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ARTICLES

On a Diophantine equation with factorials

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Abstract. The double factorial of a positive integer x, denoted $x!!$, is defined as the product of all positive integers up to x that have the same parity as x. In this note it is shown that there are precisely three triples (a, b, c) of positive integers with $b \leq c$ satisfying the equation $a! = b!! + c!!$.

Keywords: double factorial, Legendre's formula, Bertrand's postulate

MSC: 11D99, 11D45,11B65

Any reader of this journal is familiar with the factorial, defined by $x! = 1 \cdot 2 \cdots x$ for positive integers x and $0! = 1$. This function can be extended to all reals in various ways, but the most significant generalization has been derived by Daniel Bernoulli via an integral formula $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$ for $x > 0$. The improper integral is convergent even for complex numbers with positive real part and can be extended by analytic continuation to a meromorphic function that is holomorphic in the whole complex plane except the non-positive integers. The gamma function is encountered in many fields of mathematics, physics, and engineering.

Besides the analytical vein, variations of the factorial with combinatorial flavor can be introduced. Such an example is the so-called *double factorial*, denoted $x!!$ and defined for positive integers x as the product of all positive integers up to x that have the same parity as x. The explicit formula

$$x!! = \begin{cases} x(x-2)\cdots 4 \cdot 2 & \text{for } x \text{ even,} \\ x(x-2)\cdots 3 \cdot 1 & \text{for } x \text{ odd,} \end{cases}$$

can be written equivalently

$$(2x)!! = 2^x \cdot x!, \quad (2x+1)!! = \frac{(2x+1)!}{2^x \cdot x!} \quad \text{for integer } x \geq 0. \tag{1}$$

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Several dozens of combinatorial configurations counted by double factorials are referred to in [4], especially on pages dedicated to sequences A000165, A001147, A006882, A114488, and A143280.

Here we ignore all developments mentioned above and deal with a Diophantine equation described by its proposer as 'one factorial equal to two double factorials'. The user nnkken posted on April 3, 2023 the following request on [3]:

> Suppose $a! = b!! + c!!$, prove that $a + b + c < 2022$.
>
> I can only prove that both b, c are odd numbers.
>
> I wrote a program to search for solutions, which only gives $(2, 1, 1)$, $(3, 3, 3)$, $(5, 5, 7)$. But I can't prove that they are the only solutions.

The aim of this note is to prove we know all solutions, so the maximum for the sum of entries in a solution is 17.

Theorem 1. *The only solutions for the equation*

$$a! = b!! + c!!, \quad a, b, c \in \mathbb{N}, \tag{2}$$

are $(a, b, c) = (2, 1, 1), (3, 3, 3), (5, 5, 7), (5, 7, 5)$.

It is clear that the right-hand side of equation (2) is at least 2. Hence, in any solution one has $a \geq 2$, so the left side of equation (2) is even. This implies. in particular, that the values for b and c are congruent modulo 2. We discuss separately the solutions according to the parity of b in the next two sections. Although there are many ideas common in these two cases, the differences are sufficiently marked that it is better to write down the details in distinct sections.

One common ingredient in arguments is a well-known formula for the exponent $v_p(x!)$ with which a prime p appears in the factorization of the number $x!$. For reader's convenience, here is its statement.

Lemma 2. *(Legendre's formula)*

$$v_p(x!) = \left\lfloor \frac{x}{p} \right\rfloor + \left\lfloor \frac{x}{p^2} \right\rfloor + \left\lfloor \frac{x}{p^3} \right\rfloor + \cdots .$$

1. EVEN CASE

Throughout this section we denote by (a, b, c) a solution to equation (2) with $b = 2s$ and $c = 2t$ for some integers $1 \leq s \leq t$. For the sake of simplicity, we shall put $R = b!! + c!!$.

A moment thought reveals that either $R = 4$ or $R \geq 8$, whence $a \geq 4$. This in turn implies $t \geq 3$. The assumption $b \leq c$ entails $b!!$ divides $c!!$, in other words, $R = A \cdot b!!$ for some integer $A \geq 2$.

Our first lemma provides comparison of the entries of the solution under consideration.

Lemma 3. *$a \geq \max\{2s, t+2\}$ and $t \leq s+2$.*

Proof. We argue by contradiction. Assume first that $a \leq 2s-1$. Then Legendre's formula in conjunction with (1) yield

$$v_2(a!) \leq s-1+\sum_{k\geq 2}\left\lfloor\frac{2s-1}{2^k}\right\rfloor \leq s-1+v_2(s!) < v_2(b!!) \leq v_2(R).$$

Thus, the equality $a! = R$ cannot hold.

Suppose now $a \leq t+1$. Then, by (1) and (2) one has $(t+1)! \geq a! > c!! = 2^t \cdot t!$, in contradiction with the inequality $2^x \geq x+1$ valid for all integers $x \geq 1$.

The final part for the conclusion of our lemma follows from the elementary fact that the product of three consecutive integers is divisible by 3. More precisely, if $t \geq s+3$, then from $R = 2^s \cdot s!\big(1+2^{t-s}(s+1)\cdots t\big)$ one gets $v_3(R) = v_3(s!) < v_3(t!)$, while from $a > t$ one obtains $v_3(a!) \geq v_3(t!)$, whence the contradiction $a! \neq R$. □

In view of what we just proved, we have to further consider the Diophantine equations

$$\begin{aligned} a! &= 2^{s+1} \cdot s!, \\ a! &= 2^s(2s+3) \cdot s!, \\ a! &= 2^s(2s+3)^2 \cdot s!. \end{aligned}$$

The prime divisors of the right-hand side of each of them are all less than or equal to s, with the possible exception of $2s+3$. This contradicts the so-called Bertrand's postulate, proved by Chebyshev in [2], asserting that for any $x \geq 2$ there exists a prime p satisfying $x \leq p < 2x$. Indeed, any prime between s and $2s$ divides $a!$ because $a \geq 2s$.

This contradiction ends the proof of the main result of the section.

Proposition 4. *There are no solutions for equation* (2) *with even b.*

2. ODD CASE

The most visible source of differences between the reasoning employed in the previous section and that used below is the existence of solutions. Their presence requires deeper investigation than above. We try to decide the existence of solutions under various additional hypotheses, whence a more intricate structure of the proof and longer arguments.

From a technical point of view, the explanation for the additional challenges we have to overcome is found in formula (1): $x!!$ is a product of 'simple' factors when x is even and a quotient thereof when x is odd. This makes the evaluation of multiplicities with which relevant odd prime numbers appear in various terms much more problematic.

Throughout (a, b, c) will denote a solution to equation (2) having b and c odd, while R stands for $b!! + c!!$.

Lemma 5. *The only solutions with $b = c$ are $(a, b, c) = (2, 1, 1), (3, 3, 3)$.*

Proof. This is seen by noticing that the only factorials congruent to 2 modulo 4 are $2! = 2$ and $3! = 6$. □

Lemma 6. *The only solution with $b = 1$ is $(a, b, c) = (2, 1, 1)$.*

Proof. For $c \geq 3$ one has the right-hand side of the equation at least 4, so $a \geq 3$, whence the contradiction 3 divides $a! - c!! = 1$. □

From now on we may suppose $3 \leq b < c$. Consequently, $a! \geq 3!! + 5!! = 18$, so that $a \geq 4$, and $a! < 2 \cdot c!! < c!$, whence $a < c$.

The next observation points out a simple yet effective condition on the odd values b, c.

Lemma 7. *There exist positive integers s, t such that $\{b, c\} = \{8s \pm 1, 8t + 5\}$. In particular, b, $c \in \{5, 7, 9, 13, 15, 17, 21, 23, 25, 29, \ldots\}$.*

Proof. It is sufficient to notice that $c \geq 5$ entails $a \geq 4$, so $a!$, as well as R, must be multiple of 8, and then to look at residue classes modulo 8 of the double factorials. □

Up to now we have used the p-adic valuation of an integer x and a prime p, denoted $v_p(x)$, simply as a short-hand for the largest integer e such that p^e divides x. Next we shall need a well known property of this function, viz., $v_p(x + y) \geq \min\{v_p(x), v_p(y)\}$, with equality attained if $v_p(x) \neq v_p(y)$.

Lemma 8. *$b = 5$ only for $(a, b, c) = (5, 5, 7)$.*

Proof. Indeed, for $b = 5$ and $c \geq 9$, one has $v_3(5!!) = 1$, $v_3(c!!) \geq v_3(9!!) = 3$, so that $v_3(R) = 1$. This implies $a \leq 5$, in contradiction with $a! > 9!! > 6!$. □

The argument from the previous paragraph shows that no solution has $b = 7$.

Suppose now that $b = 9$. By Lemma 7, either $c = 13$ or $c \geq 21$. The later possibility is excluded since then $v_3(R) = 3$, which is a value not attained by $v_3(a!)$. If $b = 9$ and $c = 13$, then $R = 144 \cdot 9!!$, so that $v_2(a!) = 4$, whence $a \leq 7$. Thus we reached the contradiction $5040 \geq a! = R = 144 \cdot 945$.

We can continue to similarly examine small values of b, concluding, for instance, that in any other solution one necessarily has $b \geq 23$. A patient reader will even obtain the lower bound $b \geq 53$. A less patient reader could use a computer for excluding many more values for b. Adopting a conservative standpoint, we agree to assume $b \geq 53$, which entails $a \geq 34$.

The arguments involve two main ideas: comparison of the 2-adic and/or 3-adic valuations of $a!$ and R, on the one hand, and of the sizes of $a!$ and $c!!$, on the other hand. In order to perform such tasks for larger values of a, b, c, one needs to refine the ideas employed in the previous section. One of them, which served to complete the proof of Proposition 4, was to make appeal to Bertrand's postulate. This is the first of a series of results of the type: For fixed real $\varepsilon > 0$, there exists $N(\varepsilon)$ with the property that for any $x \geq N(\varepsilon)$, there exits a prime p satisfying $x \leq p < (1+\varepsilon)x$. For our purposes, it is sufficient the next result, established in [1].

Lemma 9. *(Nagura's analogue of Bertrand's postulate) There exists a prime between x and $1.2x$ for any $x \geq 25$.*

For the sake of concision, we introduce a new notation. For an integer $x \geq 2$, let $q(x)$ denote the largest prime which is less than or equal to x.

Here are some consequences derived for a hypothetical solution (a, b, c) restricted as follows:

$$53 \leq b < c, \quad 34 \leq a < c. \tag{3}$$

Lemma 10. *$q(a) = q(b)$.*

Proof. The inequality $q(a) \geq q(b)$ is obvious from the fact that $q(b)$ divides both $b!!$ and $c!!$ (because $b < c$), so it divides $b!! + c!! = a!$. The converse inequality follows by noticing that $q(a)$ divides both $a!$ and $c!!$ (recall that $a < c$), and hence their difference, which is $b!!$. □

In view of Nagura's bound, as soon as $\min\{a, b\} \geq 29$ one obtains $b < 1.2q(b) = 1.2q(a) \leq 1.2a$, whence

Lemma 11. *$5b + 1 \leq 6a$ for $b \geq 53$.*

Proof. Indeed, $b \geq 53$ together with Lemma 10 entail $a \geq 53$, so the condition for using Nagura's result for a is fulfilled. □

After we showed that a and b can not be too distant one from another, we prove that c can not be very close to b.

Lemma 12. *For any solution satisfying the standing hypothesis* (3) *one has $c - b \geq 6$ and $v_3(b!! + c!!) = v_3(b!!)$.*

Proof. Since the product of three consecutive odd numbers is divisible by 3, the second part of the conclusion is a direct consequence of the first one. In view of Lemma 7, we need to consider the following cases.

Case $b = 8s - 1$. Then $c = 8t + 5$ for some integer $t \geq s$, so that $c - b \geq 6$.

Case $b = 8s+1$. Assuming $c = 8s+5$, one obtains $b!!+c!! = (8s+4)^2 \cdot b!!$, whence $v_2(a!) = v_2(R) = 4$. Hence $a \leq 7$, in contradiction with (3). Thus, $c = 8t + 5$ for some integer $t > s$, which readily gives $c - b \geq 12$.

Case $b = 8s + 5$. Note that one necessarily has $s \geq 6$. It suffices to show that none of the possibilities $c = 8s + 7$, $c = 8s + 9$ occurs. When $c = 8s + 7$, then $R = 8(s + 1) \cdot b!!$. Put $s = 2^e u - 1$ for some odd u to obtain $v_2(R) = e + 3$. With Lemma 11 and Legendre's formula one gets

$$v_2(a!) \geq \sum_{k \geq 1} \left\lfloor \frac{40s + 26}{6 \cdot 2^k} \right\rfloor \geq (3s + 4) + (s + 5) + 5 + 2 + 1 > 2^{e+2}u \geq e + 3.$$

If $c = 8s + 9$, then $R = 64(s + 1)^2 \cdot b!!$ and $v_2(R) = 2e + 6$. Proceeding as in the previous paragraph, one arrives again at $v_2(a!) > 4(s + 1)$. Since the inequality $4(s+1) > 2e+6$ is obvious for $e \leq 10$ and is a consequence of $2^x > x + 2$ for $e \geq 11$, we reached again the contradiction $v_2(a!) > v_2(R)$.

It results that in this case one has $c - b \geq 10$. □

In the last step in the search of solutions satisfying condition (3) we compare the 3-adic valuations of the two sides of Equation (2). The outcome of the study is the next result, in whose proof we use Legendre's formula and the obvious inequality $\lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$.

Lemma 13. *For any solution* (a, b, c) *of* (2) *that satisfies* (3) *one also has* $v_3(a!) > v_3(b!!)$.

Proof. Put $b = 2s + 1$ and let u be the unique integer identified by the requirements $3^u \leq b < 3^{u+1}$. Note that $b \geq 53$ entails $u \geq 3$. By formula (1), one gets

$$v_3(b!!) = v_3((2s + 1)!) - v_3(s!) = \sum_{k=1}^{u} \left(\left\lfloor \frac{2s + 1}{3^k} \right\rfloor - \left\lfloor \frac{s}{3^k} \right\rfloor \right).$$

Since, for any integer $k \geq 1$, from Lemma 11 one gets

$$\left\lfloor \frac{a}{3^k} \right\rfloor \geq \left\lfloor \frac{5s + 3}{3^{k+1}} \right\rfloor = \left\lfloor \frac{2s + 1 - \frac{s}{3}}{3^k} \right\rfloor \geq \left\lfloor \frac{2s + 1}{3^k} \right\rfloor - \left\lfloor \frac{s}{3^{k+1}} \right\rfloor - 1,$$

we obtain

$$v_3(a!) - v_3(b!!) \geq \sum_{k=1}^{u} \left(\left\lfloor \frac{s}{3^k} \right\rfloor - \left\lfloor \frac{s}{3^{k+1}} \right\rfloor - 1 \right) = \left\lfloor \frac{s}{3} \right\rfloor - u \geq \frac{3^{u-1} - 1}{2} - u > 0$$

(remember that $u \geq 3$). □

Comparison of the last two lemmas results in the main result of this section.

Proposition 14. *No solution of equation* (2) *satisfies condition* (3).

Now, Theorem 1 follows from Propositions 4 and 14 together with Lemmas 5, 6, and 8.

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A cover property

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Abstract. We discuss the possible size (in the Lebesgue measure sense) of certain union of open intervals centered at the rational numbers.

Keywords: Cover, Lebesgue measure, enumeration.

MSC: Primary 28A05, Secondary 54A25.

1. Introduction

The starting point of this work is a statement from [2] used in the proof of the fact that every infinite set of real numbers contains a countable subset which is dense in the original set. The statement is that if (r_n) is an enumeration of the rationals then

$$\bigcup_{n=1}^{\infty}\left(r_n-\frac{1}{n}, r_n+\frac{1}{n}\right)=\mathbb{R}.$$

While this equality is possible, it is definitely not true for an arbitrary enumeration and, in fact, the union above can be rather small. We will find the infimum of the Lebesgue measures of all possible enumerations, will show how to get the equality above, will give a proof of the statement about sets independent of this covering property and will discuss two generalizations of this problem.

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2. The Covers

By an enumeration of a countable set A we understand an invertible function $\phi : \mathbb{N} \to A$. We will denote by $\mathcal{E}(A)$ the set of all such enumerations.

Let μ be the Lebesgue measure on the real line.

Theorem 1.

$$\inf_{\phi\in\mathcal{E}(\mathbb{Q})} \mu\left(\bigcup_{n=1}^{\infty}\left(\phi(n)-\frac{1}{n},\phi(n)+\frac{1}{n}\right)\right)=2.$$

Proof. Since

$$(\phi(1)-1,\phi(1)+1)\subseteq\bigcup_{n=1}^{\infty}\left(\phi(n)-\frac{1}{n},\phi(n)+\frac{1}{n}\right)$$

and

$$\mu\left((\phi(1)-1,\phi(1)+1)\right)=2,$$
$$\mu\left(\bigcup_{n=1}^{\infty}\left(\phi(n)-\frac{1}{n},\phi(n)+\frac{1}{n}\right)\right)\geq 2$$

and thus the infimum above is ≥ 2.

We will construct a particular enumeration of $\mathbb{Q}$, ψ.

Let $\psi(1) = 0$ and a natural number $k > 1$. Let $A =$ the rational numbers in $(-1/2, 1/2)$ except for 0 and B the rest of the rational numbers. Since B is countable we consider some enumeration $\theta \in \mathcal{E}(B)$.

For $n \geq 1$ we define

$$\psi(2^{k+n}) = \theta(n).$$

Since A is countable there is some enumeration $\beta \in \mathcal{E}(A)$.

Let C be the set of all natural numbers except $1, 2^{k+1}, 2^{k+2}, \ldots$. Since C is countable there is some enumeration $\phi \in \mathcal{E}(C)$.

We define $\psi(\phi(n)) = \beta(n)$. Notice that the least element of C is 2 and thus

$$\frac{1}{\phi(n)}\leq\frac{1}{2}.$$

Since $\beta(n) \in (-1/2, 1/2)$ we have

$$\left(\psi(\phi(n))-\frac{1}{\phi(n)},\psi(\phi(n))+\frac{1}{\phi(n)}\right)=\left(\beta(n)-\frac{1}{\phi(n)},\beta(n)+\frac{1}{\phi(n)}\right)\subseteq(-1,1).$$

Therefore for the enumeration ψ we have

$$\bigcup_{n=1}^{\infty}\left(\psi(n)-\frac{1}{n},\psi(n)+\frac{1}{n}\right)\subseteq(-1,1)\cup\bigcup_{n=1}^{\infty}\left(\psi(2^{k+n})-\frac{1}{2^{k+n}},\psi(2^{k+n})+\frac{1}{2^{k+n}}\right).$$

Here

$$\mu\left((-1,1)\cup\bigcup_{n=1}^{\infty}\left(\psi(2^{k+n})-\frac{1}{2^{k+n}},\psi(2^{k+n})+\frac{1}{2^{k+n}}\right)\right)$$

$$\leq \mu\left((-1,1)\right)+\sum_{n=1}^{\infty}\mu\left(\left(\psi(2^{k+n})-\frac{1}{2^{k+n}},\psi(2^{k+n})+\frac{1}{2^{k+n}}\right)\right)$$

$$=2+\sum_{n=1}^{\infty}\frac{2}{2^{k+n}}=2+\frac{1}{2^{k-1}}.$$

This implies that

$$\mu\left(\bigcup_{n=1}^{\infty}\left(\psi(n)-\frac{1}{n},\psi(n)+\frac{1}{n}\right)\right)\leq 2+\frac{1}{2^{k-1}}$$

and thus the infimum in the statement is $\leq 2+\frac{1}{2^{k-1}}$. Since $k>1$ was an arbitrary natural number we get that the infimum is ≤ 2.

This implies the conclusion. □

Proposition 2. *There is ϕ, an enumeration of the rationals, such that*

$$\bigcup_{n}\left(\phi(n)-\frac{1}{n},\phi(n)+\frac{1}{n}\right)=\mathbb{R}.$$

Proof. Let $\phi(1)=0$, and, for $n\geq 1$,

$$\phi(4n-1)=\frac{1}{3}+\frac{1}{7}+\cdots+\frac{1}{4n-1}$$

and

$$\phi(4n+1)=-\frac{1}{5}-\frac{1}{9}-\cdots-\frac{1}{4n+1}.$$

Let A be the rest of the rational numbers. Since A is countable, there is $\psi\in\mathcal{E}(A)$.

We define $\phi(2n)=\psi(n)$.

We will show that in this case

$$\bigcup_{n}\left(\phi(n)-\frac{1}{n},\phi(n)+\frac{1}{n}\right)=\mathbb{R}.$$

In fact

$$\bigcup_{n=1}^{\infty}\left(\phi(2n-1)-\frac{1}{2n-1},\phi(2n-1)+\frac{1}{2n-1}\right)=\mathbb{R}.$$

Let $x\in\mathbb{R}$.

If $x\in(-1,1)$ then

$$x\in\left(\phi(1)-1,\phi(1)+1\right).$$

Suppose that $x\geq 1$. Since

$$\lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{7}+\cdots+\frac{1}{4n-1}\right)=\infty,$$

there is a natural number n such that

$$\frac{1}{3}+\frac{1}{7}+\cdots+\frac{1}{4n-1}\leq x<\frac{1}{3}+\frac{1}{7}+\cdots+\frac{1}{4n-1}+\frac{1}{4n+3}.$$

Therefore

$$\begin{aligned} x &\in \left[\phi(4n-1), \phi(4n-1)+\frac{1}{4n+3}\right) \\ &\subseteq \left(\phi(4n-1)-\frac{1}{4n-1}, \phi(4n-1)+\frac{1}{4n-1}\right). \end{aligned}$$

Suppose now that $x \leq -1$. Since

$$\lim_{n\to\infty}\left(-\frac{1}{5}-\frac{1}{9}-\cdots-\frac{1}{4n+1}\right)=-\infty,$$

there is a natural number $n \geq 2$ such that

$$-\frac{1}{5}-\frac{1}{9}-\cdots-\frac{1}{4n-3}-\frac{1}{4n+1}<x\leq -\frac{1}{5}-\frac{1}{9}-\cdots-\frac{1}{4n-3}.$$

Therefore

$$\begin{aligned} x &\in \left(\phi(4n-3)-\frac{1}{4n+1}, \phi(4n-3)\right] \\ &\subseteq \left(\phi(4n-3)-\frac{1}{4n-3}, \phi(4n-3)+\frac{1}{4n-3}\right). \end{aligned}$$

Thus

$$x \in \bigcup_{n=1}^{\infty}\left(\phi(2n-1)-\frac{1}{2n-1}, \phi(2n-1)+\frac{1}{2n-1}\right)$$

and so

$$\mathbb{R} \subseteq \bigcup_{n=1}^{\infty}\left(\phi(2n-1)-\frac{1}{2n-1}, \phi(2n-1)+\frac{1}{2n-1}\right).$$

Since the opposite inclusion is obvious,

$$\bigcup_{n=1}^{\infty}\left(\phi(2n-1)-\frac{1}{2n-1}, \phi(2n-1)+\frac{1}{2n-1}\right)=\mathbb{R}.$$

□

3. The Subset Problem

We will show here a proof of the statement at the beginning of this paper which does not use the covering property above and which we consider more in the spirit of real analysis.

Recall that a set X is dense in a set Y if every element of Y is the limit of a sequence of elements in X. In other words, if $Y \subseteq \overline{X}$, where $\overline{X}$ is the closure of the set X.

Proposition 3. *If A is an infinite subset of real numbers, then there is a countable set B such that $B \subseteq A$ and B is dense in A.*

Proof. If A is a countable set then we can just take $B = A$. Thus, we can assume, without loss of generality, that A is uncountable.

We will assume first that the set A is bounded, $A \subseteq [u, v]$.

Let $b_1 \in A$. If $A \setminus \{b_1\} \cap [u, (u + v)/2] \neq \emptyset$ then we chose $b_2 \in A \setminus \{b_1\} \cap [u, (u + v)/2]$. If $A \setminus \{b_1\} \cap [u, (u + v)/2] = \emptyset$ we choose an arbitrary $b_2 \in A \setminus \{b_1\}$.

If $A \setminus \{b_1, b_2\} \cap [(u + v)/2, v] \neq \emptyset$ then we chose $b_3 \in A \setminus \{b_1, b_2\} \cap [(u + v)/2, v]$. If $A \setminus \{b_1\} \cap [(u+v)/2, v] = \emptyset$ we choose an arbitrary $b_3 \in A \setminus \{b_1, b_2\}$.

We continue the procedure next dividing the interval $[u, v]$ into 3 intervals of equal length and choosing 3 more elements, then 4 and so on.

Let $B = (b_n)$. Then B is countable and $B \subseteq A$. It suffices to show that B is dense in $A \setminus B$.

Let $a \in A \setminus B$.

At step n the interval $[u, v]$ was divided into n sub intervals of length $(v - u)/n$, at least one of them containing a. Since $a \notin B$, when the intersection with that sub interval was considered it was not empty an we chose some $b_{p_n} \in B$ from that subinterval.

This means that

$$|a - b_{p_n}| \leq \frac{v - u}{n}$$

and so a is in the closure of B.

This completes the proof in this case.

For the general case, if A is an unbounded set, for all integers n let $A_n = A \cap [n, n + 1]$.

If A_n is at most countable then we take $B_n = A_n$. If A_n is uncountable then, by the first case, there is $B_n \subseteq A_n$, B_n countable, B_n dense in A_n.

Let $B = \bigcup_n B_n$. Then $B \subseteq A$ and B is countable. Moreover,

$$\overline{\bigcup_n B_n} \supset \bigcup_n \overline{B_n} \supset \bigcup_n A_n = A.$$

The first inclusion in the chain above is a general property of closures of sets. See, for example, the statement 5 on page 126 of [1]. □

4. A General Case

We will consider here a more general problem. Let (r_n) be a decreasing (not necessarily strictly) sequence with limit 0. We will do a computation similar to the one in Section 2 where $r_n = 1/n$.

Theorem 4. *If (x_n) is a sequence decreasing to 0 and X is a countable dense set of real numbers, then*

$$\text{(a)} \inf_{\phi\in\mathcal{E}(X)} \mu\left(\bigcup_{n=1}^{\infty}(\phi(n)-r_n,\phi(n)+r_n)\right) = 2r_1.$$

$$\text{(b)} \sup_{\phi\in\mathcal{E}(X)} \mu\left(\bigcup_{n=1}^{\infty}(\phi(n)-r_n,\phi(n)+r_n)\right) = 2\sum_{n=1}^{\infty} r_n.$$

Proof. Let α be the infimum and β be the supremum.

Since, for every $\phi \in \mathcal{E}(X)$ we have

$$(\phi(1)-r_1,\phi(1)+r_1) \subseteq \bigcup_{n=1}^{\infty}(\phi(n)-r_n,\phi(n)+r_n)$$

we get

$$\mu\left(\bigcup_{n=1}^{\infty}(\phi(n)-r_n,\phi(n)+r_n)\right) \geq \mu((\phi(1)-r_1,\phi(1)+r_1)) = 2r_1.$$

Therefore $\alpha \geq 2x_1$.

We will consider that $r_1 = r_2 = \cdots = r_m > r_{m+1}$.

Let $k > 1$.

We will take $\phi(1)$ arbitrary and for $2 \leq n \leq m$ we choose

$$\phi(n) \in \left(\phi(1),\phi(1)+\frac{1}{2^{n+k}}\right) \cap (X \setminus \{\phi(1),\phi(2),\ldots,\phi(n-1)\}).$$

The choice above is possible because X is dense and so is any subset of X of finite complement in X.

Let

$$A = \big(X \cap (\phi(1)+r_{m+1}-r_1,\phi(1)+r_1-r_{m+1})\big) \setminus \{\phi(1),\phi(2),\ldots,\phi(m)\}$$

and

$$B = X \setminus \big(A \cup \{\phi(1),\phi(2),\ldots,\phi(m)\}\big).$$

Because X is countable and dense both A and B are countable sets.

Since r_n has limit 0 there is a subsequence (k_n) with $k_1 > m$ and $k_{j+1} > k_j + 2$ such that $r_{k_n} < 1/2^{k+1+n}$. Let $C = \{n : n \geq m+1, n \neq k_j, 1 \leq j\}$. Then C is countable.

Let $\theta \in \mathcal{E}(A)$ and $\gamma \in \mathcal{E}(C)$. We define $\phi(\gamma(n)) = \theta(n)$.

Let $\psi \in \mathcal{E}(B)$. We define $\phi(k_n) = \psi(n)$.

Then

$$\bigcup_{n=1}^{\infty}(\phi(n)-r_n,\phi(n)+r_n)$$

$$= (\phi(1) - r_1, \phi(1) + r_1) \cup \bigcup_{j=2}^{m} (\phi(j) - r_j, \phi(j) + r_j)$$

$$\cup \bigcup_{i=1}^{\infty} (\phi(k_i) - r_{k_i}, \phi(k_i) + r_{k_i}) \cup \bigcup_{j=1}^{\infty} (\phi(\gamma(j)) - r_{\gamma(j)}, \phi(\gamma(j)) + r_{\gamma(j)}).$$

Let $j \geq 1$. Then

$$\phi(\gamma(j)) = \theta(j) \in (\phi(1) + r_{m+1} - r_1, \phi(1) + r_1 - r_{m+1})$$

and $r_{\gamma(j)} \leq r_{m+1}$.

Therefore

$$\phi(1) + r_{m+1} - r_1 < \phi(\gamma(j)) < \phi(1) + r_1 - r_{m+1}$$

which implies that

$$\phi(1) - r_1 \leq \phi(1) + r_{m+1} - r_{\gamma(j)} - r_1 < \phi(\gamma(j)) - r_{\gamma(j)}$$

and

$$\phi(\gamma(j)) + r_{\gamma(j)} < \phi(1) + r_1 - r_{m+1} + r_{\gamma(j)} \leq \phi(1) + r_1$$

Hence

$$\bigcup_{j=1}^{\infty} (\phi(\gamma(j)) - r_{\gamma(j)}, \phi(\gamma(j)) + r_{\gamma(j)}) \subseteq (\phi(1) - r_1, \phi(1) + r_1)$$

and so

$$\bigcup_{n=1}^{\infty} (\phi(n) - r_n, \phi(n) + r_n)$$

$$= (\phi(1) - r_1, \phi(1) + r_1) \cup \bigcup_{j=2}^{m} (\phi(j) - r_j, \phi(j) + r_j) \cup \bigcup_{i=1}^{\infty} (\phi(k_i) - r_{k_i}, \phi(k_i) + r_{k_i})$$

$$= (\phi(1) - r_1, \phi(1) + r_1) \cup \bigcup_{j=2}^{m} (\phi(j) - r_1, \phi(j) + r_1) \cup \bigcup_{i=1}^{\infty} (\phi(k_i) - r_{k_i}, \phi(k_i) + r_{k_i}).$$

For $2 \leq j \leq m$

$$\phi(1) < \phi(j) < \phi(1) + \frac{1}{2^{k+j}}$$

and so

$$\phi(1) - r_1 < \phi(j) - r_1$$

and

$$\phi(j) + r_1 < \phi(1) + r_1 + \frac{1}{2^{k+j}} \leq \phi(1) + r_1 + \frac{1}{2^{k+2}}.$$

Hence

$$\bigcup_{j=2}^{m} (\phi(j) - r_1, \phi(j) + r_1) \subseteq \left(\phi(1) - r_1, \phi(1) + r_1 + \frac{1}{2^{k+2}}\right)$$

and thus

$$\bigcup_{n=1}^{\infty}(\phi(n)-r_n,\phi(n)+r_n)\subseteq\left(\phi(1)-r_1,\phi(1)+r_1+\frac{1}{2^{k+2}}\right)$$
$$\cup\bigcup_{i=1}^{\infty}(\phi(k_i)-r_{k_i},\phi(k_i)+r_{k_i}).$$

Therefore

$$\mu\left(\bigcup_{n=1}^{\infty}(\phi(n)-r_n,\phi(n)+r_n)\right)$$
$$\leq\mu\left(\left(\phi(1)-r_1,\phi(1)+r_1+\frac{1}{2^{k+2}}\right)\cup\bigcup_{i=1}^{\infty}(\phi(k_i)-r_{k_i},\phi(k_i)+r_{k_i})\right)$$
$$\leq 2r_1+\frac{1}{2^{k+2}}+2\sum_{i=1}^{\infty}r_{k_i}<2r_1+\frac{1}{2^k}+2\sum_{i=1}^{\infty}1/2^{k+1+i}=2r_1+\frac{1}{2^{k-1}}.$$

This means that $\alpha\leq 2r_1+\frac{1}{2^{k-1}}$ and since k was an arbitrary natural number then $\alpha\leq 2r_1$.

Therefore $\alpha=2x_1$.

For any $\phi\in\mathcal{E}(X)$ we have

$$\mu\left(\bigcup_{n=1}^{\infty}(\phi(n)-r_n,\phi(n)+r_n)\right)\leq\sum_{n=1}^{\infty}\mu\left((\phi(n)-r_n,\phi(n)+r_n)\right)=2\sum_{n=1}^{\infty}r_n.$$

This implies that $\beta\leq 2\sum_{n=1}^{\infty}r_n$.

Let $m>1$.

We define $\phi(1)$ arbitrary, and we choose $\phi(2)\in X\cap(\phi(1)+r_1+r_2,\infty)$, $\phi(3)\in X\cap(\phi(2)+r_2+r_3,\infty)$ and so on, $\phi(m)\in X\cap(\phi(m-1)+r_{m-1}+r_m,\infty)$.

In this case

$$\bigcup_{j=1}^{m}(\phi(j)-r_j,\phi(j)+r_j)$$

is a disjoint union.

Let $\psi\in\mathcal{E}(X\setminus\{\phi(1),\phi(2),\ldots,\phi(m)\}$. We define, for $n>m,\phi(n)=\psi(n-m)$.

Since

$$\bigcup_{j=1}^{m}(\phi(j)-r_j,\phi(j)+r_j)\subseteq\bigcup_{n\geq 1}(\phi(n)-r_n,\phi(n)+r_n)$$

then

$$\mu\left(\bigcup_{n\geq 1}(\phi(n)-r_n,\phi(n)+r_n)\right)\geq\mu\left(\bigcup_{j=1}^{m}(\phi(j)-r_j,\phi(j)+r_j)\right)=2\sum_{j=1}^{m}r_j.$$

This implies that $\beta \geq 2\sum_{j=1}^{m} r_j$ and since m was arbitrary we get that $\beta \geq \sum_{n=1}^{\infty} r_n$.

Therefore $\beta = \sum_{n=1}^{\infty} r_n$. □

5. An Even More General Case

One more step we can take is to consider a sequence in which some terms repeat infinitely many times.

Theorem 5. *If* (r_n) *is a bounded sequence such that* $\liminf_n r_n = 0$ *and* $\sup_n r_n = r$ *and* X *is a countable dense set of real numbers, then*

$$\inf_{\phi\in\mathcal{E}(X)} \mu\left(\bigcup_{n=1}^{\infty}(\phi(n)-r_n,\phi(n)+r_n)\right) = 2r.$$

Proof. Let α be the infimum. Since for every $\phi \in \mathcal{E}(X)$

$$(\phi(m)-r_m,\phi(m)+r_m) \subseteq \bigcup_{n=1}^{\infty}(\phi(n)-r_n,\phi(n)+r_n)$$

we get that

$$\mu\left(\bigcup_{n=1}^{\infty}(\phi(n)-r_n,\phi(n)+r_n)\right) \geq \mu\left((\phi(m)-r_m,\phi(m)+r_m)\right) = 2r_m.$$

Hence $\alpha \geq 2r_m$ for every m and so $\alpha \geq 2r$.

Let $\varepsilon > 0$. Since $\liminf_n r_n = 0$, there is some subsequence (r_{k_n}) such that $r_{k_n} < \varepsilon/2^{n+1}$ for every n.

Let $A = X \cap (-\varepsilon/4, \varepsilon/4)$, $B = X \setminus A$ and $C = \mathbb{N} \setminus \{k_{2n}\}$.

Let $\theta \in \mathcal{E}(B)$. We define $\phi(k_{2n}) = \theta(n)$.

Let $\sigma \in \mathcal{E}(A)$. Since C is countable, there is an invertible function $\tau : C \to \mathbb{N}$.

For $m \in C$ we define $\phi(m) = \sigma(\tau(m))$.

If $m \in C$,

$$(\phi(m)-r_m,\phi(m)+r_m) = (\sigma(\tau(m))-r_m,\sigma(\tau(m)+r_m) \subseteq (-r-\varepsilon/4, r+\varepsilon/4)$$

and thus the measure of the union of all these intervals is $\leq 2x + \varepsilon/2$.

For $\phi(k_{2n})$ the union of all intervals $(\phi(k_{2n}) + r_{k_{2n}}, \phi(k_{2n}) + r_{k_{2n}})$ has measure $\leq 2\sum_{n=1}^{\infty} \varepsilon/2^{2n+1} < \varepsilon/2$.

Therefore the measure of all intervals for these enumeration is $\leq 2r+\varepsilon$, which, since ε was arbitrary, implies that $\alpha \leq 2r$

Thus $\alpha = 2r$. □

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Solutions to two open problems involving odd harmonic numbers

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Abstract. In this paper we solve two open problems recently considered by Ovidiu Furdui and Alina Sîntămărian concerning the calculation of a cubic and a quadratic series involving the odd harmonic number $O_n = H_{2n} - \frac{1}{2}H_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}$. Our proof involves the use of some identities due to De-Yin Zheng and Cornel Ioan Vălean.

Keywords: Riemann Zeta function, generalized harmonic numbers, infinite summation formulas.

MSC: Primary 40A25; Secondary 11M06.

1. INTRODUCTION AND THE MAIN RESULT

In this paper we calculate the following harmonic series

$$\sum_{n=1}^{\infty} \frac{O_n^3}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{O_n^2}{n^3}$$

that were considered as an open problem by Furdui and Sîntămărian in [1, p. 11].

Throughout this paper H_n denotes the nth harmonic number defined by $H_n = \sum_{k=1}^{n} \frac{1}{k}$, O_n is the nth odd harmonic number given by $O_n = \sum_{k=1}^{n} \frac{1}{2k-1}$, and $O_n^{(r)}$ is the notation for the generalized nth odd harmonic number of order r defined by $O_n^{(r)} = \sum_{k=1}^{n} \frac{1}{(2k-1)^r}$. We also mention that $\zeta(s)$ is the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $\Re(s) > 1$ and $\operatorname{Li}_n(x)$ denotes the polylogarithm function, defined for $|x| \le 1$ by $\operatorname{Li}_n(x) = \sum\limits_{k=1}^{\infty} \frac{x^k}{k^n}$, $n \in \mathbb{N}$, $n \ge 2$.

The main result of this paper is the following theorem.

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Theorem 1. *The following identities hold:*

(a) $$\sum_{n=1}^{\infty}\frac{O_n^3}{n^2}=\frac{21}{8}\zeta(2)\zeta(3);$$

(b) $$\sum_{n=1}^{\infty}\frac{O_n^2}{n^3}=\frac{7}{4}\zeta(2)\zeta(3)-\frac{31}{16}\zeta(5).$$

Proof. (a) Using the Dougall–Dixon summation theorem [5, p. 56], De-Yin Zheng [2, (3.6d)] has evaluated the series

$$\sum_{n=1}^{\infty}\frac{2O_n^3+O_n^{(3)}}{n^2}=\frac{93}{8}\zeta(5).$$

In [4, (75)] J. Braun, D. Romberger, and H. J. Bentz proved that

$$\sum_{n=1}^{\infty}\frac{O_n^{(3)}}{n^2}=\frac{93}{8}\zeta(5)-\frac{21}{4}\zeta(2)\zeta(3)$$

and it follows, based on the previous equality, that

$$\sum_{n=1}^{\infty}\frac{O_n^3}{n^2}=\frac{21}{8}\zeta(2)\zeta(3). \tag{1}$$

(b) Using the Dougall–Dixon summation theorem [5, p. 56], De-Yin Zheng [2, (3.6c), (3.7c)] has evaluated the following two series

$$\sum_{n=1}^{\infty}\frac{(H_{n-1}+H_n)O_n^2}{n^2}=\frac{93}{8}\zeta(5) \tag{2}$$

and

$$\sum_{n=1}^{\infty}\frac{H_{n-1}H_nO_n}{n^2}=\frac{7}{4}\zeta(2)\zeta(3)+\frac{31}{8}\zeta(5). \tag{3}$$

Using the identities $H_{n-1}=H_n-\frac{1}{n}$ and $O_n=H_{2n}-\frac{1}{2}H_n$ on left side side of (3), we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_{n-1}H_nO_n}{n^2}&=\sum_{n=1}^{\infty}\frac{\left(H_n-\frac{1}{n}\right)H_n\left(H_{2n}-\frac{1}{2}H_n\right)}{n^2}\\&=\sum_{n=1}^{\infty}\frac{\left(H_n^2-\frac{1}{n}H_n\right)\left(H_{2n}-\frac{1}{2}H_n\right)}{n^2}\\&=\sum_{n=1}^{\infty}\frac{H_{2n}H_n^2}{n^2}-\sum_{n=1}^{\infty}\frac{H_n^3}{2n^2}-\sum_{n=1}^{\infty}\frac{H_{2n}H_n}{n^3}+\sum_{n=1}^{\infty}\frac{H_n^2}{2n^3}\\&=\frac{7}{4}\zeta(2)\zeta(3)+\frac{31}{8}\zeta(5).\end{aligned}$$

In [3] it is proved that the following formula holds

$$\sum_{n=1}^{\infty}\frac{H_{2n}H_n}{n^3}=\frac{307}{16}\zeta(5)-\frac{1}{2}\zeta(2)\zeta(3)+\frac{8}{3}\log^3(2)\zeta(2)-7\log^2(2)\zeta(3)\\-\frac{8}{15}\log^5(2)-16\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-16\operatorname{Li}_5\left(\frac{1}{2}\right).$$

We have, based on the above equality, that

$$\sum_{n=1}^{\infty}\frac{H_{2n}H_n^2}{n^2}-\sum_{n=1}^{\infty}\frac{H_n^3}{2n^2}+\sum_{n=1}^{\infty}\frac{H_n^2}{2n^3}=\frac{5}{4}\zeta(2)\zeta(3)+\frac{369}{16}\zeta(5)+\frac{8}{3}\log^3(2)\zeta(2)\\-7\log^2(2)\zeta(3)-\frac{8}{15}\log^5(2)-16\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-16\operatorname{Li}_5\left(\frac{1}{2}\right).$$

In the previous identity using Euler sums (see [6, (3c)] and [6, (3b)])

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^3}=-\zeta(2)\zeta(3)+\frac{7}{2}\zeta(5)\quad\text{and}\quad\sum_{n=1}^{\infty}\frac{H_n^3}{n^2}=\zeta(2)\zeta(3)+10\zeta(5)$$

we get

$$\sum_{n=1}^{\infty}\frac{H_{2n}H_n^2}{n^2}=\frac{9}{4}\zeta(2)\zeta(3)+\frac{421}{16}\zeta(5)+\frac{8}{3}\log^3(2)\zeta(2)-7\log^2(2)\zeta(3)\\-\frac{8}{15}\log^5(2)-16\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-16\operatorname{Li}_5\left(\frac{1}{2}\right).\tag{4}$$

Using the identity $O_n=H_{2n}-\frac{1}{2}H_n$ on the left hand side of (1), one has that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{O_n^3}{n^2}&=\sum_{n=1}^{\infty}\frac{\left(H_{2n}-\frac{1}{2}H_n\right)^3}{n^2}\\&=\sum_{n=1}^{\infty}\frac{3H_{2n}H_n^2}{4n^2}-\sum_{n=1}^{\infty}\frac{H_n^3}{8n^2}-\sum_{n=1}^{\infty}\frac{3H_{2n}^2H_n}{2n^2}+\sum_{n=1}^{\infty}\frac{H_{2n}^3}{n^2}\\&=\frac{21}{8}\zeta(2)\zeta(3).\end{aligned}$$

We observe that

$$\sum_{n=1}^{\infty}\frac{H_{2n}^3}{n^2}=4\sum_{n=1}^{\infty}\frac{H_{2n}^3}{(2n)^2}=2\left(\sum_{n=1}^{\infty}\frac{H_n^3}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^nH_n^3}{n^2}\right)$$

and it follows

$$\sum_{n=1}^{\infty}\frac{3H_{2n}H_n^2}{4n^2}-\sum_{n=1}^{\infty}\frac{H_n^3}{8n^2}-\sum_{n=1}^{\infty}\frac{3H_{2n}^2H_n}{2n^2}+2\left(\sum_{n=1}^{\infty}\frac{H_n^3}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n H_n^3}{n^2}\right) \tag{5}$$
$$=\frac{21}{8}\zeta(2)\zeta(3).$$

Using the following relation (see [7, p. 865])

$$\sum_{n=1}^{\infty}\frac{H_n^3}{n^2}(-1)^n=\frac{9}{4}\zeta(5)+\frac{27}{16}\zeta(2)\zeta(3)+\log^3(2)\zeta(2)-\frac{21}{8}\log^2(2)\zeta(3)$$
$$-\frac{1}{5}\log^5(2)-6\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-6\operatorname{Li}_5\left(\frac{1}{2}\right),$$

formula (4) and the Euler sum $\sum_{n=1}^{\infty}\frac{H_n^3}{n^2}=\zeta(2)\zeta(3)+10\zeta(5)$, in the identity (5) one obtains that

$$\sum_{n=1}^{\infty}\frac{H_nH_{2n}^2}{n^2}=\frac{23}{8}\zeta(2)\zeta(3)+\frac{917}{32}\zeta(5)+\frac{8}{3}\log^3(2)\zeta(2)-7\log^2(2)\zeta(3)$$
$$-\frac{8}{15}\log^5(2)-16\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-16\operatorname{Li}_5\left(\frac{1}{2}\right).$$

Using the identities $H_{n-1}=H_n-\frac{1}{n}$ and $O_n=H_{2n}-\frac{1}{2}H_n$ on the left side of (2), we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(H_{n-1}+H_n)O_n^2}{n^2}&=\sum_{n=1}^{\infty}\frac{\left(2H_n-\frac{1}{n}\right)O_n^2}{n^2}\\&=\sum_{n=1}^{\infty}\frac{2H_nO_n^2}{n^2}-\sum_{n=1}^{\infty}\frac{O_n^2}{n^3}\\&=\sum_{n=1}^{\infty}\frac{2H_{2n}^2H_n}{n^2}-\sum_{n=1}^{\infty}\frac{2H_{2n}H_n^2}{n^2}+\sum_{n=1}^{\infty}\frac{H_n^3}{2n^2}-\sum_{n=1}^{\infty}\frac{O_n^2}{n^3}\\&=\frac{93}{8}\zeta(5).\end{aligned}$$

It follows, based on the preceding equality and the previous formulae, that

$$\sum_{n=1}^{\infty}\frac{O_n^2}{n^3}=\frac{7}{4}\zeta(2)\zeta(3)-\frac{31}{16}\zeta(5),$$

and the theorem is proved. □

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Traian Lalescu national mathematics contest for university students, 2023 edition

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Abstract. We present the problems from the 2023 edition of the Traian Lalescu National Mathematics Contest for University Students, hosted by the Gheorghe Asachi Technical University of Iaşi between May 4 and 6.

Keywords: Functional equations, trace, rank, determinant, eigenvalues, normal matrix, integral inequalities, power series, gamma function, Fermat's Little Theorem.

MSC: 11A25, 15A18, 15A21, 26A03, 26D15, 39B22, 40A05.

The 2023 edition of the Traian Lalescu National Mathematics Contest for University Students and the National Session of Student Scientific Communications were organized between May 4 and 6 by the Department of Mathematics and Informatics of Gheorghe Asachi Technical University of Iaşi, with the support of the Traian Lalescu Foundation and under auspices of the Ministry of Education and the Romanian Mathematical Society.

A total of 101 participating students represented 11 universities from Braşov, Bucureşti, Cluj–Napoca, Iaşi, and Timişoara:

- Section A — first- and second-year students from faculties of Mathematics;
- Section B — first-year students that follow some specialization in Electrical Engineering, or in Computer Science;

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- Section C — first-year students from technical faculties with a specialization outside the field of Electrical Engineering;
- Section D + E — for second-year students from technical faculties.
- Scientific Communications, Mathematics Section;
- Scientific Communications, Computer Science Section.

The awarded prizes were substantial, being offered by the Traian Lalescu Foundation, the Ministry of Education, the Romanian Mathematical Society as well as by the sponsor, the Amazon company.

The contest was held in the famous library of Gheorghe Asachi Technical University of Iaşi and afterwards the students had the opportunity to visit the beautiful city of Iaşi.

The interested reader may find additional details at the competition's website: `http://math.etti.tuiasi.ro/TL2023/`.

We present the statements and solutions of the problems given at Sections A and B of the contest.

1. Section A

Problem 1. Solve the following equation $x^{2023} = \widehat{38}$ in $\mathbb{Z}_{2023}$.

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Author's solution. Since $(38, 2023) = 1$, $\widehat{38}$ is invertible in $\mathbb{Z}_{2023}$ and, consequently, any solution x of the equation is invertible in $\mathbb{Z}_{2023}$.

Therefore, using Euler's Theorem, $x^{\varphi(2023)} = x^{1632} = \widehat{1}$, so the equation is equivalent to $x^{391} = \widehat{38}$, which can be written

$$x^{17\cdot 23} = \widehat{38}.$$

According to Chinese Remainder Theorem, this congruence is equivalent to the system

$$\begin{cases} x^{17\cdot 23} \equiv 38 \pmod 7 \\ x^{17\cdot 23} \equiv 38 \pmod{17^2}, \end{cases}$$

which, taking into account that $x^6 \equiv 1 \pmod 7$ and $x^{16\cdot 17} = x^{\varphi(17^2)} \equiv 1 \pmod{17^2}$, can be written as follows:

$$\begin{cases} x \equiv 3 \pmod 7, \\ x^{17\cdot 7} \equiv 38 \pmod{17^2}. \end{cases} \tag{1}$$

From $x^{17\cdot 7} \equiv 38 \pmod{17^2}$ we get $x^{17\cdot 7} \equiv 38 \pmod{17}$. Since $x^{16} \equiv 1 \pmod{17}$ by Fermat's Little Theorem, we obtain $x^7 \equiv 4 \pmod{17}$.

Raising this last congruence to the power of 7, we get $x \equiv 4^7 \pmod{17}$, that is, $x \equiv -4 \pmod{17}$. Hence, there exists $\lambda \in \{0, 1, \ldots, 16\}$ such that $x \equiv -4 + 17\lambda \pmod{17^2}$; it is verified by calculation that all these values are solutions of congruence $x^{17\cdot 7} \equiv 38 \pmod{17^2}$. According to Chinese Remainder Theorem, the solution of the system (1) is $x \equiv 3 \cdot 1156 + 868 \cdot (-4+17\lambda) \pmod{2023}$, that is $x \equiv -4+595\lambda \pmod{2023}$, $\lambda \in \{0, 1, \ldots, 16\}$.

Concluding, the solutions of the initial equation are:

$$\{\widehat{-4+595}\lambda : \lambda \in \{0,1,\ldots,16\}\} = \{\widehat{-4+119}\lambda : \lambda \in \{0,1,\ldots,16\}\}.$$

Problem 2. Let $A, B \in \mathcal{M}_n(\mathbb{C})$ such that there exist $a, b \in \mathbb{C}$ and $c \in \mathbb{R}\setminus\left\{\frac{1}{2}\right\}$ for which

$$aA + bB = (1-c)AB + cBA.$$

Prove that $(AB - BA)^n = O_n$.

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Authors' solution. The relation in the statement can be written

$$(A - bI_n)(aI_n - B) + abI_n = c(BA - AB)$$

and, respectively,

$$(aI_n - B)(A - bI_n) + abI_n = (c-1)(BA - AB).$$

The matrices $(A - bI_n)(aI_n - B) + abI_n$ and $(aI_n - B)(A - bI_n) + abI_n$ have the same characteristic polynomial, so they have the same eigenvalues. Then the matrices $c(BA - AB)$ and $(c-1)(BA - AB)$ have the same spectrum.

If $c = 1$, the conclusion is obvious. If $c \neq 1$, we deduce that $BA - AB$ and $\frac{c}{c-1}(BA - AB)$ have the same eigenvalues. Let λ be an eigenvalue of the matrix $BA - AB$. Then $\frac{c}{c-1}\lambda$ is also an eigenvalue and, by induction, $\left(\frac{c}{c-1}\right)^k \lambda$ is an eigenvalue for the matrix $BA - AB$, for any integer $k \geq 1$.

Because $BA - AB$ has a finite number of eigenvalues and $\frac{c}{c-1} \neq \pm 1$, one has $\lambda = 0$. All the eigenvalues of the matrix $BA - AB$ are therefore zero, hence the conclusion.

Problem 3. Let $f : [0,1] \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that $f(0) = f(1) = 0$. Prove that

$$\left(\int_0^1 x^2 f(x)\,\mathrm{d}x\right)^2 \leq \frac{1}{112}\int_0^1 \left(f'(x)\right)^2 \mathrm{d}x.$$

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Author's solution. We consider

$$\int_0^1 \left(\frac{x^3+c}{3}\right)' f(x)\,\mathrm{d}x = \left(\frac{x^3+c}{3}\right) f(x)\Big|_0^1 - \int_0^1 \left(\frac{x^3+c}{3}\right) f'(x)\,\mathrm{d}x$$
$$= -\int_0^1 \left(\frac{x^3+c}{3}\right) f'(x)\,\mathrm{d}x.$$

By applying the CBS inequality, we obtain

$$\left(\int_0^1 \left(\frac{x^3+c}{3}\right) f'(x)\,\mathrm{d}x\right)^2 \leq \int_0^1 \left(\frac{x^3+c}{3}\right)^2 \mathrm{d}x \cdot \int_0^1 \left(f'(x)\right)^2 \mathrm{d}x.$$

We compute

$$\int_0^1 \left(\frac{x^3+c}{3}\right)^2 \mathrm{d}x = \int_0^1 \frac{1}{9}\left(x^6+2cx^3+c^2\right)\mathrm{d}x = \frac{1}{9}\left(\frac{1}{7}+2c\frac{1}{4}+c^2\right).$$

The function $g(c) = \frac{1}{9}c^2 + \frac{1}{18}\cdot c + \frac{1}{63}$ attains for $c = -\frac{1}{4}$ its minimum value, which is $\frac{1}{112}$.

Problem 4. Let $f : (0,\infty) \to (0,\infty)$ be an increasing function such that

$$\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = 1.$$

Prove that $\lim\limits_{x\to\infty} \frac{f(x)}{a^x} = 0$, for any $a > 1$.

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Author's solution. Let $a > 1$ be arbitrary. Suppose, by contradiction, that the function $x \mapsto \frac{f(x)}{a^x}$ does not have the limit equal to 0 when $x \to \infty$. Then there exists $\rho > 0$ such that, for any $r > 0$, there exists $x_r \geq r$ such that

$$\frac{f(x_r)}{a^{x_r}} \geq \rho.$$

In particular, for any $n \in \mathbb{N}$, there exists $x_n \geq n$ such that $f(x_n) \geq \rho a^{x_n}$. Obviously, $x_n \to \infty$.

Fix now $\varepsilon \in (0, a-1)$. From

$$\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = 1,$$

we get the existence of $b \in \mathbb{R}$ such that, for any $x \in [b,\infty)$,

$$1 \leq \frac{f(x+1)}{f(x)} < 1+\varepsilon.$$

By induction, we deduce that

$$f(x+k) < (1+\varepsilon)^k f(x), \quad \forall x \in [b,\infty), \quad \forall k \in \mathbb{N}^*.$$

In particular, $f(b+k) < (1+\varepsilon)^k f(b)$, for any $k \in \mathbb{N}^*$.

Let $n \in \mathbb{N}$ such that $x_n > b$. We denote $k = \lfloor x_n - b \rfloor + 1$. Then $k > x_n - b$, or $b + k > x_n$. We have

$$\rho a^{x_n} \leq f(x_n) \leq f(b+k) < (1+\varepsilon)^k f(b) < (1+\varepsilon)^{x_n-b+2} f(b),$$

hence we deduce

$$f(b) > \rho(1+\varepsilon)^{b-2}\left(\frac{a}{1+\varepsilon}\right)^{x_n},$$

for any $n \in \mathbb{N}$ with $x_n > b$. By passing to the limit for $n \to \infty$, we obtain that $f(b) \geq \infty$. Contradiction.

Alternative solution. *The following solution was given by Florin Grigore, from Babeş–Bolyai University of Cluj–Napoca (contestant).*

Using the assumption of the problem, we deduce that $\lim\limits_{n\to\infty} \frac{f(n+1)}{f(n)} = 1$.

Consider the sequence (y_n) with $y_n := \frac{f(n)}{a^n}, \forall n \geq 1$. We have that

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \lim_{n\to\infty} \frac{f(n+1)}{a^{n+1}} \cdot \frac{a^n}{f(n)} = \frac{1}{a} < 1,$$

hence $y_n \to 0$ for $n \to \infty$.

For arbitrary $x > 0$, denote $n := \lfloor x \rfloor$, so $n \leq x < n+1$ and $f(n) \leq f(x) \leq f(n+1)$, hence

$$y_n \cdot \frac{1}{a} = \frac{f(n)}{a^{n+1}} \leq \frac{f(x)}{a^x} \leq \frac{f(n+1)}{a^n} = y_{n+1} \cdot a.$$

The conclusion follows easily.

Alternative solution. *The following solution was given by Marian Vasile, from West University of Timişoara (contestant).*

Define $g : (0, \infty) \to \mathbb{R}$ by $g(x) := \ln f(x)$. Using the monotony of f, we get that g is increasing.

The assumption of the problem implies that

$$\lim_{x\to\infty} (g(x+1) - g(x)) = 0.$$

Then, by Stolz–Cesàro Theorem, we obtain

$$\lim_{n\to\infty} \frac{g(n)}{n} = \lim_{n\to\infty} \frac{g(n+1) - g(n)}{1} = 0.$$

Let $x > 0$ and $n = \lfloor x \rfloor$. Then, since g is increasing, $n \leq x < n+1$ and $g(n) \leq g(x) \leq g(n+1)$, hence

$$\frac{g(n)}{n} \cdot \frac{n}{n+1} = \frac{g(n)}{n+1} \leq \frac{g(x)}{x} \leq \frac{g(n+1)}{n} = \frac{g(n+1)}{n+1} \cdot \frac{n+1}{n},$$

hence $\lim\limits_{x\to\infty} \frac{g(x)}{x} = 0$.

Now, take arbitrary $a > 1$ and fix $\varepsilon \in (0, \ln a)$. We know that there exists $\delta > 0$ such that, for any $x > \delta$, $g(x) < \varepsilon x$. Then, for any $x > \delta$, we have $g(x) - x \ln a < x(\varepsilon - \ln a)$, hence $\lim\limits_{x\to\infty} (g(x) - x \ln a) = -\infty$, and

$$\lim_{x\to\infty} \frac{f(x)}{a^x} = e^{\lim\limits_{x\to\infty} (\ln f(x) - x \ln a)} = e^{-\infty} = 0.$$

2. Section B

Problem 1. Let $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $AB = A$ and $BA = B$. We consider the following functions:

$$\begin{aligned} f &: \mathbb{C} \to \mathbb{C}, \quad f(z) = \operatorname{Tr}((1-z)A + zB), \quad \text{for any } z \in \mathbb{C}, \\ g &: \mathbb{C} \to \mathbb{N}, \quad g(z) = \operatorname{rank}((1-z)A + zB), \quad \text{for any } z \in \mathbb{C}, \\ h &: \mathbb{C} \to \mathbb{C}, \quad h(z) = \det((1-z)A + zB), \quad \text{for any } z \in \mathbb{C}. \end{aligned}$$

Prove that the functions f, g, and h are constant.

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Authors' solution. For any $z \in \mathbb{C}$, we denote $C_z = (1-z)A + zB$. We have

$$A^2 = A \cdot A = AB \cdot A = A \cdot BA = AB = A,$$
$$B^2 = B \cdot B = BA \cdot B = B \cdot AB = BA = B,$$

and

$$\begin{aligned} C_z^2 &= ((1-z)A + zB)^2 = (1-z)^2 A^2 + z(1-z)(AB + BA) + z^2 B^2 \\ &= (1-z)^2 A + z(1-z)(A+B) + z^2 B \\ &= \left((1-z)^2 + z(1-z)\right) A + \left(z^2 + z(1-z)\right) B \\ &= (1-z)A + zB = C_z. \end{aligned}$$

a) Since $\operatorname{Tr} AB = \operatorname{Tr} BA$, it follows that $\operatorname{Tr} A = \operatorname{Tr} B$, hence

$$f(z) = \operatorname{Tr} C_z = (1-z)\operatorname{Tr} A + z \operatorname{Tr} B = \operatorname{Tr} A$$

for any $z \in \mathbb{C}$, so f is constant.

b) Any idempotent matrix M has the Jordan canonical form of the form $J_r = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$, where $r = \operatorname{rank} M$, hence $\operatorname{Tr} M = \operatorname{rank} M$. Since C_z is an idempotent matrix, it follows that

$$g(z) = \operatorname{rank} C_z = \operatorname{Tr} C_z = f(z) \text{ for any } z \in \mathbb{C},$$

so $g = f$, thus g is constant.

c) Since $C_z^2 = C_z$, it follows that $\det C_z \in \{0, 1\}$, for any $z \in \mathbb{C}$. Let $z \in \mathbb{C}$ be fixed. We define the function $h_z : [0,1] \to \{0,1\}$, $h_z(t) = h(tz) = \det C_{tz}$. Obviously, the function h_z is continuous, so h_z must to be constant, wherefrom

$$h(z) = h_z(1) = h_z(0) = h(0) = \det A,$$

so h is constant.

Alternative solution. *The following solution of the fact that the functions f and h are constant was given by Petru–Vlad Ionescu, from University Politehnica of Bucharest (contestant).*

Since $\operatorname{Tr} AB = \operatorname{Tr} BA$, we have that $\operatorname{Tr} A = \operatorname{Tr} B$, so

$$\begin{aligned} f(z) &= \operatorname{Tr}((1-z)A + zB) = \operatorname{Tr}(A - z(A-B)) \\ &= \operatorname{Tr} A - z \operatorname{Tr}(A-B) = \operatorname{Tr} A - z(\operatorname{Tr} A - \operatorname{Tr} B) = \operatorname{Tr} A. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} h(z) &= \det((1-z)A + zB) = \det((1-z)A + zBA) \\ &= \det((1-z)I_n + zB)\det A = \det A \det((1-z)I_n + zB) \\ &= \det((1-z)A + zAB) = \det((1-z)A + zA) = \det A \in \{0,1\}. \end{aligned}$$

Remark. The matrices A, B, and C_z are idempotent. Moreover, $\operatorname{rank} A = \operatorname{rank} B$ and since the function g is constant $(g(0) = g(1) = g(z) = k)$ we obtain that the matrices A, B, and C_z are all similar to the matrix $J_k = \begin{pmatrix} I_k & O \\ O & O \end{pmatrix}$.

With this argument, since the trace, rank, and determinant are invariant to the similarity of matrices, we deduce that

$$\begin{aligned} f(z) &= \operatorname{Tr} C_z = \operatorname{Tr} J_k = k, \\ g(z) &= \operatorname{rank} C_z = \operatorname{rank} J_k = k, \\ h(z) &= \det C_z = \det J_k = \begin{cases} 0, & k < n, \\ 1, & k = n. \end{cases} \end{aligned}$$

Problem 2. Let $A \in \mathcal{M}_n(\mathbb{C})$ be a matrix of rank 1 such that $\operatorname{Tr} A \in \mathbb{R}$ and there exist $a, b \in \mathbb{C}$ with $|a| = 1$ such that

$$A^2 + aAA^* + aA^*A + b(A^*)^2 = O_n,$$

where $A^* = \bar{A}^T$. Prove that there exists $u \in \mathbb{C}^n$ such that $A = \pm u \cdot \bar{u}^T$.

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Author's solution. Since $\operatorname{rank} A = 1$ it follows that A can be written in the form $A = x \cdot \bar{y}^T$, with $x, y \in \mathbb{C}^n \setminus \{0\}$. Moreover, $A^2 = \langle x, y \rangle \cdot A = \operatorname{Tr} A \cdot A$. Obviously, because $\operatorname{Tr}(A^*) = \overline{\operatorname{Tr} A}$ and A^* also has rank 1, we have that $(A^*)^2 = \overline{\operatorname{Tr} A} \cdot A^*$.

Let's note, first, that if we had $\operatorname{Tr} A = 0$, then also $\operatorname{Tr} A^* = \overline{\operatorname{Tr} A} = 0$ and, passing to the trace in the relation from the statement, we obtain, taking into account the previous relations,

$$\begin{aligned} (\operatorname{Tr} A)^2 + a\operatorname{Tr}(AA^*) + a\operatorname{Tr}(A^*A) + b(\operatorname{Tr} A^*)^2 = 0 &\Leftrightarrow 2a\operatorname{Tr}(AA^*) = 0 \\ &\Leftrightarrow A = O_n, \end{aligned}$$

which contradicts the fact that $\operatorname{rank} A = 1$. So, $\operatorname{Tr} A = \operatorname{Tr} A^* \in \mathbb{R} \setminus \{0\}$.

We rewrite the relation from the statement, successively, as follows

$$\begin{aligned} (A + aA^*)(A + aA^*) + (b - a^2)(A^*)^2 &= O_n, \\ (A + aA^*)(A + aA^*) &= (a^2 - b)(A^*)^2, \\ a(\bar{a}A + A^*)(A + aA^*) &= (a^2 - b)(A^*)^2, \\ (A + aA^*)^*(A + aA^*) &= (a - \bar{a}b)(A^*)^2. \end{aligned}$$

If $b = a^2$ or, equivalently, $a - \bar{a}b = 0$, from the last relation it follows that $A + aA^* = O_n$ and from here, taking into account that $\operatorname{Tr} A = \operatorname{Tr} A^* \neq 0$, it follows that $a = -1$, which implies that $A = A^*$.

If $b \neq a^2$, from the equality $(A + aA^*)^*(A + aA^*) = (a - \bar{a}b)(A^*)^2$ it also follows that $[(A + aA^*)^*(A + aA^*)]^* = \left[(a - \bar{a}b)(A^*)^2\right]^*$ is equivalent to $(A + aA^*)^*(A + aA^*) = (\bar{a} - a\bar{b})A^2$, therefore

$$(A^*)^2 = \frac{\bar{a} - a\bar{b}}{a - \bar{a}b}A^2 \Leftrightarrow \operatorname{Tr} A \cdot A^* = \frac{\bar{a} - a\bar{b}}{a - \bar{a}b} \cdot \operatorname{Tr} A \cdot A \Leftrightarrow A^* = \frac{\bar{a} - a\bar{b}}{a - \bar{a}b} \cdot A.$$

Passing to the trace, it follows that $\dfrac{\bar{a} - a\bar{b}}{a - \bar{a}b} = 1$, therefore $A = A^*$.

We conclude that $A = A^*$.

Taking into account that $A^* = \left((x\bar{y}^T)\right)^* = y\bar{x}^T$, the equality $A = A^*$ implies, multiplying scalar by x,

$$x\bar{y}^T = y\bar{x}^T \Leftrightarrow \langle x, y\rangle \cdot x = \|x\|^2 \cdot y.$$

Therefore x and y are linearly dependent, $x = \alpha y$, with $\alpha = \frac{\|x\|^2}{\langle x,y\rangle} = \pm\frac{\|x\|}{\|y\|} \in \mathbb{R} \setminus \{0\}$. Then, it follows that

$$A = \alpha \cdot x \cdot \bar{x}^T = \pm u \cdot \bar{u}^T, \quad \text{with } u = \sqrt{\frac{\|x\|}{\|y\|}} \cdot x.$$

Alternative solution. *The following solution was given by Roland Nicholas Cornoc, from Technical University of Cluj-Napoca (contestant).*

Let $\alpha = \operatorname{Tr} A \in \mathbb{R}$. Because $A^2 = \operatorname{Tr} A \cdot A$, we have that $A^2 = \alpha A$. On the other hand, we can write $(A^*)^2 = \operatorname{Tr}(A^*) \cdot A^* = \operatorname{Tr}(A) \cdot A^* = \alpha A^*$.

If $\alpha = 0$, then $A^2 = (A^*)^2 = O_n$. Passing to the trace in the relation from the statement, we obtain $a \operatorname{Tr}(AA^*) + a \operatorname{Tr}(A^*A) = 0$, so $2a \operatorname{Tr}(AA^*) = 0$. Since $|a| = 1$, we find that $\operatorname{Tr}(AA^*) = 0$, which means that $A = O_n$, contradiction with $\operatorname{rank} A = 1$.

Thus $\alpha \neq 0$. The relation from the statement is equivalent to

$$\alpha A + aAA^* + aA^*A + b\alpha A^* = O_n. \tag{2}$$

Multiplying relation (2) to the right by the matrix A^* we get

$$\alpha AA^* + a\alpha AA^* + aA^*AA^* + b\alpha^2 A^* = O_n.$$

On the other hand, multiplying relation (2) on the left by the matrix A^* we get

$$\alpha A^*A + aA^*AA^* + a\alpha A^*A + b\alpha^2 A^* = O_n.$$

Subtracting the last relation from the previous one, we obtain

$$\alpha (AA^* - A^*A) + a\alpha (AA^* - A^*A) = O_n,$$

so $(a + 1)(AA^* - A^*A) = O_n$.

Multiplying relation (2) to the right by the matrix A we obtain

$$\alpha^2 A + aAA^*A + a\alpha A^*A + b\alpha A^*A = O_n.$$

On the other hand, multiplying (2) on the left by the matrix A we get

$$\alpha^2 A + \alpha aAA^* + aAA^*A + b\alpha AA^* = O_n.$$

Subtracting the last relation from the previous one we obtain

$$\alpha a\left(A^*A - AA^*\right) + b\alpha\left(A^*A - AA^*\right) = O_n,$$

so $(a+b)\left(A^*A - AA^*\right) = O_n$.

If $A^*A \neq AA^*$, we obtain that $a = -1$, $b = 1$. The relation from the statement implies that $A^2 - AA^* - A^*A + (A^*)^2 = O_n$, so $(A - A^*)^2 = O_n$, wherefrom $(A - A^*)(A^* - A) = O_n$, so $(A - A^*)(A - A^*)^* = O_n$ and we get that $A - A^* = O_n$, therefore $A = A^*$, so $AA^* = A^*A$, contradiction.

So $AA^* = A^*A$, which means that the matrix A is normal. It follows that the matrix A is unitarily diagonalizable. Since $\operatorname{rank} A = 1$ and $\operatorname{Tr} A \in \mathbb{R}$, there exists a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$, such that

$$A = U \begin{pmatrix} \lambda & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} U^*,$$

with $\lambda \in \mathbb{R} \setminus \{0\}$.

If $\lambda > 0$, we can write

$$A = U \begin{pmatrix} \sqrt{\lambda} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} \sqrt{\lambda} & 0 & \dots & 0 \end{pmatrix} U^*.$$

With the choice

$$u = U \begin{pmatrix} \sqrt{\lambda} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

we have $A = u \cdot \overline{u}^T$.

If $\lambda < 0$, we can write

$$A = -U \begin{pmatrix} \sqrt{-\lambda} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} \sqrt{-\lambda} & 0 & \dots & 0 \end{pmatrix} U^*.$$

With the choice

$$u = U \begin{pmatrix} \sqrt{-\lambda} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

we have $A = -u \cdot \overline{u}^T$.

Problem 3. Let $f : [0,1] \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that $f(0) = f(1) = 0$. Prove that

$$\left(\int_0^1 x^2 f(x)\, \mathrm{d}x\right)^2 \leq \frac{1}{112} \int_0^1 \left(f'(x)\right)^2 \mathrm{d}x.$$

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Solution. See Problem 3 from Section A.

Problem 4. Consider the continuous and strictly increasing function $f : [0,+\infty) \to [0,+\infty)$ which satisfies $f(0) = 0$ and $f(\mathbb{N}) \subset \mathbb{N}$.

a) Prove that f is bijective and the power series $\sum\limits_{n=0}^{\infty} x^{f(n)}$ converges for any $x \in (-1,1)$.

b) We denote the inverse of f by g, and the sum of the previous power series by h. Suppose that

$$f(xy) = f(x) f(y), \quad \forall x, y \geq 0.$$

Compute $\lim\limits_{\substack{x \to 1 \\ x < 1}} g(1-x) \cdot h(x)$.

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Author's solution. a) Since f is strictly increasing, it is injective, and from the continuity of f and the fact that $f(\mathbb{N}) \subset \mathbb{N}$ it follows that $\lim\limits_{x \to \infty} f(x) = +\infty$. Since $f(0) = 0$, it follows that f is bijective.

Observe that the power series can be written as $\sum\limits_{n=1}^{\infty} a_k x^k$, where

$$a_k = \begin{cases} 1, & \text{for } k = f(n), \\ 0, & \text{otherwise.} \end{cases}$$

Then $\limsup\limits_{k \to \infty} \sqrt[k]{|a_k|} = 1$, hence the radius of convergence of the power series is 1.

b) By denoting $u(x) = \ln f(e^x)$, we observe that u is continuous and satisfies the Cauchy functional equation

$$u(x+y) = u(x) + u(y), \quad \forall x, y \in \mathbb{R}.$$

We have, successively,

$$u(x) = u(1)\,x, \quad \text{for any } x \in \mathbb{R}.$$
$$f(e^x) = e^{u(1)x}, \quad \text{for any } x \in \mathbb{R},$$
$$f(x) = x^{u(1)}, \quad \text{for any } x > 0.$$

Since $f(\mathbb{N}) \subset \mathbb{N}$, we get that there exists $p \in \mathbb{N}^*$ such that $f(x) = x^p$, for any $x \geq 0$.

Moreover, since f is increasing and $x \in (0,1)$, we have for $y \in [n, n+1]$ that

$$x^{f(n+1)} \leq x^{f(y)} \leq x^{f(n)},$$

hence

$$x^{f(n+1)} \leq \int_n^{n+1} x^{f(y)} \mathrm{d}y \leq x^{f(n)}.$$

It follows that

$$\sum_{n=0}^{N} x^{f(n+1)} \leq \int_0^{N+1} x^{f(y)} \mathrm{d}y \leq \sum_{n=0}^{N} x^{f(n)},$$

and, since the series converges for $x \in (0,1)$, we obtain that the integral $\int_0^\infty x^{f(y)} \mathrm{d}y$ converges and

$$\begin{aligned} \lim_{x \to 1_-} g(1-x)\,h(x) &= \lim_{x \to 1_-} (1-x)^{\frac{1}{p}} \int_0^\infty x^{f(y)} \mathrm{d}y \\ &= \lim_{x \to 1_-} (1-x)^{\frac{1}{p}} \int_0^\infty e^{y^p \cdot \ln x} \mathrm{d}y. \end{aligned} \tag{3}$$

By the change of variable $y^p \cdot \ln x = -t$, we get

$$\begin{aligned} \int_0^\infty e^{y^p \cdot \ln x} \mathrm{d}y &= \int_0^\infty e^{-t} \cdot \frac{1}{p} \left(\frac{t}{-\ln x} \right)^{\frac{1}{p}-1} \cdot \frac{1}{-\ln x}\, \mathrm{d}t \\ &= \frac{1}{p(-\ln x)^{\frac{1}{p}}} \int_0^\infty e^{-t} \cdot t^{\frac{1}{p}-1} \mathrm{d}t \\ &= \frac{1}{(-\ln x)^{\frac{1}{p}}} \cdot \frac{1}{p} \cdot \Gamma\left(\frac{1}{p}\right). \end{aligned}$$

Using relation (3), we obtain

$$\lim_{x \to 1_-} g(1-x) \cdot h(x) = \lim_{x \to 1_-} \left(\frac{1-x}{-\ln x} \right)^{\frac{1}{p}} \frac{1}{p} \cdot \Gamma\left(\frac{1}{p}\right) = \frac{1}{p} \Gamma\left(\frac{1}{p}\right).$$

MATHEMATICAL NOTES

Maximal number of diameters of finite planar lattice point sets

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Abstract. Given an integer $n \geq 2$, we prove that the largest possible number of diameters a planar set of n lattice points may have is $n-1$.

Keywords: Finite planar lattice point set, diameter, diameter graph.
MSC: Primary 52C05; Secondary 11H06.

1. Introduction

Erdős addressed a large number of various topics in combinatorial geometry, such as allowed or forbidden subconfigurations of point configurations, extremal numbers of different types of distances in finite point configurations, to name a few. Amongst these latter, he dealt with the maximal number of occurrences of the largest distance (diameter) in a finite point configuration in the Euclidean plane [4]. This topic goes back to [7, 12].

A *diameter* of a finite planar set is any line segment of maximal Euclidean length having both end points in that set.

Fix an integer $n \geq 2$ once and for all. If $n \geq 3$, the largest possible number of diameters a planar n-point set may have is n [4, 7, 10, 12]. The standard proof is by induction on n. The key point is that, by the triangle inequality, every two diameters intersect: Either they both emanate from the same point or they both cross at some interior point.

Consider the *diameter graph* on n pairwise distinct points in the plane, i. e., the geometric graph on those points, whose edges are the diameters of the configuration. In the continuous realm, it may very well happen that the diameter graph be 2-*regular*, i. e., vertices all have degree two. A 2-regular graph consists of one or more (disconnected) cycles. Since every two diameters intersect, this is possible if and only if n is odd, in which case the diameter graph is a star-shaped self-crossing n-cycle $A_1A_2A_3\dots A_n$ whose polygonal convex hull reads around the boundary $A_1A_3A_5\dots A_nA_2A_4A_6\dots A_{n-1}$. A star-shaped regular polygon with an odd number of vertices is one such. However, this configuration is not unique, at least for $n=5$ [2].

A *lattice point* in the Cartesian plane is one whose coordinates are both integral. It seems natural to ask what is the largest possible number of diameters a planar set of n lattice points may have. Our purpose here is to prove that this maximum is $n-1$.

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Theorem 1. *The largest possible number of diameters a planar set of n lattice points may have is $n-1$.*

The inductive argument goes verbatim along the lines in the standard proof [4, 7, 10, 12], and will therefore not be repeated here. The only difference is that this time the diameter graph cannot be 2-regular. Roughly speaking, there are always 'loose ends'. The base case, $n = 2$, being clear, this is precisely what makes induction work, to force the upper bound $n-1$ on the number of diameters.

It is therefore sufficient to rule out the virtually possible case of a 2-regular diameter graph. Ball's theorem [1] settles the case: There are no equilateral lattice polygons (possibly, self-crossing) with an odd number of vertices. In particular, the diameter graph is acyclic.

Before proceeding any further to prove the upper bound achieved, we take time out for a brief digression. A 2-*distance* set (configuration) of points in an Euclidean space is one whose non-zero distance set has size two. The vertex set of a non-equilateral isosceles triangle, or of a square, or of a regular pentagon are obvious examples in the Euclidean plane. The full collection of 2-distance sets in the Euclidean plane is described in [3, 13].

In the lattice point realm, Ball's theorem [1] rules out most of these configurations, to leave only non-equilateral isosceles lattice triangles and lattice squares. The size of the vertex set can equally well be directly obtained from the theorem.

Corollary 2. *A* 2*-distance set of lattice points in the plane has size* 3 *or* 4.

Proof. Consider a 2-distance set of n lattice points in the plane. By the theorem, there are at most $n-1$ diameters, so there are at least $\frac{1}{2}n(n-1) - (n-1) = \frac{1}{2}(n-1)(n-2)$ minimal distances. The maximal vertex degree in the associated minimal distance graph is then at least $n-2$.

Let a be a vertex of maximal degree, and choose $n-2$ neighbors of a to form a set S. Every two distinct points of S are the end points of a diameter; otherwise, a and some pair of points from S would form an equilateral lattice triangle, which is impossible. This forces $n-2 = |S| \leq 2$, since otherwise some triple of points from S would form an equilateral lattice triangle, which is again impossible. Consequently, $n = 3$ or 4 and the corollary follows. □

In Section 2 below we exhibit a configuration of n lattice points in the plane with $n-1$ diameters.

2. Achieving the Upper Bound

To obtain planar configurations of lattice points that achieve the upper bound for the number of diameters, consider the Diophantine equation

$$x^2 + y^2 = 5^N, \tag{$*$}$$

where N is a non-negative integer [9, 11].

By Jacobi's two-square theorem [5, 6, 8, 9, 11], the number of pairwise distinct solutions of the Diophantine equation $x^2 + y^2 = M$, where M is a positive integer, is equal to four times the excess of the number of divisors of M that are congruent to 1 modulo 4 over those congruent to 3 modulo 4.

In the case at hand, $(*)$ has exactly $4(N+1)$ pairwise distinct solutions. This can equally well be established directly [9], by noticing that $1 \pm 2\sqrt{-1}$ are Gaussian primes, and $\frac{1}{\pi}\arg(1 + 2\sqrt{-1})$ is irrational; explicitly, the solutions are $u(1 + 2\sqrt{-1})^k(1 - 2\sqrt{-1})^{N-k}$, where $u = \pm 1$ or $\pm\sqrt{-1}$, and $k = 0, 1, \ldots, N$.

Exactly $N + 1$ of these solutions lie in the first quadrant, $x > 0$ and $y \geq 0$; and since 5 is odd, exactly $\left\lceil \frac{1}{2}(N+1) \right\rceil$ of these, say, (x_i, y_i), $i = 1, 2, \ldots, \left\lceil \frac{1}{2}(N+1) \right\rceil$, satisfy $x > y \geq 0$.

The $\left\lceil \frac{1}{2}(N+1) \right\rceil$ lattice points (x_i, y_i) are all exactly $5^{N/2}$ away from the origin, and every two are (strictly) less than $5^{N/2}$ distance apart.

Consequently, the origin and the (x_i, y_i) form a planar configuration of $\left\lceil \frac{1}{2}(N+3) \right\rceil$ lattice points with exactly $\left\lceil \frac{1}{2}(N+1) \right\rceil$ diameters of length $5^{N/2}$ each. Setting $N = 2n - 3$ completes the argument.

We end by describing a related configuration. Consider an even integer $N \geq n$. The (x_i, y_i) above and the $(5^{N/2} - x_i, y_i)$ form a configuration of $N + 2$ lattice points with exactly $N + 1$ diameters: $\frac{1}{2}N + 1$ of these join $(0, 0)$ to each (x_i, y_i), and another $\frac{1}{2}N$ join $(5^{N/2}, 0)$ to each $(5^{N/2} - x_i, y_i)$ with a positive y_i. Deletion of any $N - n + 2$ points with both coordinates positive then settles the case.

Remark 3. For each $n \leq 6$, there exists a configuration of n lattice points whose diameter graph is a path: Let (a, b, c) be a Pythagorean triple, where $a < b < c < 2a$, e.g., $(a, b, c) = (3, 4, 5)$, and let $A_1 = (0, 0)$, $A_2 = (a, b)$, $A_3 = (a - b, b - a)$, $A_4 = (a - b + c, b - a)$, $A_5 = (a - 2b + c, b)$, and $A_6 = (2a - 2b + c, 0)$. The diameters are $A_iA_{i+1} = c$, $i = 1, 2, \ldots, 5$, so every n consecutive A_i form a configuration whose diameter graph is a path.

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A note on the sequence A064222

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Abstract. We determine the general formula of the sequence A064222 in The On-Line Encyclopedia of Integer Sequences.

Keywords: OEIS A064222, sequence, general formula, digits.
MSC: 11B83, 11B37, 11A67.

1. Introduction

The sequence A064222 in The On-Line Encyclopedia of Integer Sequences introduced on 31st of October 2007 by Reinhard Zumkeller, is defined by $x_0 = 0$ and, for each $n \in \mathbb{N}$, by setting x_n to be the number obtained from $1+x_{n-1}$ by rearranging its digits so that they are written in decreasing order.

Starting at x_1, the terms of the sequence are:
1, 2, 3, 4, 5, 6, 7, 8, 9,
10, 11, 21, 22, 32, 33, 43, 44, 54, 55, 65, 66, 76, 77, 87, 88, 98, 99,
100, 110, 111, 211, 221, 222, 322, ..., 887, 888, 988, 998, 999,
1000, 1100, 1110, 1111, 2111, 2211, ..., 9888, 9988, 9998, 9999,
....,
where the terms are grouped according to their number of digits. We see that this sequence is strictly increasing because every term is by at least one unit larger than the previous one.

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2. Number of terms with a given number of digits

Looking at the terms written above, we see that the (nonzero) terms with k digits start with 10^{k-1} and end with $10^k - 1$. We will actually prove this by induction. From the list above we see that it is verified by the terms having one digit. By considering that the statement holds true for the terms with k digits, it means that the last term having k digits is $10^k - 1$, therefore the first term with $k+1$ digits is $10^k = 1\underbrace{00\ldots0}_{k}$.

It follows that the next terms are:

$11\underbrace{00\ldots0}_{k-1} = 10^k + 10^{k-1}$

$111\underbrace{00\ldots0}_{k-2} = 10^k + 10^{k-1} + 10^{k-2}$

....

$111\ldots1 = 10^k + 10^{k-1} + 10^{k-2} + \cdots + 1$

$2\underbrace{11\ldots1}_{k-1} = 2 \cdot 10^k + 10^{k-1} + \cdots + 1$

....

$99\ldots98 = 9 \cdot 10^k + 9 \cdot 10^{k-1} + \cdots + 9 \cdot 10 + 8$

$99\ldots9 = 9 \cdot \left(10^k + 10^{k-1} + \cdots + 1\right)$

The last term written above is followed by terms having more than $k+1$ digits, because as we have said, the sequence is strictly increasing. We thus proved by induction that the terms with k digits start with 10^{k-1} and end with $10^k - 1$.

Moreover, looking at the list from the induction step above, we see k terms starting with 1, k terms starting with 2 and, in fact, k terms starting with c for every digit $c \neq 0$. Thus, the number of terms having exactly k digits is $9k$

For $k \geq 1$ let us denote by $\alpha(k)$ the number of terms of $(x_n)_{n\geq1}$ that have at most k digits. Since we will need it for uniformity of writing, let $\alpha(0) = 0$. From the above, we immediately obtain that for all nonnegative integers k

$$\alpha(k) = \frac{9k(k+1)}{2}.$$

Using this notation, the first term of $(x_n)_{n\geq1}$ that has k digits is

$$x_{\alpha(k-1)+1} = 10^{k-1},$$

whilst the last one is

$$x_{\alpha(k)} = 10^k - 1.$$

3. Number of digits of a term in terms of its index

At this point, it is only natural to ask how many digits the terms have.

Given an index $n \in \mathbb{N}$, let us assume that x_n has k_n digits. Then,

the first term that has k_n digits $\leq x_n \leq$ *the last term that has k_n digits,*

that is,

$$x_{\alpha(k_n-1)+1} \leq x_n \leq x_{\alpha(k_n)}.$$

The sequence $(x_n)_n$ being strictly increasing, these inequalities also hold for the indices:

$$\alpha(k_n-1)+1 \leq n \leq \alpha(k_n),$$

which means

$$\frac{9k_n(k_n-1)}{2}+1 \leq n \leq \frac{9k_n(k_n+1)}{2}.$$

This implies

$$\frac{9k_n(k_n-1)}{2} \leq n-1 < \frac{9k_n(k_n+1)}{2},$$

or

$$4k_n(k_n-1) \leq \frac{8(n-1)}{9} < 4k_n(k_n+1),$$

otherwise written as

$$(2k_n-1)^2 \leq \frac{8n+1}{9} < (2k_n+1)^2,$$

which means

$$k_n \leq \frac{\sqrt{8n+1}+3}{6} < k_n+1,$$

whence

$$k_n = \left\lfloor \frac{\sqrt{8n+1}+3}{6} \right\rfloor. \tag{1}$$

We now want to find the formula of x_n based on its index $\mathcal{P}_n$ in the ordered list of terms that have the same number of digits. Let us notice that

$$\mathcal{P}_n = n - \alpha\big(k_n-1\big). \tag{2}$$

Also, for ease of writing, for a positive integer s we will denote by $\psi(s)$ the number

$$\underbrace{11\ldots11}_{s} = \sum_{i=0}^{s-1} 10^i = \frac{10^s-1}{9}.$$

According to the description we made above as to what the terms of the sequence look like, we will have

$$x_n = \underbrace{\overbrace{aa\ldots aa}^{k_n-m}\overbrace{bb\ldots bb}^{m}}_{k_n}$$

for some $a \in \{1, 2, \ldots, 9\}$ and some $m \in \{0, 1, \ldots, k_n - 1\}$. Here b was used to denote the digit $a - 1$.
Then,

$$x_{n+m} = \underbrace{aa \ldots aa}_{k_n},$$

and this means we can write

$$x_n = a\psi(k_n) - \psi(m),$$

where

$$ak_n - m = \mathcal{P}_n.$$

Since this last relation may be written $-\mathcal{P}_n = -ak_n + m$, we notice that $-a = \left\lfloor -\dfrac{\mathcal{P}_n}{k_n} \right\rfloor$, so $a = \left\lceil \dfrac{\mathcal{P}_n}{k_n} \right\rceil$, and $m = \big(-\mathcal{P}_n\big) \mod k_n$. Therefore,

$$x_n = \left\lceil \frac{\mathcal{P}_n}{k_n} \right\rceil \psi(k_n) - \psi\Big(\big(-\mathcal{P}_n\big) \mod k_n\Big). \tag{3}$$

4. General formula

Now relation (4) gives us a nice formula for x_n, but it is not very explicit, since it is written in terms of the rather intricate k_n and $\mathcal{P}_n$. If we move to discard these, plugging $\mathcal{P}_n$ from (2) and k_n from (1) into (3), we get

$$x_n = \left\lceil \frac{n - \alpha\big(k_n - 1\big)}{k_n} \right\rceil \psi\big(k_n\big) - \psi\Big(\Big(\alpha\big(k_n - 1\big) - n\Big) \mod k_n\Big). \tag{4}$$

But

$$\frac{n - \alpha(k_n - 1)}{k_n} = \frac{n}{k_n} - \frac{9(k_n - 1)}{2}$$

and

$$\alpha(k_n - 1) \mod k_n = \frac{9k_n(k_n - 1)}{2} \mod k_n = \frac{k_n}{(2, k_n)} \mod k_n$$

$$= \frac{[2, k_n]}{2} \mod k_n = \frac{1}{2}\left[2, \left\lfloor \frac{\sqrt{8n+1}+3}{6} \right\rfloor\right] \mod \left\lfloor \frac{\sqrt{8n+1}+3}{6} \right\rfloor,$$

so we obtain the following general formula for $(x_n)_{n\geq 1}$:

$$\left\lceil \frac{n}{\left\lfloor \frac{\sqrt{8n+1}+3}{6} \right\rfloor} - \frac{9\left\lfloor \frac{\sqrt{8n+1}-3}{6} \right\rfloor}{2} \right\rceil \cdot \underbrace{11\ldots11}_{\left\lfloor \frac{\sqrt{8n+1}+3}{6} \right\rfloor} - \underbrace{11\ldots\ldots\ldots\ldots\ldots\ldots\ldots11}_{\frac{\left[2, \left\lfloor \frac{\sqrt{8n+1}+3}{6} \right\rfloor\right]}{2} - n \bmod \left\lfloor \frac{\sqrt{8n+1}+3}{6} \right\rfloor} \tag{5}$$

If we choose to replace the expression of the form $\underbrace{11\ldots11}_{s}$ in formula (5) by their $\dfrac{10^s - 1}{9}$ counterparts, the general formula of the sequence $(x_n)_{n\geq 1}$ becomes

$$\left\lceil \frac{n}{\left\lfloor \frac{\sqrt{8n+1}+3}{6} \right\rfloor} - \frac{9\left\lfloor \frac{\sqrt{8n+1}-3}{6} \right\rfloor}{2} \right\rceil \cdot \frac{10^{\left\lfloor \frac{\sqrt{8n+1}+3}{6} \right\rfloor} - 1}{9}$$

$$- \frac{10^{\frac{\left[2, \left\lfloor \frac{\sqrt{8n+1}+3}{6} \right\rfloor\right]}{2} - n \bmod \left\lfloor \frac{\sqrt{8n+1}+3}{6} \right\rfloor} - 1}{9}.$$

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[1] The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A064222.

PROBLEMS

Authors should submit proposed problems to `gmaproblems@rms.unibuc.ro`. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of May 2024**.

PROPOSED PROBLEMS

544. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function with continuous derivative on $[0, 1]$ such that $f(0) = f(1/2) = f(1) = 0$. Show that

$$\int_0^1 (f'(x))^2 \,\mathrm{d}x \geq 48 \left(\int_0^1 f(x) \,\mathrm{d}x \right)^2 .$$

Proposed by Robert Dragomirescu, Stanford University, USA, and Cezar Lupu, Yanqi Lake Beijing Institute of Mathematical Sciences and Applications (BIMSA) and Tsinghua University, P. R. China

545. Let $A, B \in M_n(\mathbb{R})$ such that $A^2 = -I_n$, $\det B \neq 0$, and $AB = -BA$. Prove that n is even and the sign of $\det B$ is $(-1)^{n/2}$.

Proposed by Mihai Opincariu, Brad, and Vasile Pop, Cluj-Napoca, Romania.

546. Let $X, Y \in M_n(\mathbb{C})$ such that $Y^2 = YX - XY$ and the rank of $X + Y$ is 1. Prove that $Y^3 = YXY = O_n$.

Proposed by Stănescu Florin, Șerban Cioculescu School, Găești, Romania.

547. Prove that

$$\int_0^\infty \frac{|\sin x|}{1 + x^2} \,\mathrm{d}x = \frac{e^2 - 1}{2e} \ln\left(\frac{e + 1}{e - 1} \right).$$

Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.

548. Let $A \in M_n(\mathbb{C})$ such that $(I_n - AA^*)^2 = I_n - A^*A$. Prove that $A^2A^* = A$.

Here A^* denotes the conjugate transpose of A, $A^* = \bar{A}^t$.

Proposed by Mihai Opincariu, Brad, and Vasile Pop, Cluj-Napoca, Romania.

549. Let $n \geq 3$ and let $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$ be pairwise distinct. We denote by s_1, s_2, s_3 the first symmetric sums in the variables $a_1, \ldots, a_n$, i.e., $s_1 = \sum_i a_i$, $s_2 = \sum_{i<j} a_i a_j$, and $s_3 = \sum_{i<j<k} a_i a_j a_k$. Prove that

$$(n-2)s_1 \left(s_2 - \frac{n(n-1)(n+1)}{12} \right) \geq 3ns_3.$$

When does equality hold?

Proposed by Leonard Giugiuc, Traian National College, Drobeta-Turnu Severin, Romania.

550. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$, and let $E_n(k) = a_1^k + a_2^k + \cdots + a_n^k$.

(a) Prove that $E_5(k) \geq 5$ for $k \geq \frac{5}{4}$.

(b)* Prove or disprove that $E_7(k) \geq 7$ for $k \geq \frac{3}{2}$. (This is an open problem. At this time, the author doesn't have a solution.)

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania.

551. Solve in $\mathcal{M}_2(\mathbb{R})$ the equation $A^{2024} = -A^T$, where A^T denotes the transpose of A.

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Romania.

552. Let $f : [0,1] \to [0,1]$ be a continuous function which is derivable on $(0,1]$, such that $f'(x) < 0$ for $x \in (0,1]$, $f(1) = 0$ and $f'(1) < 0$. Prove that for every integer $n \geq 1$ the equation $f(x) = x^n$ has a unique solution in the interval $(0,1)$, denoted by a_n, and $\lim\limits_{n\to\infty} \frac{n}{\ln n}(a_n - 1) = -1$.

Proposed by Dumitru Popa, Department of Mathematics, Ovidius University of Constanța, Romania.

SOLUTIONS

529. Prove that

$$\prod_{n=2}^{\infty} e^{n^2+1/2}\left(1-\frac{1}{n^2}\right)^{n^4} = \pi \exp\left(\frac{-7}{4}+\frac{3\zeta(3)}{\pi^2}\right).$$

`Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.`

Solution by the author. The infinite product we are asked to compute is the limit of $P_N = \prod_{n=2}^{N} e^{n^2+1/2}\left(1-\frac{1}{n^2}\right)^{n^4}$ as $N \to \infty$.

We have

$$\begin{aligned}\sum_{n=2}^{N}(n^2+1/2) = \sum_{n=1}^{N}(n^2+1/2) - 3/2 &= N(N+1)(2N+1)/6 + N/2 - 3/2 \\ &= \frac{N^3}{3}+\frac{N^2}{2}+\frac{2N}{3}-\frac{3}{2}.\end{aligned}$$

Hence $\prod_{n=2}^{N} e^{n^2+1/2} = \exp\left(\frac{N^3}{3}+\frac{N^2}{2}+\frac{2N}{3}-\frac{3}{2}\right)$.

Next, we have

$$\prod_{n=2}^{N}\left(1-\frac{1}{n^2}\right)^{n^4} = \prod_{n=2}^{N}\left(\frac{(n-1)(n+1)}{n^2}\right)^{n^4}.$$

But

$$\prod_{n=2}^{N}(n-1)^{n^4} = \prod_{n=1}^{N-1} n^{(n+1)^4} = N^{-(N+1)^4}\prod_{n=2}^{N} n^{(n+1)^4},$$

$$\prod_{n=2}^{N}(n+1)^{n^4} = \prod_{n=3}^{N+1} n^{(n-1)^4} = 2^{-1}(N+1)^{N^4}\prod_{n=2}^{N} n^{(n-1)^4}.$$

Consequently,

$$\begin{aligned}\prod_{n=2}^{N}\left(1-\frac{1}{n^2}\right)^{n^4} &= \frac{(N+1)^{N^4}}{2N^{(N+1)^4}}\prod_{n=2}^{N} n^{(n+1)^4+(n-1)^4-2n^4} \\ &= \frac{(N+1)^{N^4}}{2N^{(N+1)^4}}\prod_{n=2}^{N} n^{12n^2+2} = \frac{(N+1)^{N^4}(N!)^2}{2N^{(N+1)^4}}\prod_{n=2}^{N} n^{12n^2}.\end{aligned}$$

Hence $P_N = Q_N R_N$, where

$$Q_n = \frac{(N+1)^{N^4}(N!)^2}{2N^{(N+1)^4}}\exp\left(\frac{N^3}{3}+\frac{N^2}{2}+\frac{2N}{3}-\frac{3}{2}\right),$$

$$R_n = \prod_{n=2}^{N} n^{12n^2}.$$

We have

$$\begin{aligned} N^4 \log\left(1+\frac{1}{N}\right) &= N^4\left(\frac{1}{N}-\frac{1}{2N^2}+\frac{1}{3N^3}-\frac{1}{4N^4}+O(N^{-5})\right) \\ &= N^3 - N^2/2 + N/3 - 1/4 + o(1). \end{aligned}$$

After taking exponentials, we get

$$\left(1+\frac{1}{N}\right)^{N^4} = e^{N^3-N^2/2+N/3-1/4+o(1)} \cong e^{N^3-N^2/2+N/3-1/4}.$$

It follows that

$$(N+1)^{N^4} = N^{N^4}\left(1+\frac{1}{N}\right)^{N^2} \cong N^{N^4} e^{N^3-N^2/2+N/3-1/4}.$$

Also, by Stirling's approximation formula,

$$(N!)^2 \cong 2\pi N^{2N+1} e^{-2N}.$$

It follows that

$$\begin{aligned} Q_N &\cong \frac{N^{N^4} e^{N^3-N^2/2+N/3-1/4}\cdot 2\pi N^{2N+1}e^{-2N}}{2N^{(N+1)^4}} \exp\left(\frac{N^3}{3}+\frac{N^2}{2}+\frac{2N}{3}-\frac{3}{2}\right) \\ &\cong \pi N^{-4N^3-6N^2-2N}\exp\left(\frac{4N^3}{3}-N-\frac{7}{4}\right). \end{aligned}$$

To estimate R_N, we apply the Euler–Maclaurin formula for

$$\log R_N = 12\sum_{n=2}^{N} n^2 \log n$$

and obtain

$$\begin{aligned} \log R_N &= 12\left(\frac{N^3\log N}{3}-\frac{N^3}{9}+\frac{N^2\log N}{2}+\frac{N\log N}{6}+\frac{N}{12}+C+O\left(\frac{1}{N}\right)\right) \\ &= (4N^3+6N^2+2N)\log N - \frac{4N^3}{3} + N + 12C + o(1), \end{aligned}$$

where, by Maple, $C = -\zeta'(-2) = \frac{\zeta(3)}{4\pi^2}$ and so $12C = \frac{3\zeta(3)}{\pi^2}$. By taking exponentials, we get

$$R_N \cong N^{4N^3+6N^2+2N}\exp\left(-\frac{4N^3}{3}+N+\frac{3\zeta(3)}{\pi^2}\right).$$

From the asymptotic values of Q_N and R_N we get that

$$P_N = Q_N R_N \cong \pi\exp\left(\frac{-7}{4}+\frac{3\zeta(3)}{\pi^2}\right)$$

when $N \to \infty$, which concludes the proof.

Editor's notes. The author's solution is somewhat incomplete, as the key relation $C = -\zeta'(-2) = \frac{\zeta(3)}{4\pi^2}$ is given without a proof, only with a reference to the Maple software. We now give a proper proof for this relation.

We start by reminding the reader some basic results regarding the Euler–Maclaurin formula. (A good source for Euler-Maclaurin is wikipedia, at `https://en.wikipedia.org/wiki/Euler-Maclaurin_formula`.)
Given $m, n \in \mathbb{Z}$, $m < n$, and p-differentiable function $f : [m, n] \to \mathbb{C}$, then

$$\sum_{j=m+1}^{n} f(j) - \int_m^n f(x)\,\mathrm{d}x = \sum_{k=1}^{p} \frac{B_k}{k!}(f^{(k-1)}(n) - f^{(k-1)}(m)) + R_p(m, n),$$

where B_k is the kth Bernoulli number and the remainder $R_p(m, n)$ is bounded by $|R_p(m, n)| \le \frac{2\zeta(p)}{(2\pi)^p} \int_m^n |f^{(p)}(x)|\mathrm{d}x$.

As a consequence, if $n_0 \in \mathbb{Z}$ and $x_N := \sum_{j=n_0+1}^{N} f(j) - \int_{n_0}^{N} f(x)\mathrm{d}x - \sum_{k=1}^{p} \frac{B_k}{k!} f^{(k-1)}(N)$, then $x_m - x_n = R_p(m, n)$. Hence if we have that $\lim_{m,n\to\infty} R_p(m, n) = 0$, then the sequence x_N is convergent. Moreover, if $\lim x_N = \ell$, then $x_N - \ell = \lim_{n\to\infty} R_p(N, n)$. In particular, $|x_N - \ell| \le \frac{2\zeta(p)}{(2\pi)^p} \int_N^\infty |f^{(p)}(x)|\,\mathrm{d}x$.

In our case, we take $f(x) = x^2 \log x$, $n_0 = 1$, and $p = 4$. We have $\int f(x)\,\mathrm{d}x = \frac{x^3 \log x}{3} - \frac{x^3}{9} + \mathcal{C}$, $f'(x) = 2x \log x + x$, $f''(x) = 2 \log x + 3$, $f'''(x) = 2x^{-1}$, and $f^{\prime v}(x) = -2x^{-2}$. Then $x_N = \sum_{n=2}^{N} n^2 \log n - \frac{N^3 \log N}{3} + \frac{N^3}{9} - \frac{1}{9} - \frac{N^2 \log N}{2} - \frac{N \log N}{6} - \frac{N}{12} + \frac{1}{360N}$ and $\frac{2\zeta(4)}{(2\pi)^4} \int_N^\infty |f^{\prime v}(x)|\,\mathrm{d}x = \frac{4\zeta(4)}{(2\pi)^4 N}$. Then we get the estimate from the author's solution, with $C = \ell + \frac{1}{9}$.

To determine the constant ℓ, and so C, we extend the problem by introducing the function $f(x, z) = x^{-z} \log x$, where $x \in [1, \infty)$ and the parameter z belongs to the domain $D = \{z \in \mathbb{C} : \Re z \ge -5/2,\ z \ne 1\}$. The resulting sequence will be denoted by $x_n(z)$, the remainder by $R(m, n, z)$, and the limit by $\ell(z)$. When we take $z = -2$ we get the original problem, i.e., $f(x, -2) = f(x)$, $x_n(-2) = x_n$, and $\ell(-2) = \ell$. We have $\int f(x, z)\,\mathrm{d}x = \frac{x^{1-z} \log x}{1-z} - \frac{x^{1-z}}{(1-z)^2} + \mathcal{C}$ and, by induction, $f^{(k)}(x, z) = P_k(z)x^{-k-z} \log x + Q_k(z)x^{-k-z}$ for some polynomials $P_k, Q_k \in \mathbb{R}[X]$. (Here the derivatives are with respect to x.) Then

$$\begin{aligned} x_N(z) = \sum_{n=2}^{N} n^{-z} \log n - \frac{N^{1-z} \log N}{1-z} + \frac{N^{1-z}}{(1-z)^2} - \frac{1}{(1-z)^2} - \frac{N^{-z} \log N}{2} \\ - \frac{1}{6}\left(P_1(z)N^{-1-z} \log N + Q_1(z)N^{-1-z}\right) \\ + \frac{1}{720}\left(P_3(z)N^{-3-z} \log N + Q_3(z)N^{-3-z}\right) \end{aligned}$$

and $R_4(m,n,z) \le \frac{2\zeta(4)}{(2\pi)^4}\int_m^n |P_4(z)x^{-4-z}\log x + Q_4(z)x^{-4-z}|\,\mathrm{d}x$. For every $z \in D$ we have $\Re z > -5/2$, so $-4-z < -3/2$. Hence $|x^{-4-z}| \le x^{-3/2}$ for every $x \ge 1$. For every $R > 0$ let $D_R = \{z \in D : |z| < R\}$. Let $M_R = \frac{2\zeta(4)}{(2\pi)^4}\max\{|P_4(z)|, |Q_4(z)| : |z| \le R\}$. Then for every $z \in D_R$ we have

$$\begin{aligned}|x_n(z) - x_m(z)| &= |R_4(m,n,z)| \\ &\le \int_m^n M_R(x^{-3/2}\log x + x^{-3/2})\,\mathrm{d}x \\ &< \int_m^\infty M_R(x^{-3/2}\log x + x^{-3/2})\,\mathrm{d}x \\ &= M_R(2m^{-1/2}\log m + 6m^{-1/2}).\end{aligned}$$

It follows that the sequence $x_N(z)$ converges uniformly on D_R to a function $\ell(z)$. Since each $x_N(z)$ is holomorphic, so is $\ell(z)$. Since $D = \bigcup_{R>0} D_R$, this result holds on D.

If $\Re z > 1$, then the exponents $1-z$, $-z$, $-1-z$, and $-3-z$ of N from the formula for $x_N(z)$ have negative real parts, so we have

$$\ell(z) = \lim_{N\to\infty} x_N(z) = \lim_{N\to\infty}\sum_{n=2}^{N} n^{-z}\log n - \frac{1}{(1-z)^2} = -\zeta'(z) - \frac{1}{(1-z)^2}.$$

(We differentiate $\zeta(z) = 1 + \sum_{n\ge 2} n^{-z}$ and we get $\zeta'(z) = -\sum_{n\ge 2} n^{-z}\log z$.)

Since $\ell(z)$ is holomorphic and D is a connected domain, the formula $\ell(z) = -\zeta'(z) - \frac{1}{(1-z)^2}$ holds on D. We take $z = -2$ and we get $\ell = \ell(-2) = -\zeta'(-2) - 1/9$, which implies that $C = \ell + 1/9 = -\zeta'(-2)$, as claimed.

Finally, the relation $-\zeta'(-2) = \frac{\zeta(3)}{4\pi^2}$ is a particular case of the well known formula $\zeta'(-2n) = (-1)^n\frac{(2n)!}{2(2\pi)^{2n}}\zeta(2n+1)$ $\forall n \ge 1$, which follows from the functional relation for the zeta function, written in the form $\zeta(s) = \sin\left(\frac{\pi s}{2}\right) f(s)$, where $f(s) = 2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)$. We then have $\zeta'(s) = \frac{\pi}{2}\cos\left(\frac{\pi s}{2}\right) f(s) + \sin\left(\frac{\pi s}{2}\right) f'(s)$. If $s = -2n$, then $\sin\left(\frac{\pi s}{2}\right) = 0$ and $\cos\left(\frac{\pi s}{2}\right) = (-1)^n$. It follows that

$$\zeta'(-2n) = (-1)^n\frac{\pi}{2}f(-2n) = (-1)^n\frac{\pi}{2}\cdot 2^{-2n}\pi^{-2n-1}\Gamma(2n+1)\zeta(2n+1).$$

Since $\Gamma(2n+1) = (2n)!$, we get the claimed formula.

Solution by Vlad-Ioan Țîr, physics student, University of Oxford, UK. We denote the left-hand side of the equation by L and take its natural logarithm. We have

$$\ln L = \sum_{n=2}^{\infty}\left(n^2 + \frac{1}{2} + n^4\ln\left(1 - \frac{1}{n^2}\right)\right).$$

Using the Taylor series $\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$, we can rewrite $\ln L$:

$$\ln L = \sum_{n=2}^{\infty} \left(n^2 + \frac{1}{2} - \sum_{k=1}^{\infty} \frac{n^{4-2k}}{k} \right).$$

We immediately see that the $k = 1$ and $k = 2$ terms in the expansion cancel the n^2 and $\frac{1}{2}$ terms, respectively. Thus we are left with (replacing the running index k by $k+2$)

$$\ln L = -\sum_{n=2}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(k+2)n^{2k}}.$$

After switching the order of summation, we get

$$\ln L = -\sum_{k=1}^{\infty} \frac{1}{k+2} \sum_{n=2}^{\infty} \frac{1}{n^{2k}} = -\sum_{k=1}^{\infty} \frac{\zeta(2k)-1}{k+2}$$

To simplify the expression above, we will make use of some results known in literature. Firstly, by [1] we have that

$$\ln \Gamma(2+t) = (1-\gamma)t + \sum_{k=2}^{\infty} (-1)^k \frac{(\zeta(k)-1)t^k}{k},$$

where γ is the Euler–Mascheroni constant. Plugging in $-t$ in the expression above also gives

$$\ln \Gamma(2-t) = -(1-\gamma)t + \sum_{k=2}^{\infty} \frac{(\zeta(k)-1)t^k}{k}.$$

Summing up the two expressions above cancels the odd k terms and leads to

$$\ln \Gamma(2+t) + \ln \Gamma(2-t) = \sum_{k=1}^{\infty} \frac{(\zeta(2k)-1)t^{2k}}{k}.$$

Differentiating with respect to t gives

$$\psi(2+t) - \psi(2-t) = 2 \sum_{k=1}^{\infty} (\zeta(2k)-1)t^{2k-1},$$

where $\psi(x)$ is the digamma function, defined as $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$.

Multiplying the expression above by t^4 and integrating with respect to t from 0 to 1 gives

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)-1}{k+2} = \int_0^1 (t^4\psi(2+t) - t^4\psi(2-t))\,\mathrm{d}t = -\ln L. \tag{1}$$

Reference [2] gives us an analytic expression for indefinite integrals of the type above:

$$\int t^n \psi(a+bt)\,\mathrm{d}t = n! \sum_{k=0}^{n} \frac{(-1)^k}{b^{k+1}(n-k)!} t^{n-k} \psi^{(-k-1)}(a+bt),$$

where $\psi^{(k)}(x)$ is the polygamma function. (It satisfies $\psi^{(k)}(z) = \frac{d}{dz}\psi^{(k-1)}(z)$ and $\psi^{(0)}(z) = \psi(z)$.)

Relevant to us are the cases $n = 4$, $a = 2$, and $b = \pm 1$ which give:

$$\begin{aligned}\int t^4 \psi(2+t)\,\mathrm{d}t = {} & t^4 \psi^{(-1)}(2+t) - 4t^3 \psi^{(-2)}(2+t) + 12t^2 \psi^{(-3)}(2+t) \\ & - 24t\psi^{(-4)}(2+t) + 24\psi^{(-5)}(2+t),\end{aligned}$$

$$\begin{aligned}\int t^4 \psi(2-t)\,\mathrm{d}t = {} & -t^4 \psi^{(-1)}(2-t) - 4t^3 \psi^{(-2)}(2-t) - 12t^2 \psi^{(-3)}(2-t) \\ & - 24t\psi^{(-4)}(2-t) - 24\psi^{(-5)}(2-t).\end{aligned}$$

Therefore the definite integrals evaluate to

$$\begin{aligned}\int_0^1 t^4 \psi(2+t)\,\mathrm{d}t = {} & \psi^{(-1)}(3) - 4\psi^{(-2)}(3) + 12\psi^{(-3)}(3) - 24\psi^{(-4)}(3) \\ & + 24\psi^{(-5)}(3) - 24\psi^{(-5)}(2),\end{aligned}$$

$$\begin{aligned}\int_0^1 t^4 \psi(2-t)\,\mathrm{d}t = {} & -\psi^{(-1)}(1) - 4\psi^{(-2)}(1) - 12\psi^{(-3)}(1) - 24\psi^{(-4)}(1) \\ & - 24\psi^{(-5)}(1) + 24\psi^{(-5)}(2).\end{aligned}$$

Substituting in (1) gives

$$\begin{aligned}-\ln L = {} & \psi^{(-1)}(3) - 4\psi^{(-2)}(3) + 12\psi^{(-3)}(3) - 24\psi^{(-4)}(3) + 24\psi^{(-5)}(3) \\ & + \psi^{(-1)}(1) + 4\psi^{(-2)}(1) + 12\psi^{(-3)}(1) + 24\psi^{(-4)}(1) + 24\psi^{(-5)}(1) \\ & - 48\psi^{(-5)}(2).\end{aligned}$$

The relevant values of the polygamma function are presented in Table 1. Putting everything together we obtain

$$\ln L = \ln \pi - \frac{7}{4} + \frac{3\zeta(3)}{\pi^2}.$$

After exponentiating, we get

$$L = \pi e^{-\frac{7}{4} + \frac{3\zeta(3)}{\pi^2}}.$$

TABLE 1. $\psi^{(k)}(t)$, A is the Glaisher–Kinkelin constant

$\psi^{(-1)}(1)$	0
$\psi^{(-2)}(1)$	$\frac{1}{2}(\ln(2)+\ln(\pi))$
$\psi^{(-3)}(1)$	$\ln(A)+\frac{1}{4}(\ln(2)+\ln(\pi))$
$\psi^{(-4)}(1)$	$\frac{\ln(A)}{2}+\frac{\zeta(3)}{8\pi^2}+\frac{1}{12}(\ln(2)+\ln(\pi))$
$\psi^{(-5)}(1)$	$\frac{\ln(A)}{6}-\frac{1}{6}\zeta'(-3)+\frac{\zeta(3)}{16\pi^2}-\frac{11}{4320}+\frac{1}{48}(\ln(2)+\ln(\pi))$
$\psi^{(-5)}(2)$	$\frac{4\ln(A)}{3}-\frac{1}{3}\zeta'(-3)+\frac{\zeta(3)}{4\pi^2}-\frac{397}{4320}+\frac{1}{3}(\ln(\pi)+\ln(2))$
$\psi^{(-1)}(3)$	$\ln(2)$
$\psi^{(-2)}(3)$	$-3+\frac{7\ln(2)}{2}+\frac{3\ln(\pi)}{2}$
$\psi^{(-3)}(3)$	$3\ln(A)-\frac{15}{4}+\frac{17\ln(2)}{4}+\frac{9\ln(\pi)}{4}$
$\psi^{(-4)}(3)$	$\frac{9\ln(A)}{2}+\frac{3\zeta(3)}{8\pi^2}-\frac{11}{4}+\frac{43\ln(2)}{12}+\frac{9\ln(\pi)}{4}$
$\psi^{(-5)}(3)$	$\frac{9\ln(A)}{2}-\frac{1}{2}\zeta'(-3)+\frac{9\zeta(3)}{16\pi^2}-\frac{89}{60}+\frac{113\ln(2)}{48}+\frac{27\ln(\pi)}{16}$

REFERENCES

[1] M. Abramowitz, I. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* National Bureau of Standards, 1964.

[2] O. Espinosa, V. Moll, On some integrals involving the Hurwitz zeta function. II, *The Ramanujan Journal* **6** (2002), no. 4, 449–468.

Solution by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA. In his solution Brian Bradie denotes the infinite product by P and, same as in the solution by V.I. Țîr, he proves that $\ln P = -\sum_{j=1}^{\infty}\frac{\zeta(2j)-1}{j+2}$. Then

$$\ln P = -\int_0^1 \sum_{j=1}^{\infty}(\zeta(2j)-1)x^{j+1}\,\mathrm{d}x.$$

From here the proof goes as follows. From the generating function

$$\sum_{j=1}^{\infty}\zeta(2j)x^j = \frac{1}{2}-\frac{\pi\sqrt{x}}{2}\cot(\pi\sqrt{x})$$

we get

$$\sum_{j=1}^{\infty}\left[\zeta(2j)-1\right]x^{j+1} = 1+\frac{3x}{2}-\frac{\pi x\sqrt{x}}{2}\cot(\pi\sqrt{x})-\frac{1}{1-x}.$$

It follows that

$$\begin{aligned}\ln P &= -\frac{7}{4} + \int_0^1 \left(\frac{\pi x \sqrt{x}}{2} \cot(\pi \sqrt{x}) + \frac{1}{1-x} \right) \mathrm{d}x \\ &= -\frac{7}{4} + \int_0^\pi \left(\frac{u^4}{\pi^4} \cot u + \frac{2u}{\pi^2 - u^2} \right) \mathrm{d}u \\ &= -\frac{7}{4} + \left[\frac{u^4}{\pi^4} \ln(\sin u) - \ln(\pi^2 - u^2) \right] \Bigg|_0^\pi - \frac{4}{\pi^4} \int_0^\pi u^3 \ln(\sin u)\, \mathrm{d}u.\end{aligned}$$

Now,

$$\lim_{u \nearrow \pi} \left[\frac{u^4}{\pi^4} \ln(\sin u) - \ln(\pi^2 - u^2) \right] = -\ln(2\pi)$$

and

$$\lim_{u \searrow 0} \left[\frac{u^4}{\pi^4} \ln(\sin u) - \ln(\pi^2 - u^2) \right] = -2\ln \pi.$$

Using the Fourier series

$$\ln(\sin u) = -\ln 2 - \sum_{k=1}^\infty \frac{\cos 2ku}{k},$$

we find

$$\begin{aligned}\int_0^\pi u^3 \ln(\sin u)\, \mathrm{d}u &= -\frac{\pi^4}{4} \ln 2 - \sum_{k=1}^\infty \frac{1}{k} \int_0^\pi u^3 \cos 2ku \, \mathrm{d}u \\ &= -\frac{\pi^4}{4} \ln 2 - \frac{3\pi^2}{4} \sum_{k=1}^\infty \frac{1}{k^3} \text{ (after integration by parts)} \\ &= -\frac{\pi^4}{4} \ln 2 - \frac{3\pi^2 \zeta(3)}{4}.\end{aligned}$$

Finally,

$$\ln P = -\frac{7}{4} - \ln 2 + \ln \pi - \frac{4}{\pi^4} \left(-\frac{\pi^4}{4} \ln 2 - \frac{3\pi^2\zeta(3)}{4} \right) = -\frac{7}{4} + \ln \pi + \frac{3\zeta(3)}{\pi^2},$$

which yields

$$\prod_{n=2}^\infty e^{n^2+1/2} \left(1 - \frac{1}{n^2} \right)^{n^4} = \pi \exp\left(-\frac{7}{4} + \frac{3\zeta(3)}{\pi^2} \right)$$

upon exponentiation.

Remarks. In his proof, Brian Bradie leaves to the reader two limits that are not very obvious. First we have

$$\lim_{u\nearrow\pi}\left[\frac{u^4}{\pi^4}\ln(\sin u)-\ln(\pi^2-u^2)\right]=\lim_{u\nearrow\pi}\left(\frac{u^4}{\pi^4}-1\right)\ln(\sin u)$$
$$+\lim_{u\nearrow\pi}\left(\ln(\sin u)-\ln(\pi^2-u^2)\right).$$

For the first limit in the right-hand side note that $\frac{u^4}{\pi^4}-1=O(\pi-u)$ and $\ln(\sin u)=\ln(\sin(\pi-u))\cong\ln(\pi-u)$ when u is close to π. Thus $\left(\frac{u^4}{\pi^4}-1\right)\ln(\sin u)=O((\pi-u)\ln(\pi-u))=o(1)$ and so the first limit is zero. The second limit writes as

$$\lim_{u\nearrow\pi}\ln\frac{\sin u}{\pi^2-u^2}=\lim_{u\nearrow\pi}\ln\left(\frac{\sin(\pi-u)}{\pi-u}\cdot\frac{1}{\pi+u}\right)=\ln\left(1\cdot\frac{1}{2\pi}\right)=-\ln(2\pi).$$

Also as u approaches 0 we have $\frac{u^4}{\pi^4}\ln(\sin u)\cong\frac{u^4}{\pi^4}\ln u=o(1)$, so

$$\lim_{u\searrow 0}\left[\frac{u^4}{\pi^4}\ln(\sin u)-\ln(\pi^2-u^2)\right]=-\ln\pi^2=-2\ln\pi.$$

The generating function for $\zeta(2j)$ follows from the Euler's formula for the cotangent $\pi\cot\pi x=\frac{1}{x}-\sum_{n=1}^{\infty}\frac{2x}{n^2-x^2}$, obtained by taking the logarithmic derivative of the sine function written in the Euler's product form $\sin\pi x=\pi x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right)$. But $\frac{2x}{n^2-x^2}=\frac{2}{x}\cdot\frac{x^2}{n^2}\cdot\frac{1}{1-x^2/n^2}=\frac{2}{x}\sum_{j\ge 1}\frac{x^{2j}}{n^{2j}}$. It follows that

$$\pi\cot\pi x=\frac{1}{x}-\frac{2}{x}\sum_{n,j\ge 1}\frac{x^{2j}}{n^{2j}}=\frac{1}{x}-\frac{2}{x}\sum_{j\ge 1}\zeta(2j)x^{2j}$$

and so $\sum_{j\ge 1}\zeta(2j)x^{2j}=\frac{1}{2}-\frac{\pi x}{2}\cot\pi x$. When we replace x by $\sqrt{x}$, we get the claimed formula.

And the Fourier series for $\ln(\sin x)$ follows from the fact that the relation $\ln(1-z)=-\sum_{k\ge 1}\frac{z^k}{k}$ can be extended to $\{z\in\mathbb{C}\,:\,|z|\le 1,\ z\ne 1\}$. We take $z=\exp(2ui)$, where u is not a multiple of π, and we get

$$\ln(1-\exp(2ui))=-\sum_{k\ge 1}\frac{\exp(2kui)}{k}.$$

When we take the real parts, the right side becomes $-\sum_{k\ge 1}\frac{\cos(2ku)}{k}$, while the left side becomes $\ln|1-\exp(2ui)|=\ln|-\exp(ui)(\exp(ui)-\exp(-ui))|=\ln|-2i\exp(ui)\sin u|=\ln|2\sin u|=\ln 2+\ln|\sin u|$. Hence $\ln 2+\ln|\sin u|=-\sum_{k\ge 1}\frac{\cos(2ku)}{k}$, which concludes the proof.

530. Let $a,b,c,d,e\in\mathbb{R}$ be such that $a\ge b\ge c\ge d\ge e\ge 0$ and

$$ab+bc+cd+de+ea=5.$$

Prove that

$$(3-a)^2+(3-b)^2+(3-c)^2+(3-d)^2+(3-e)^2 \geq 20.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania.

Solution by the author. We prove the following more general result.

If $a_1, a_2, \ldots, a_n \quad (n \geq 3)$ *are real numbers such that* $a_1 \geq a_2 \geq \cdots \geq a_n$ *and*

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 = n,$$

then

$$(3-a_1)^2 + (3-a_2)^2 + \cdots + (3-a_n)^2 \geq 4n.$$

Proof. Let

$$a = \frac{a_2 + a_3 + \cdots + a_{n-1}}{n-2}.$$

By Jensen's inequality applied to the convex function $f(x) = (3-x)^2$, we have

$$(3-a_2)^2 + (3-a_3)^2 + \cdots + (3-a_{n-1})^2 \geq (n-2)(3-a)^2.$$

Therefore it suffices to show that

$$(3-a_1)^2 + (3-a_n)^2 + (n-2)(3-a)^2 \geq 4n.$$

Using the substitutions

$$A = \frac{a+a_1}{2}, \qquad B = \frac{a+a_n}{2},$$

the inequality writes successively as follows:

$$(3+a-2A)^2 + (3+a-2B)^2 + (n-2)(3-a)^2 \geq 4n,$$

$$4(A^2+B^2) - 4(3+a)(A+B) + 2(3+a)^2 + (n-2)(3-a)^2 - 4n \geq 0,$$

$$4(A+B)^2 - 4(3+a)(A+B) + 2(3+a)^2 + (n-2)(3-a)^2 - 4n \geq 8AB,$$

$$(2A+2B-3-a)^2 + (3+a)^2 + (n-2)(3-a)^2 - 4n \geq 8AB.$$

It is therefore enough to prove that

$$(3+a)^2 + (n-2)(3-a)^2 - 4n \geq 8AB.$$

By Lemma below, we have

$$4AB + (n-4)a^2 \leq n.$$

So, it suffices to show that

$$(3+a)^2 + (n-2)(3-a)^2 - 4n \geq 2n - 2(n-4)a^2,$$

which is equivalent to the obviously true inequality

$$3(n-3)(a-1)^2 \geq 0.$$

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

Lemma. *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are real numbers such that $a_1 \ge a_2 \ge \cdots \ge a_n$, then*

$$(a + a_1)(a + a_n) + (n-4)a^2 \le a_1a_2 + a_2a_3 + \cdots + a_na_1,$$

where

$$a = \frac{a_2 + a_3 + \cdots + a_{n-1}}{n-2}.$$

Proof. For $n = 3$, the inequality is an identity. For $n > 3$, since the sequences $(a_2, \ldots, a_{n-2})$ and $(a_3, \ldots, a_{n-1})$ are decreasing, by Chebyshev's inequality, we have

$$(n-3)(a_2a_3 + \cdots + a_{n-2}a_{n-1}) \ge (a_2 + \cdots + a_{n-2})(a_3 + \cdots + a_{n-1}).$$

Thus, the desired inequality is true if

$$(a_1+a_n)a+(n-3)a^2 \le a_1a_2+a_{n-1}a_n+\frac{1}{n-3}(a_2+\cdots+a_{n-2})(a_3+\cdots+a_{n-1}),$$

which is equivalent to

$$a_1(a-a_2)+a_n(a-a_{n-1})+(n-3)a^2 \le \frac{1}{n-3}[(n-2)a-a_{n-1}][(n-2)a-a_2].$$

Since $a - a_2 \le 0$ and $a_1 \ge a_2$ we have $a_1(a-a_2) \le a_2(a-a_2)$. Since $a - a_{n-1} \ge 0$ and $a_n \le a_{n-1}$, we have $a_n(a-a_{n-1}) \le a_{n-1}(a-a_{n-1})$. Hence it suffices to show that

$$a_2(a-a_2)+a_{n-1}(a-a_{n-1})+(n-3)a^2 \le \frac{1}{n-3}[(n-2)a-a_{n-1}][(n-2)a-a_2],$$

which, after multiplication by $n-3$, becomes

$$(2n-5)a^2 - (2n-5)(a_2+a_{n-1})a + (n-3)(a_2^2 + a_{n-1}^2) + a_2a_{n-1} \ge 0,$$

$$(2n-5)\left(a - \frac{a_2+a_{n-1}}{2}\right)^2 + \frac{2n-7}{4}(a_2-a_{n-1})^2 \ge 0.$$

Remark 1. The inequality

$$(k-a_1)^2 + (k-a_2)^2 + \cdots + (k-a_n)^2 \ge n(k-1)^2$$

does not hold for $k > 3$. Indeed, by choosing $a_2 = \cdots = a_{n-1} = 1$, the constraint $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 = n$ becomes $a_1a_n + a_1 + a_n = 3$, while the inequality can be written as follows:

$$(k-a_1)^2 + (k-a_n)^2 \ge 2(k-1)^2, \quad \text{i.e.,} \quad a_1^2 + a_n^2 - 2k(a_1+a_n) + 4k - 2 \ge 0.$$

To complete the square, we add $2(a_1a_n + a_1 + a_n - 3) = 0$ and the inequality becomes

$$(a_1+a_n)^2+(2-2k)(a_1+a_n)+4k-8 \ge 0, \quad \text{i.e.,} \quad (a_1+a_n-2)(a_1+a_n+4-2k) \ge 0.$$

We have $a_1 \geq 1 \geq a_n$. Then $a_1 = 1 + t$, with $t \geq 0$, and the constraint $a_1 a_n + a_1 + a_n = 3$ becomes $(2+t)a_n + 1 + t = 3$, i.e., $a_n = \frac{2-t}{2+t} = 1 - \frac{2t}{2+t}$, and so $a_1 + a_n = 2 + \frac{t^2}{2+t}$. We take $t > 0$ small. Then the required condition $a_1 \geq 1 \geq a_n$ holds and, moreover, $a_1 + a_n > 2$ and $a_1 + a_n$ is close to 2. Then $(a_1 + a_n - 2)(a_1 + a_n + 4 - 2k) \geq 0$ is equivalent to $a_1 + a_n \geq 2k - 4$. But when we take $t \to 0$ we get $a_1 + a_n \to 2$. So we get the necessary condition $2 \geq 2k - 4$, that is $k \leq 3$.

Remark 2. Using the substitutions $a_i = x_i + 1$ for $i = 1, 2, \ldots, n$, the constraints become

$$x_1 \geq x_2 \geq \cdots \geq x_n, \qquad x_1x_2 + x_2x_3 + \cdots + x_nx_1 + 2(x_1 + x_2 + \cdots + x_n) = 0$$

and the desired inequality becomes

$$x_1^2 + x_2^2 + \cdots + x_n^2 - 4(x_1 + x_2 + \cdots + x_n) \geq 0,$$

i.e.,

$$x_1^2 + x_2^2 + \cdots + x_n^2 + 2(x_1x_2 + \cdots + x_{n-1}x_n + x_nx_1) \geq 0.$$

By homogenizing, one proves that $x_1^2 + x_2^2 + \cdots + x_n^2 + 2(x_1x_2 + \cdots + x_{n-1}x_n + x_nx_1) > 0$ if $x_1 + \cdots + x_n \neq 0$. (We cannot have equality because this would imply $x_1 = \cdots = x_n = 0$, which contradicts $x_1 + \cdots + x_n \neq 0$.) Then, by continuity, one gets the following result.

If $x_1, x_2, \ldots, x_n$ $(n \geq 3)$ are real numbers such that $x_1 \geq x_2 \geq \cdots \geq x_n$ and $x_1x_2 + x_2x_3 + \cdots + x_nx_1 + 2(x_1 + x_2 + \cdots + x_n) = 0$, then

$$x_1^2 + x_2^2 + \cdots + x_n^2 + 2(x_1x_2 + \cdots + x_nx_1) \geq 0,$$

with possible equality only if $x_1 + \cdots + x_n = 0$.

A direct approach gives a more precise result. We denote $x = (x_2 + \cdots + x_{n-1})/(n-2)$. By the Jensen's inequality applied to the map $u \mapsto u^2$, we get $x_2^2 + \cdots + x_{n-1}^2 \geq (n-2)x^2$, with equality if and only if $x_2 = \cdots = x_{n-1} = x$, and, by the Lemma, $x_1x_2 + \cdots + x_nx_1 \geq (x + x_1)(x + x_n) + (n-4)x^2$. Putting $x_{n+1} = x_1$, we conclude that

$$\begin{aligned}\sum_{j=1}^{n}(x_j^2 + 2x_jx_{j+1}) &\geq x_1^2 + x_n^2 + (n-2)x^2 + 2(x + x_1)(x + x_n) + (2n-8)x^2 \\ &= (x_1 + x_n)^2 + 2x(x_1 + x_n) + (3n-8)x^2 \\ &= (x_1 + x_n + x)^2 + (3n-9)x^2 \geq 0.\end{aligned}$$

For the equality to hold, if $n = 3$ we must have $x_1 + x_3 + x = 0$, i.e., $x_1 + x_2 + x_3 = 0$. If $n \geq 4$ we must have $x = 0$, so $x_2 = \cdots = x_{n-1} = 0$, and $x_1 + x_n + x = 0$, i.e., $x_1 + x_n = 0$. One checks that these conditions are also sufficient, so the equality holds if and only if either $n = 3$ and $x_1 + x_2 + x_3 = 0$ or $n \geq 4$, $x_2 = \cdots = x_{n-1} = 0$, and $x_1 + x_n = 0$. $\square$

We also received solutions from Leonard Giugiuc and Marian Cucoaneș. In both these solutions they prove a slightly stronger result, where negative numbers are allowed, i.e., where the condition $a \geq b \geq c \geq d \geq e \geq 0$ from the hypothesis is replaced by $a \geq b \geq c \geq d \geq e$.

531. If $n \geq 2$ is an integer, it is known that if $k \nmid n \ \forall k \leq \sqrt{n}$, then n is a prime.

Let $\theta \in [1/3, 1/2]$. Determine the probability for a number $n \geq 2$ with $k \nmid n \ \forall k \leq n^\theta$ to be a prime. That is, determine the limit

$$\lim_{x\to\infty} \frac{\#\{p \leq x \,:\, p \text{ prime}\}}{\#\{n \leq x \,:\, k \nmid n \forall k \leq n^\theta\}}.$$

`Constantin-Nicolae Beli, IMAR, București, România.`

Solution by the author. Let $P(x) = \{p \leq x : p \text{ prime}\}$ and let $A_\theta(x) = \{n \leq x : k \nmid n \forall k \leq n^\theta\}$. Note that if $\theta = 1/2$, then $A_\theta(x) = B(x)$, so our limit is 1. Therefore we will assume that $\theta > 1/2$. By the prime number theorem, $\#P(x) =: \pi(x) \sim x/\log x$. Hence we must determine $\lim_{x\to\infty} \frac{x/\log x}{\#A_\theta(x)}$.

If $n \in A_\theta(x)$ factors into primes as $n = p_1 \cdots p_s$, then $p_i > n^\theta \ \forall i$ so $n > n^{\theta s}$. Thus $\theta s < 1$ and so $s < 1/\theta \leq 3$. Hence $A_\theta(x) = P(x) \cup B_\theta(x)$, where $B_\theta(x)$ is the set of all $n \in A_\theta(x)$ that write as $n = pq$, where $p \leq q$ are primes.

If $p \leq q$ are primes, then $n = pq$ belongs to $B_\theta(x)$ iff $p > (pq)^\theta$, i.e. $q < p^{1/\theta-1}$, and $pq \leq x$, i.e. $q \leq x/p$. Hence q must belong to both $[p, p^{1/\theta-1})$ and $[p, x/p]$. The first interval is always non-empty, since $\theta < 1/2$, so $1/\theta - 1 > 1$ and $p < p^{1/\theta-1}$. The second interval is nonempty iff $p \leq x/p$, i.e. iff $p \leq \sqrt{x}$. It follows that

$$\#B_\theta(x) = \sum_{p\leq\sqrt{x}} \#\{q \in [p, p^{1/\theta-1}) \cap [p, x/p]\},$$

where p and q run through prime numbers. The interval $[p, p^{1/\theta-1}) \cap [p, x/p]$ is equal to $[p, p^{1/\theta-1})$ or $[p, x/p]$ corresponding to $p^{1/\theta-1} \leq x/p$ or $p^{1/\theta-1} > x/p$, respectively. But $p^{1/\theta-1} > x/p$ is equivalent to $p > x^\theta$. Hence $\#B_\theta(x) = S_1 + S_2$, where

$$S_1 = \sum_{p\leq x^\theta} \#\{q \in [p, p^{1/\theta-1})\} \qquad \text{and} \qquad S_2 = \sum_{x^\theta<p\leq\sqrt{x}} \#\{q \in [p, x/p]\}.$$

We have $\#\{p \leq x^\theta\} = \pi(x^\theta)$, so S_1 is a sum of $\pi(x^\theta)$ terms. For each term we have $p \leq x^\theta$ so $p^{1/\theta-1} \leq x^{1-\theta}$. Hence $\#\{q \in [p, p^{1/\theta-1})\} \leq \{q \leq x^{1-\theta}\} = \pi(x^{1-\theta})$. It follows that

$$S_1 \le \pi(x^\theta)\pi(x^{1-\theta}) \sim \frac{x^\theta}{\log x^\theta} \cdot \frac{x^{1-\theta}}{\log x^{1-\theta}} = \frac{x}{\theta(1-\theta)\log^2 x}.$$

Thus $S_1 = O(x/\log^2 x)$.

For S_2 we note that there are $\pi(x/p)$ primes $\le x/p$ and there are $\pi(p)-1$ primes $< p$. Hence $\#\{q \in [p, x/p]\} = \pi(x/p) - (\pi(p)-1)$ and we have $S_2 = S_3 - S_4$ where

$$S_3 = \sum_{x^\theta < p \le \sqrt{x}} \pi(x/p) \qquad \text{and} \qquad S_4 = \sum_{x^\theta < p \le \sqrt{x}} (\pi(p)-1).$$

We have $\#\{x^\theta < p \le \sqrt{x}\} \le \#\{p \le \sqrt{x}\} = \pi(\sqrt{x})$, so S_4 is the sum of $\le \pi(\sqrt{x})$ terms. For each term we have $p \le \sqrt{x}$ so $\pi(p) - 1 < \pi(\sqrt{x})$. Hence

$$S_4 < \pi(\sqrt{x})^2 \sim \left(\frac{\sqrt{x}}{\log\sqrt{x}}\right)^2 = \frac{4x}{\log^2 x}.$$

Hence $S_4 = O(x/\log^2 x)$.

For S_3 we note that for $x^\theta < p \le \sqrt{x}$ we have $x/p \ge \sqrt{x} \gg 1$ so $\pi(x/p) \sim f(p)$, where $f(t) = \frac{x/t}{\log(x/t)} = \frac{x}{t(\log x - \log t)}$. It follows that $S_3 \sim S_3'$, where $S_3' = \sum_{x^\theta < p \le \sqrt{x}} f(p)$. To estimate S_3' we employ Riemann-Stieltjes integrals, a technique widely used in the analytic number theory. We note that $\pi(x)$ is a step function with jump discontinuities of 1 at prime numbers. More precisely, we have $\pi(p^-) = \pi(p) - 1$ and $\pi(p^+) = \pi(p)$ for every prime p. Since also f is continuous, we have $S_3' = \int_{x^\theta}^{\sqrt{x}} f(t)\mathrm{d}\pi(t)$. We integrate by parts and we get

$$S_3' = f(t)\pi(t)\Big|_{x^\theta}^{\sqrt{x}} - \int_{x^\theta}^{\sqrt{x}} \pi(t)\mathrm{d}f(t) = f(t)\pi(t)\Big|_{x^\theta}^{\sqrt{x}} - \int_{x^\theta}^{\sqrt{x}} \pi(t)f'(t)\mathrm{d}t.$$

For $0 < \alpha < 1$ we have $f(x^\alpha) = \frac{x/x^\alpha}{\log(x/x^\alpha)} = \frac{x^{1-\alpha}}{(1-\alpha)\log x}$ and $\pi(x^\alpha) \sim \frac{x^\alpha}{\log x^\alpha} = \frac{x^\alpha}{\alpha\log x}$. Hence $f(x^\alpha)\pi(x^\alpha) \sim \frac{x}{\alpha(1-\alpha)\log^2 x}$. It follows that $f(t)\pi(t)\Big|_{x^\theta}^{\sqrt{x}} = O(x/\log^2 x)$.

Since $f(t) = xt^{-1}(\log x - \log t)^{-1}$, by taking differential logarithms, we get

$$f'(t)/f(t) = -\frac{1}{t} - \frac{-1/t}{\log x - \log t} \sim -\frac{1}{t} \qquad \text{when} \qquad x^\theta < t \le \sqrt{x}.$$

(We have $\log x - \log t \ge \log x - \log\sqrt{x} = \frac{1}{2}\log x \gg 1$ so $\frac{1}{t} \gg \frac{1/t}{\log x - \log t}$.)

Hence $f'(t) \sim -\frac{1}{t}f(t) = -\frac{x}{t^2\log(x/t)}$, which, together with $\pi(t) \sim \frac{t}{\log t}$, implies $\pi(t)f'(t) \sim -\frac{x}{t\log t\log(x/t)}$. Hence

$$-\int_{x^\theta}^{\sqrt{x}} \pi(t)f'(t)\mathrm{d}t \sim \int_{x^\theta}^{\sqrt{x}} \frac{x}{\log t\log(x/t)}\mathrm{d}t/t.$$

We make the substitution $t = x^u$. Then $\mathrm{d}t/t = \log x\,\mathrm{d}u$ and the last integral writes as

$$\int_\theta^{1/2} \frac{x}{\log x^u\log(x/x^u)}\log x\,\mathrm{d}u = \int_\theta^{1/2} \frac{x}{u(1-u)\log x}\mathrm{d}u = \frac{x}{\log x}\log\frac{u}{1-u}\bigg|_\theta^{1/2}$$
$$= \log(1/\theta - 1)\frac{x}{\log x}.$$

Since also $f(t)\pi(t)\bigg|_{x^\theta}^{\sqrt{x}} = O(x/\log^2 x)$, we get $S_3 \sim S_3' \sim \log(1/\theta - 1)x/\log x$. We also have that both S_1 and S_4 are $O(x/\log^2 x)$. Hence $\#B_\theta(x) = S_1 + S_2 = S_1 + S_3 - S_4 \sim \log(1/\theta - 1)x/\log x$. Together with $\#P(x) \sim x/\log x$, this implies that $\#A_\theta(x) = \#P(x) + \#B_\theta(x) \sim (1 + \log(1/\theta - 1))x/\log x$. It follows that our limit is

$$\lim_{x\to\infty} \frac{x/\log x}{\#A_\theta(x)} = \frac{1}{1 + \log(1/\theta - 1)}.$$

532. Let $(S, \cdot)$ be a semigroup with the property that for every $x \in S$ there is a unique $x' \in S$ such that $(xx')^2 = xx'$.

Prove that S is a group.

`Proposed by Gheorghe Andrei, Constanța, and Mihai Opincariu, Brad, Romania.`

Solution by the authors. Let $x \in S$ be arbitrary and let $x' \in S$ such that $(xx')^2 = xx'$. If $y = x'xx'$, then $xy = xx'xx' = (xx')^2$, so $(xy)^2 = (xx')^4 = ((xx')^2)^2 = (xx')^2 = xy$. By the unicity of x', we get $x'xx' = x'$.

Consequently, $(x'x)^2 = (x'xx')x = x'x$. Hence x'', the unique element of S with the property that $(x'x'')^2 = x'x''$, coincides with x. Then, if we replace x by x' in the formula $x'xx' = x'$, we get $x''x'x'' = x''$, i.e., $xx'x = x$.

Note that if $xyx = x$, then $(xy)^2 = (xyx)y = xy$, so $y = x'$. Hence $y = x'$ is the only element of S with the property that $xyx = x$.

Let now $x, y \in S$ be arbitrary. Put $a = x(yx)'y$. By using the formulas $x'xx' = x'$ and $xx'x = x$, we get

$$a(xx')a = x(yx)'y(xx'x)(yx)'y = x(yx)'yx(yx)'y = x(yx)'y = a$$

and

$$a(y'y)a = x(yx)'(yy'y)x(yx)'y = x(yx)'yx(yx)'y = x(yx)'y = a.$$

But the only element $b \in S$ with the property that $aba = a$ is $b = a'$. Hence we must have $xx' = y'y = a'$. Thus we have proved that $xx' = y'y$ for every $x, y \in S$. When we take x arbitrary and $y = x$, we get $xx' = x'x$. If we fix $c \in S$ and we denote $e = c'c$, then for every $x \in S$ we have $xx' = c'c = e$. Since $x'x = xx'$, we also have $x'x = e$. Then $xe = x(x'x) = xx'x = x$ and $ex = (xx')x = xx'x = x$, so e is a neutral element of S. Since $xx' = x'x = e$, each element of $x \in S$ has an inverse, namely x'. Hence S is a group.

We also received a solution from Hao Zhang, from Hunan University, P. R. China. Same as the authors, he obtains the formulas $x'xx' = x'$ and $x'' = x$. But after that he proceeds differently.

For every $x, y \in S$ we have $(xy(xy)')^2 = xy(xy)'$. By uniqueness we get

$$x' = y(xy)'.$$

We apply this property with x and y replaced by y and $(xy)'$. We get

$$y' = (xy)'(y(xy)')' = (xy)'x'' = (xy)'x.$$

By using the two displayed formulas above we get

$$x'x = y(xy)'x = yy'.$$

From here on the proof goes as in the authors' solution.

533. Prove that

$$\sum_{m=2}^{\infty} (-1)^m \left(\zeta(m) - \zeta(m+1)\right) \left(H_{\frac{m-1}{2}} - H_{\frac{m}{2}}\right) = \frac{\pi^2}{3}\left(1 - \log 2\right) - 2\gamma,$$

where $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$ is the Riemann zeta function, $H_n = \int_0^1 \frac{1-x^n}{1-x}\mathrm{d}x$, and $\gamma = \lim_{n\to\infty}\left(-\log n + \sum_{k=1}^{n} 1/k\right)$ is the Euler–Mascheroni constant.

`Proposed by Narendra Bhandari, Bajura, and Yogesh Joshi, Kailali, Nepal.`

Solution by the authors. Before proving the proposed result, we prove the following:

$$\sum_{m=2}^{\infty} \mathcal{H}_m \left(\zeta(m) - \zeta(m+1)\right) = \zeta(2) - \gamma - \log 2, \tag{1}$$

$$\mathcal{H}_m = \log 2 + \frac{(-1)^m}{2}\left(H_{\frac{m-1}{2}} - H_{\frac{m}{2}}\right), \tag{2}$$

where $\mathcal{H}_m = \sum_{k=1}^{m} (-1)^{k+1}/k$ is the mth skew-harmonic number. Using the results (1) and (2), we obtain the proposed closed from.

Proof of (1). Since the generating function of the skew-harmonic numbers is given by $\sum_{m=1}^{\infty} \mathcal{H}_m x^m = \frac{\log(1+x)}{1-x}$ for $|x| < 1$, we have:

$$\sum_{m=2}^{\infty} \mathcal{H}_m \left(\zeta(m) - \zeta(m+1)\right) = \sum_{m=2}^{\infty} \mathcal{H}_m \sum_{j=1}^{\infty} \frac{1}{j^m}\left(1 - \frac{1}{j}\right)$$
$$= \sum_{j=1}^{\infty} \left(1 - \frac{1}{j}\right) \sum_{m=2}^{\infty} \frac{\mathcal{H}_m}{j^m} = \sum_{j=2}^{\infty} \left(1 - \frac{1}{j}\right) \left(\frac{\log\left(1 + \frac{1}{j}\right)}{1 - \frac{1}{j}} - \frac{1}{j}\right)$$
$$= \sum_{j=2}^{\infty} \left(\log\left(1 + \frac{1}{j}\right) - \frac{1}{j} + \frac{1}{j^2}\right) = \sum_{j=1}^{\infty} \left(\log\left(1 + \frac{1}{j}\right) - \frac{1}{j} + \frac{1}{j^2}\right) - \log 2$$
$$= \lim_{N\to\infty} \sum_{j=1}^{N} \left(\log\left(1 + \frac{1}{j}\right) - \frac{1}{j}\right) + \sum_{j=1}^{\infty} \frac{1}{j^2} - \log 2$$
$$= \lim_{N\to\infty} \left(\log(N+1) - H_N\right) + \zeta(2) - \log 2 = \zeta(2) - \gamma - \log 2.$$

This completes the proof of result (1).

Proof of (2).

$$\mathcal{H}_m = \sum_{n=1}^{m} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - \sum_{n=m+1}^{\infty} \frac{(-1)^{n+1}}{n}$$
$$= \log 2 - \sum_{n=1}^{\infty} \frac{(-1)^{n+m+1}}{n+m} = \log 2 + (-1)^m \sum_{n=1}^{\infty} \left(\frac{1}{2n+m-1} - \frac{1}{2n+m}\right)$$
$$= \log 2 + \frac{(-1)^m}{2}\left(H_{\frac{m-1}{2}} - H_{\frac{m}{2}}\right),$$

where we used $\sum_{n=1}^{\infty} \left(\frac{1}{n+a} - \frac{1}{n+b}\right) = \psi_0(b+1) - \psi_0(a+1)$ for $a = \frac{m-1}{2}$ and $b = \frac{m}{2}$, and the formula $\psi_0(a) = H_{a-1} - \gamma$. Here $\psi_0(a)$ is the digamma function. This completes the proof of (2).

From (1) and (2), we get

$$\sum_{m=2}^{\infty} (-1)^m \left(\zeta(m) - \zeta(m+1)\right) \left(H_{\frac{m-1}{2}} - H_{\frac{m}{2}}\right)$$
$$= 2 \sum_{m=2}^{\infty} (\zeta(m) - \zeta(m+1))(\mathcal{H}_m - \log 2)$$
$$= 2(\zeta(2) - \gamma - \log 2) - 2(\zeta(2) - 1)\log 2 = \frac{\pi^2}{3}(1 - \log 2) - 2\gamma.$$

(Note that $\zeta(m)-\zeta(m+1)=\sum_{j=1}^{\infty}\left(1-\frac{1}{j}\right)\frac{1}{j^m}=\sum_{j=2}^{\infty}\left(1-\frac{1}{j}\right)\frac{1}{j^m}$, so

$$\sum_{m=2}^{\infty}(\zeta(m)-\zeta(m+1))=\sum_{m=2}^{\infty}\sum_{j=2}^{\infty}\left(1-\frac{1}{j}\right)\frac{1}{j^m}=\sum_{j=2}^{\infty}(1-1/j)\frac{\frac{1}{j^2}}{1-1/j}$$
$$=\sum_{j=2}^{\infty}\frac{1}{j^2}=\zeta(2)-1.$$

Also $\zeta(2)=\frac{\pi^2}{6}$.)

Editor's notes. (i) The formula for the generating function of the skew-harmonic numbers is not widely known. We provide here a proof. We have

$$\sum_{m=1}^{\infty}\mathcal{H}_m x^m=\sum_{m=1}^{\infty}\sum_{k=1}^{m}\frac{(-1)^{k+1}}{k}x^m=\sum_{k=1}^{\infty}\sum_{m=k}^{\infty}\frac{(-1)^{k+1}}{k}x^m$$
$$=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\cdot\frac{x^m}{1-x}=\frac{\log(1+x)}{1-x}.$$

(Here we used the fact that $\sum_{m=1}^{\infty}\sum_{k=1}^{m}=\sum_{m=1}^{\infty}\sum_{1\le k\le m}=\sum_{k=1}^{\infty}\sum_{m=k}^{\infty}$ and the well known formula for $\log(1+x)$.)

(ii) Formula (2), written as $H_{\frac{m-1}{2}}-H_{\frac{m}{2}}=2(-1)^m(\mathcal{H}_m-\log 2)$, may be proved by more elementary means, without resorting to digamma function. We have

$$H_{\frac{m-1}{2}}-H_{\frac{m}{2}}=\int_0^1\left(\frac{1-x^{(m-1)/2}}{1-x}-\frac{1-x^{m/2}}{1-x}\right)\mathrm{d}x$$
$$=\int_0^1\frac{x^{(m-1)/2}(\sqrt{x}-1)}{1-x}\mathrm{d}x$$
$$=-\int_0^1\frac{x^{(m-1)/2}}{1+\sqrt{x}}\mathrm{d}x.$$

We make the change of variables $u=\sqrt{x}$, i.e., $x=u^2$. We have $\mathrm{d}x=2u\mathrm{d}u$, so

$$H_{\frac{m-1}{2}}-H_{\frac{m}{2}}=-2\int_0^1\frac{u^m}{1+u}\mathrm{d}u=2(-1)^m\int_0^1\left(\frac{1-(-u)^m}{1+u}-\frac{1}{1+u}\right)\mathrm{d}x.$$

The integral of the first term writes as

$$\int_0^1\left(\sum_{k=0}^{m-1}(-1)^k u^k\right)\mathrm{d}u=\sum_{k=0}^{m-1}(-1)^k/(k+1)=\mathcal{H}_m$$

and the integral of the second term as $-\log(1+u)\big|_0^1=-\log 2$. Hence the claimed formula.

Solution by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. We denote the desired sum by S. Note that S writes as

$$S = \zeta(2)\left(H_{\frac{1}{2}} - H_1\right) + \sum_{m=3}^{\infty}(-1)^m\zeta(m)\left(H_{\frac{m}{2}-1} - H_{\frac{m}{2}}\right).$$

Now,

$$H_{\frac{1}{2}} = \int_0^1 \frac{1-\sqrt{x}}{1-x}\,\mathrm{d}x = \int_0^1 \frac{1}{1+\sqrt{x}}\,\mathrm{d}x = 2\int_0^1 \frac{u}{1+u}\,\mathrm{d}u = 2(1-\log 2),$$

so $H_{\frac{1}{2}} - H_1 = 2(1-\log 2) - 1 = 1 - 2\log 2$, and

$$H_{\frac{m}{2}-1} - H_{\frac{m}{2}} = \int_0^1 \frac{x^{\frac{m}{2}} - x^{\frac{m}{2}-1}}{1-x}\,\mathrm{d}x = -\int_0^1 (x^{\frac{m}{2}-1})\,\mathrm{d}x = -\frac{2}{m}.$$

Thus,

$$\begin{aligned}
S &= \zeta(2)\,(1-2\log 2) - 2\sum_{m=3}^{\infty}\frac{(-1)^m\zeta(m)}{m} \\
&= \zeta(2)(1-2\log 2) - 2\left(\sum_{m=2}^{\infty}\frac{(-1)^m\zeta(m)}{m} - \frac{\zeta(2)}{2}\right) \\
&= 2\zeta(2)(1-\log 2) - 2\sum_{m=2}^{\infty}\frac{(-1)^m\zeta(m)}{m} \\
&= \frac{\pi^2}{3}(1-\log 2) - 2\sum_{m=2}^{\infty}\frac{(-1)^m\zeta(m)}{m}.
\end{aligned}$$

For the remaining sum on the right side,

$$\begin{aligned}
\sum_{m=2}^{\infty}\frac{(-1)^m\zeta(m)}{m} &= \sum_{m=2}^{\infty}\frac{(-1)^m}{m}\sum_{k=1}^{\infty}\frac{1}{k^m} = \sum_{k=1}^{\infty}\sum_{m=2}^{\infty}\frac{(-1)^m}{mk^m} \\
&= \sum_{k=1}^{\infty}\left(\frac{1}{k} + \sum_{m=1}^{\infty}\frac{(-1)^m}{mk^m}\right) \\
&= \sum_{k=1}^{\infty}\left(\frac{1}{k} - \log\left(1+\frac{1}{k}\right)\right) \\
&= \lim_{n\to\infty}\left(-\log(n+1) + \sum_{k=1}^{n}\frac{1}{k}\right) = \gamma.
\end{aligned}$$

In conclusion, $S = \frac{\pi^2}{3}(1-\log 2) - 2\gamma$.

Editor's note. For the first displayed formula in Brian Bradie's solution one must first take the partial sum

$$\sum_{m=2}^{N}(-1)^m\left(\zeta(m)-\zeta(m+1)\right)\left(H_{\frac{m-1}{2}}-H_{\frac{m}{2}}\right)=\zeta(2)\left(H_{\frac{1}{2}}-H_1\right)$$
$$+\sum_{m=3}^{N}(-1)^m\zeta(m)\left(H_{\frac{m}{2}-1}-H_{\frac{m}{2}}\right)+(-1)^N\zeta(N+1)\left(H_{\frac{N}{2}}-H_{\frac{N-1}{2}}\right).$$

To prove the claimed formula, one must show $\zeta(N+1)\left(H_{\frac{N}{2}}-H_{\frac{N-1}{2}}\right)\to 0$ as $N\to\infty$. But $\zeta(N+1)\to 1$ as $N\to\infty$ and

$$H_{\frac{N}{2}}-H_{\frac{N-1}{2}}=\int_0^1\frac{x^{\frac{N-1}{2}}-x^{\frac{N}{2}}}{1-x}\mathrm{d}x=\int_0^1\frac{x^{\frac{N-1}{2}}}{1+\sqrt{x}}\mathrm{d}x<\int_0^1 x^{\frac{N-1}{2}}\mathrm{d}x=\frac{2}{N+1},$$

which concludes the proof.

We also received a solution by G. C. Greubel, Newport News, VA, USA. The version presented below includes details missing in the original submission. In this solution, H'_n denotes the skew harmonic numbers and one uses the same formula $(-1)^n\left(H_{\frac{n-1}{2}}-H_{\frac{n}{2}}\right)=2\left(H'_n-\ln 2\right)$ from the authors' solution. With this, the series in question, denoted by S, becomes

$$S=2\sum_{m=2}^{\infty}\left(\zeta(m)-\zeta(m+1)\right)\left(H'_m-\ln 2\right).$$

Then we note that

$$\sum_{m=2}^{\infty}\left(\zeta(m)-\zeta(m+1)\right)=\lim_{N\to\infty}\sum_{m=2}^{N}\left(\zeta(m)-\zeta(m+1)\right)$$
$$=\lim_{N\to\infty}\left(\zeta(2)-\zeta(N+1)\right)=\zeta(2)-1.$$

Also

$$\begin{aligned}\sum_{m=2}^{\infty}(\zeta(m)-\zeta(m+1))H'_m &= \lim_{N\to\infty}\sum_{m=2}^{N}(\zeta(m)-\zeta(m+1))H'_m\\ &= \lim_{N\to\infty}\Bigg(\zeta(2)H'_2-\zeta(N+1)H'_N\\ &\qquad +\sum_{m=3}^{N}\zeta(m)(H'_m-H'_{m-1})\Bigg)\\ &= \zeta(2)H'_2-\ln 2+\sum_{m=3}^{\infty}\frac{(-1)^{m-1}}{m}\zeta(m)\\ &= \zeta(2)H'_2-\ln 2+\left(-\gamma+\frac{1}{2}\zeta(2)\right)\\ &= \zeta(2)-\ln 2-\gamma.\end{aligned}$$

Consequently,

$$S=2(\zeta(2)-\ln 2-\gamma)-2\ln 2(\zeta(2)-1)=2\zeta(2)(1-\ln 2)-2\gamma=\frac{\pi^2}{3}(1-\ln 2)-2\gamma.$$

Note that this solution made use of $\zeta(N+1)\to 1$ and

$$H'_N\to\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=\log 2 \quad\text{as}\quad N\to\infty,$$

$H'_n-H'_{n-1}=\frac{(-1)^{n-1}}{n}$, $H'_2=\frac{1}{2}$, and $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n}\zeta(n)=-\gamma$.

The last sum writes as

$$\sum_{n=2}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n-1}}{n}\cdot\frac{1}{m^n}=\sum_{m=1}^{\infty}\left(\log\left(1+\frac{1}{m}\right)-\frac{1}{m}\right).$$

(We reversed the order of summation and we applied the well known formula $\log(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$ to $x=\frac{1}{m}$.)

Then we have the partial sum $\sum_{m=1}^{N}\left(\log\left(1+\frac{1}{m}\right)-\frac{1}{m}\right)=\log(N+1)-\sum_{m=1}^{N}\frac{1}{m}$, whose limit as $N\to\infty$ is $-\gamma$.

534. Let $n\ge 4$.

(a) Find the smallest positive constant k_n for which the inequality

$$\frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}+k_n(\sqrt{a_1}-\sqrt{a_n})^2\ge\frac{a_1+\cdots+a_n}{n}$$

holds for all $a_1,\ldots,a_n\in\mathbb{R}$ with $a_1\ge\cdots\ge a_n>0$.

(b) Find the largest positive constant c_n for which the inequality

$$\frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}+c_n(\sqrt{a_1}-\sqrt{a_n})^2\le\frac{a_1+\cdots+a_n}{n}$$

holds for all $a_1, \ldots, a_n \in \mathbb{R}$ with $a_1 \geq \cdots \geq a_n > 0$.

Proposed by Leonard Giugiuc, Traian National College, Drobeta-Turnu Severin, and Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania.

Solution by the authors. (a) The statement is homogeneous, so we may assume that $a_n = 1$. Also the inequalities $a_2 \geq \cdots \geq a_{n-1}$ play no role since the inequality is symmetric in the variables $a_2, \ldots, a_{n-1}$. So the condition $a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n$ can be replaced by $a_2, \ldots, a_n \in [1, a_1]$. Then the inequality from (a) writes as $f(a_2, \ldots, a_{n-1}) \geq 0\ \forall (a_2, \ldots, a_{n-1}) \in [1, a_1]^n$, where

$$f(x_2, \ldots, x_{n-1}) = \frac{n^2}{1 + 1/a_1 + \sum_{i=2}^{n-1} 1/x_i} + nk_n(\sqrt{a_1} - 1)^2 - 1 - a_1 - \sum_{i=2}^{n-1} x_i.$$

If $a_1 = 1$, then $a_2 = \cdots = a_{n-1} = 1$ and we have $f(a_2, \ldots, a_{n-1}) = 0$. Therefore we may restrict ourselves to the case $a_1 > 1$.

Note that f is a concave function in each variable. Indeed, if we assume that $x_2, \ldots, \hat{x}_i, \ldots, x_{n-1}$ are constants, then $f(x_2, \ldots, x_{n-1}) = g(x_i)$, where g has the form $g(x) = \frac{a}{b+1/x} - c - x = \frac{ax}{bx+1} - c - x$ for some $a, b, c \in \mathbb{R}$, $a, b > 0$. Hence $g''(x) = -\frac{2ab}{(bx+1)^3} < 0\ \forall x \in (1, a_1)$. Then f has a minimum on $[1, a_1]^{n-2}$ and this minimum is reached for some $(x_2, \ldots, x_{n-1})$ with $x_i \in \{1, a_1\}\ \forall i$. Since f is symmetric, we only have to consider the case when $a_2 = \cdots = a_{n-m} = a_1$ and $a_{n-m+1} = \cdots = a_{n-1} = 1 = a_n$ for some $1 \leq m \leq n-1$. If we put $x = \sqrt{a_1}$, then our inequality writes as

$$\frac{n^2}{\frac{n-m}{x^2} + m} + nk_n(x-1)^2 - (n-m)x^2 - m \geq 0,$$

that is,

$$nk_n(x-1)^2 \geq \frac{m(n-m)(x^2-1)^2}{mx^2 + n - m},$$

which is equivalent to $nk_n \geq h_m(x)$, where $h_m(x) = \frac{m(n-m)(x+1)^2}{mx^2+n-m}$. This should hold for every $x > 1$ and for every $1 \leq m \leq n-1$. Therefore

$$nk_n = \max_{1 \leq m \leq n-1} \sup_{x>1} h_m(x).$$

We have $h_m'(x) = \frac{2m(n-m)(x+1)(n-m-mx)}{(mx^2+n-m)^2}$. Then h_m is increasing on the interval $(0, \frac{n-m}{m})$ and decreasing on the interval $(\frac{n-m}{m}, \infty)$, so it has a maximum on $(0, \infty)$ at $\frac{n-m}{m}$ and we have $h_m(\frac{n-m}{m}) = n$. If $\frac{n-m}{m} \in (1, \infty)$, which is equivalent to $m < \frac{n}{2}$, i.e., to $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, then $\sup_{x>1} h_m(x) = \max_{x>1} h_m(x) = n$ and this maximum is reached for $x = \frac{n-m}{m}$. If $\frac{n-m}{m} \leq 1$, then h_m is decreasing on $(1, \infty)$, so $\sup_{x>1} h_m(x) = h(1) \leq \max_{x>0} h_m(x) = n$, with equality if and only if $1 = \frac{n-m}{m}$, i.e., if and only if $m = \frac{n}{2}$.

Hence $nk_n = \max_{1\le m\le n-1}\sup_{x>1} h_m(x) = n$ and so $k_n = 1$. Equality is obtained when $a_1 = \cdots = a_n$ or when for some $m \in \{1,\dots,\lfloor\frac{n-1}{2}\rfloor\}$ $(a_1,\dots,a_n)$ is obtained after scaling the vector $\left(\frac{(n-m)^2}{m^2},\dots,\frac{(n-m)^2}{m^2},1,\dots,1\right)$, where there are $n-m$ copies of $\frac{(n-m)^2}{m^2}$ and m copies of 1, by a positive number. (Recall that $a_1 = x^2$ and $x = \frac{n-m}{m}$.) Alternatively, this means that $(a_1,\dots,a_n) = \beta v_m$ for some $1 \le m \le \lfloor\frac{n-1}{2}\rfloor$ and some $\beta > 0$, where $v_m = ((n-m)^2,\dots,(n-m)^2,m^2,\dots,m^2)$, where there are $n-m$ copies of $(n-m)^2$ and m copies of m^2.

(b) Assume that $a_1 = y > 1$ and $a_2 = \ldots = a_n = 1$. Then our inequality writes as

$$\frac{n}{1/y+n-1} + c_n(\sqrt{y}-1)^2 \le \frac{y+n-1}{n}.$$

We divide by y both sides of the inequality above and we take limits as $y\to\infty$. We get $c_n \le \frac{1}{n}$. We prove that in fact $c_n = \frac{1}{n}$. For this we must prove that

$$\frac{n}{\sum_{i=1}^n 1/a_i} + \frac{1}{n}(\sqrt{a_1}-\sqrt{a_n})^2 \le \frac{\sum_{i=1}^n a_i}{n}.$$

We will prove a stronger inequality, namely

$$\left(\prod_{i=1}^n a_i\right)^{1/n} + \frac{1}{n}(\sqrt{a_1}-\sqrt{a_n})^2 \le \frac{1}{n}\sum_{i=1}^n a_i.$$

This can be written as $\left(\prod_{i=1}^n a_i\right)^{1/n} \le \frac{1}{n}\left(\sum_{i=2}^{n-1} a_i + 2\sqrt{a_1a_n}\right)$, which is simply the AM-GM inequality applied to $a_2,\dots,a_{n-1},\sqrt{a_1a_n},\sqrt{a_1a_n}$.

535. Let $A, B \in M_n(\mathbb{C})$ be two matrices of the same rank and let $k \in \mathbb{N}$. Then $A^{k+1}B^k = A$ if and only if $B^{k+1}A^k = B$.

`Proposed by Vasile Pop, Technical University of Cluj-Napoca, and Mihai Opincariu, Avram Iancu National College, Brad, Romania.`

Solution by the authors. Our statement is symmetric in A and B, so it is enough to prove the "only if" implication. We assume that $A^{k+1}B^k = A$.

We have $\operatorname{rank} A = \operatorname{rank}(A^{k+1}B^k) \le \operatorname{rank}(A^{k+1}) \le \operatorname{rank} A$, therefore $\operatorname{rank}(A^{k+1}) = \operatorname{rank} A$. It follows that $\dim\ker(A^{k+1}) = \dim\ker A$. But $\ker A \subseteq \ker A^{k+1}$, so $\ker A = \ker A^{k+1}$.

Next, from $(A^{k+1}B^{k-1})B = A^{k+1}B^k = A$ we get that $\ker B \subseteq \ker A$. But $\operatorname{rank} A = \operatorname{rank} B$, so $\dim\ker A = \dim\ker B$. Hence $\ker B = \ker A = \ker A^{k+1}$.

We have $A^{k+1}B^kA^k = AA^k = A^{k+1}$, so $A^{k+1}(B^kA^k - I_n) = O_n$. Since $\ker A^{k+1} = \ker B$, this implies that $B(B^kA^k - I_n) = O_n$, so $B^{k+1}A^k = B$.

We received essentially the same proof from Moubinool Omarjee, from Lycée Henri IV, Paris, France, and Marian-Daniel Vasile, from West University of Timişoara, Romania.

Solution by Hao Zhang, Department of Mathematics, Hunan University, P. R. China. We assume that $A^{k+1}B^k = A$ and we prove that $B^{k+1}A^k = B$.

We have $\ker B \subseteq \ker A^{k+1}B^k = \ker A$. But $\operatorname{rank} A = \operatorname{rank} B$, so we get $\ker B = \ker A$. On the other hand, we have

$$\operatorname{rank} A = \operatorname{rank} A^{k+1}B^k \le \operatorname{rank} A^{k+1} \le \operatorname{rank} A.$$

So we have $\operatorname{rank} A^{k+1} = \cdots = \operatorname{rank} A^2 = \operatorname{rank} A$. It is clear that $\ker A \subseteq \ker A^2$, so we have $\ker A = \ker A^2$. Now we claim that $\mathbb{C}^n = \ker A \oplus \operatorname{Im} A$. In fact, it is enough to prove that $\ker A \cap \operatorname{Im} A = 0$. If $x \in \ker A \cap \operatorname{Im} A$, then there exists $y \in \mathbb{C}^n$ such that $x = Ay$, so $A^2y = Ax = 0$, i.e., $y \in \ker A^2 = \ker A$. This gives $x = Ay = 0$. So we can choose a basis such that A is of the form

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where A_1 is invertible. Since $\ker A = \ker B$, we may assume that B is of the following form under the same basis.

$$\begin{pmatrix} B_1 & 0 \\ B_3 & 0 \end{pmatrix}.$$

Finally, from $A^{k+1}B^k = A$ we see that $A_1^{k+1}B_1^k = A_1$. So we have $A_1^kB_1^k = I_r$ since A_1 is invertible. Now it is easy to see that $B^{k+1}A^k = B$.

Editor's note. The end of Hao Zhang's solution is somewhat incomplete. Explicitly, we have

$$B^{k+1}A^k = \begin{pmatrix} B_1^{k+1} & 0 \\ B_3B_1^k & 0 \end{pmatrix} \begin{pmatrix} A_1^k & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B_1^{k+1}A_1^k & 0 \\ B_3B_1^kA_1^k & 0 \end{pmatrix}.$$

(The formulas for B^{k+1} and A^k are easily proved by induction.) Since $A_1^kB_1^k = I_n$, we have $B_1^k = (A_1^k)^{-1}$, whence $B_1^kA_1^k = I_n$. It follows that $B_1^{k+1}A_1^k = B_1I_n = B_1$ and $B_3B_1^kA_1^k = B_3I_n = B_3$ and so $B^{k+1}A^k = B$.