(n^2-3n+1)((n-1)x^{n-2}-(n-2)x^{n-1})$. We prove that $h(x)\le 0$, so that $f(n-1(n-2)x)=xh(x)\le 0$. For every $x\in[0,1]$ we have

$$h'(x)=n^2-3n+3+(n-1)(n-2)(n^2-3n+1)(x^{n-3}-x^{n-2})\ge n^2-3n+3>0,$$

so h is strictly increasing. Hence, for every $x\in[0,1]$ we have $h(x)\le h(1)=0$ and so $f(n-1(n-2)x)=xh(x)\le 0$, with equality iff $x\{0,1\}$.

In conclusion, our inequality holds, with equality iff $(a_1,\ldots,a_n)$ is a permutation of $(x,\ldots,x,y)$ where either $x=1$ and $y=x=1$ or $x\in\{0,1\}$ and $y=n-1-(n-2)x$, i.e., $(x,y)=(0,n-1)$ or $(1,1)$. This concludes the proof. □

Back to the problem, we note that in all cases when we have equality in the inequality (2), we also have equality in (1). Consequently, if the inequality

(1) is strict, then so is (2). That is, if

$$a_1 + \cdots + a_n + \frac{1}{n-1}\sqrt{\frac{1}{n-1}\sum_{1\le i\le j\le n}(a_i - a_j)^2} < n,$$

then

$$a_1^2 + \cdots + a_n^2 + (n^2 - 3n + 1)\, a_1 \cdots a_n < (n-1)^2.$$

But this is equivalent to the statement of the problem.

568. Let $f : [0,1] \to \mathbb{R}$ be a twice differentiable function, with f'' continuous, such that $f(0) = f(1)$. Prove that there exists $\xi \in (0,1)$ such that

$$\xi^2 f(\xi) = 2\int_0^\xi x f(x)\,\mathrm{d}x.$$

Proposed by Cezar Lupu, Beijing Institute of Mathematical Sciences and Applications (BIMSA) and Tsinghua University, Beijing, P.R. China.

Solution by the author. Let the differentiable function $H : [0,1] \to \mathbb{R}$ be given by

$$H(t) = t\int_0^t f(x)\,\mathrm{d}x - \int_0^t x f(x)\,\mathrm{d}x.$$

We have $H'(t) = \int_0^t f(x)\,\mathrm{d}x$ and $H''(t) = f(t)$. Then, by hypothesis, $H''(0) = H''(1)$. We now state and prove the following

Lemma. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function such that $f''(a) = f''(b)$. Then there exists $c \in (a,b)$ such that*

$$f(c) - f(a) = (c-a)f'(c) - \frac{(c-a)^2}{2}f''(c).$$

Proof. From the condition $f''(a) = f''(b)$, by applying Flett's mean value theorem (see [1]), there exists $u \in (a,b)$ such that

$$(u-a)f''(u) = f'(u) - f'(a).$$

Let $g : [a,b] \to \mathbb{R}$ be given by $g(x) = (x-a)f'(x) - 2f(x) + xf'(a)$. It is easy to see that the first derivative is given by $g'(x) = (x-a)f''(x) - f'(x) - f'(a)$ and $g'(u) = g'(a) = 0$. By Flett's mean value theorem, there exists $c \in (a,u) \subset (a,b)$ such that

$$(c-a)g'(c) = g(c) - g(a),$$

which is equivalent with

$$2(c-a)f''(c) - (c-a)(f'(c) - f'(a)) = (c-a)f'(c) - 2f(c) + cf'(a)$$

and thus the lemma is proved. □

Returning to our problem, since $H''(0) = H''(1)$, by the above lemma, there exists $\xi \in (0,1)$ such that $H(\xi) - H(0) = \xi H'(\xi) - \frac{\xi^2}{2} H''(\xi)$, that is,

$$\xi \int_0^\xi f(x)\,\mathrm{d}x - \int_0^\xi xf(x)\,\mathrm{d}x = \xi \int_0^\xi f(x)\,\mathrm{d}x - \frac{\xi^2}{2} f(\xi),$$

and the conclusion follows immediately.

REFERENCES

[1] T.M. Flett, A mean value problem, *Math. Gaz.* **42** (1958), 38–39.

Solution by Henry Ricardo, Westchester Area Math Circle, New York, USA. Corollary 2.3 of the proposer's paper [1] states that since $\displaystyle\int_0^1 f'(x)\,\mathrm{d}x = f(1) - f(0) = 0$, there exists $\xi \in (0,1)$ such that $\displaystyle\int_0^\xi x^2 f'(x)\,\mathrm{d}x = 0$. But, integrating by parts, we see that

$$0 = \int_0^\xi x^2 f'(x)\,\mathrm{d}x = x^2 f(x)\Big|_0^\xi - 2\int_0^\xi xf(x)\,\mathrm{d}x = \xi^2 f(\xi) - 2\int_0^\xi xf(x)\,\mathrm{d}x,$$

which implies that $\xi^2 f(\xi) = 2\displaystyle\int_0^\xi xf(x)\,\mathrm{d}x$.

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[1] C. Lupu, Mean value problems of Flett type for a Volterra operator, *Electron. J. Differ. Equ.* **2013** (2013), Paper No. 53, 1–7.
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569. For an odd positive integer x put $x!! = x(x-2)\cdots 3\cdot 1$. Let a and b be positive real numbers. Calculate

$$\lim_{n\to\infty} \left(\sqrt[n]{a\cdot(2n+1)!!} - \sqrt[n-1]{b\cdot(2n-1)!!} \right).$$

2) Let $f, g \in \mathbb{R}[x]$ be polynomials of the same degree with the coefficient of the leading term positive. Calculate

$$\lim_{n\to\infty} \left(\sqrt[n]{(2n+1)!!f(n)} - \sqrt[n-1]{(2n-1)!!g(n-1)} \right).$$

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Romania.

Solution by the authors. It suffices to solve only the second part of the problem. We assume that f and g are polynomials of degree $k \geq 0$. Let $f(x) = f_k x^k + f_{k-1}x^{k-1} + \cdots + f_1 x + f_0$ and $g(x) = g_k x^k + g_{k-1}x^{k-1} + \cdots + g_1 x + g_0$. Since $f_k > 0$ and $g_k > 0$, we have that, when n is large enough, $f(n) > 0$ and $g(n-1) > 0$, so the sequence in part 2) is well defined.

We prove the limit in part 2) equals $\frac{2}{\mathrm{e}}\left(1+\ln\frac{f_k}{g_k}\right)$ and, in particular, the limit of the sequence in part 1) of the problem is $\frac{2}{\mathrm{e}}\left(1+\ln\frac{a}{b}\right)$.

We need the following result

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\frac{2}{\mathrm{e}}. \tag{1}$$

To prove this, we note that $\frac{\sqrt[n]{(2n-1)!!}}{n}=\sqrt[n]{a_n}$, where $a_n=\frac{(2n-1)!!}{n^n}$. Since $\frac{a_{n+1}}{a_n}=\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n\to\frac{2}{\mathrm{e}}$ when $n\to\infty$, we get, based on Cauchy–d'Alembert's criterion, that $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=\frac{2}{\mathrm{e}}$.

We have

$$\begin{aligned}
x_n &= \sqrt[n]{(2n+1)!!f(n)}-\sqrt[n-1]{(2n-1)!!g(n-1)}\\
&= \sqrt[n]{(2n-1)!!g(n-1)}\left[\left(\frac{f(n)}{g(n-1)}(2n+1)\right)^{\frac{1}{n}}\right.\\
&\qquad\left.-\left(g(n-1)(2n-1)!!\right)^{\frac{1}{n(n-1)}}\right]\\
&= \sqrt[n]{(2n-1)!!g(n-1)}\left[\mathrm{e}^{\frac{1}{n}\ln\left(\frac{f(n)}{g(n-1)}(2n+1)\right)}-\mathrm{e}^{\frac{1}{n(n-1)}\ln(g(n-1)(2n-1)!!)}\right]\\
&\overset{(*)}{=} \sqrt[n]{(2n-1)!!g(n-1)}\left[\frac{1}{n}\ln\left(\frac{f(n)}{g(n-1)}(2n+1)\right)\right.\\
&\qquad\left.-\frac{1}{n(n-1)}\ln\left(g(n-1)(2n-1)!!\right)\right]\mathrm{e}^{\theta_n},
\end{aligned}$$

where θ_n is between $\frac{1}{n}\ln\left(\frac{f(n)}{g(n-1)}(2n+1)\right)$ and $\frac{1}{n(n-1)}\ln\left(g(n-1)(2n-1)!!\right)$.

We applied at step (*) Lagrange Mean Value Theorem for the exponential function $f(x)=\mathrm{e}^x$.

Observe that

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{f(n)}{g(n-1)}(2n+1)\right) &= \lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{f(n)}{g(n-1)}\right)+\lim_{n\to\infty}\frac{\ln(2n+1)}{n}\\
&= 0,
\end{aligned}$$

since $\lim\limits_{n\to\infty}\frac{f(n)}{g(n-1)}=\frac{f_k}{g_k}$. Also,

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n(n-1)}\ln\left(g(n-1)(2n-1)!!\right) &= \lim_{n\to\infty}\frac{\ln g(n-1)}{n(n-1)}+\lim_{n\to\infty}\frac{\ln(2n-1)!!}{n(n-1)}\\
&= 0.
\end{aligned}$$

Both limits $\lim\limits_{n\to\infty}\frac{\ln g(n-1)}{n(n-1)}=0$ and $\lim\limits_{n\to\infty}\frac{\ln(2n-1)!!}{n(n-1)}=0$ can be proved by an application of Stolz–Cesaro Lemma. These imply that $\lim\limits_{n\to\infty}\theta_n=0$.

It follows that

$$x_n = \sqrt[n]{g(n-1)} \cdot \frac{\sqrt[n]{2n-1)!!}}{n} \ln \frac{\frac{f(n)}{g(n-1)}(2n+1)}{\sqrt[n-1]{g(n-1)(2n-1)!!}} \cdot \mathrm{e}^{\theta_n}$$

$$= \sqrt[n]{g(n-1)} \cdot \frac{\sqrt[n]{2n-1)!!}}{n} \cdot \mathrm{e}^{\theta_n} \times \ln \left(\frac{f(n)}{g(n-1)} \cdot \frac{2n+1}{n} \cdot \frac{n}{\sqrt[n-1]{(2n-1)!!}} \cdot \frac{1}{\sqrt[n-1]{g(n-1)}} \right).$$

This implies, based on (1) and on the limits $\lim_{n\to\infty} \frac{f(n)}{g(n-1)} = \frac{f_k}{g_k}$, $\lim_{n\to\infty} \sqrt[n]{g(n-1)} = 1$, and $\lim_{n\to\infty} \sqrt[n-1]{g(n-1)} = 1$ (which can be proved elementarily), that

$$\lim_{n\to\infty} x_n = 1 \cdot \frac{2}{\mathrm{e}} \cdot \ln \left(\frac{f_k}{g_k} \cdot 2 \cdot \frac{\mathrm{e}}{2} \cdot \frac{1}{1} \right) \cdot \mathrm{e}^0 = \frac{2}{\mathrm{e}} \left(1 + \ln \frac{f_k}{g_k} \right).$$

The problem is solved.

We also received solutions from Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain, and from Ulrich Abel and Vitaliy Kushnirevych, Technische Hochschule Mittelhessen, Friedberg, Germany.