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ARTICLES

A class of inequalities involving length elements of a triangle

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Abstract. In this paper, we establish a class of inequalities involving length elements of a triangle. With the help of a computer, we also propose some similar conjectured inequalities as open problems.

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MSC: 51M16.

1. Introduction

Given a triangle ABC with side lengths a, b, c, denote by h_a, h_b, h_c the altitudes, w_a, w_b, w_c the angle-bisectors, r_a, r_b, r_c the radii of excircles, m_a, m_b, m_c the medians, and k_a, k_b, k_c the symmedians.

In a Chinese paper [2], the author established the following two inequalities involving the altitudes and medians of a triangle ABC:

$$4m_b m_c \geq (h_b + h_c)^2, \tag{1}$$

$$(m_b + m_c)^2 \geq 2(h_b^2 + h_c^2). \tag{2}$$

In another Chinese paper [4], the author considered improvements of inequality (1) and proved the following double inequality

$$w_b^2 + w_c^2 \geq 2m_a(m_b + m_c - m_a) \geq h_b^2 + h_c^2, \tag{3}$$

in which the second inequality actually improves (1) by a simple algebraic inequality. We also gave some applications of the double inequality (3) in [4].

Note that the following known relation (cf.[7])

$$k_a = \frac{2bc}{b^2 + c^2} m_a \tag{4}$$

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is equivalent to

$$k_a = \frac{2h_b h_c}{h_b^2 + h_c^2} m_a. \tag{5}$$

We conclude that the second inequality of (3) is equivalent to

$$k_a(m_b + m_c - m_a) \geq h_b h_c. \tag{6}$$

Motivated by inequality (6), the author has recently studied similar inequalities in a triangle and proved some new results, which can be stated in the following four theorems.

Theorem 1. *In any triangle ABC the following double inequality holds*

$$h_a(m_b + m_c - h_a) \geq h_b h_c \geq h_a(h_b + h_c - w_a). \tag{7}$$

Equality holds in either (7) *if and only if triangle ABC is equilateral.*

Theorem 2. *In any triangle ABC the following two inequalities hold:*

$$h_a(m_a + r_a - w_a) \geq h_b h_c, \tag{8}$$

$$w_a(m_a + r_a - w_a) \geq w_b w_c. \tag{9}$$

Equality holds in either (8) *and* (9) *if and only if triangle ABC is equilateral.*

Theorem 3. *In any triangle ABC the following two inequalities hold:*

$$m_b m_c \geq h_a(m_b + m_c - w_a), \tag{10}$$

$$m_b m_c \geq h_a(m_a + r_a - w_a). \tag{11}$$

Equality holds in either (10) *and* (11) *if and only if triangle ABC is equilateral.*

Theorem 4. *In any triangle ABC the following two inequalities hold:*

$$m_a r_a \geq h_a(m_b + m_c - w_a), \tag{12}$$

$$m_a r_a \geq h_a(m_a + r_a - w_a). \tag{13}$$

Equality holds in (12) *if and only if triangle ABC is equilateral and equality holds in* (13) *if and only if $b = c$.*

From the inequalities given in the above theorems, we can obtain some inequality chains. For example, by (10) and the first inequality of (7) we can obtain the double inequality

$$\frac{m_b m_c}{h_a} + w_a \geq m_b + m_c \geq \frac{h_b h_c}{h_a} + h_a. \tag{14}$$

Also, it follows from inequalities (13), (8) and the second inequality of (7) that

$$\frac{m_a r_a}{h_a} + w_a \geq m_a + r_a \geq \frac{h_b h_c}{h_a} + w_a \geq h_b + h_c. \tag{15}$$

The purpose of this paper is to prove these theorems. We shall also propose similar conjectural inequalities as open problems in the last section.

2. PROOF OF THEOREM 1.1

We first give the following lemma.

Lemma 5. *In any triangle ABC it holds that*

$$(m_b+m_c)^2 \geq \frac{9}{4}a^2+h_a^2, \tag{16}$$

with equality if and only if $b=c$.

Proof. We recall that for any triangle ABC the following known inequality (cf. [3] or [5]) holds:

$$b+c \geq \sqrt{a^2+4h_a^2}. \tag{17}$$

Applying this inequality to the triangle GBC (G is the centroid of the triangle ABC) and noting that the altitude from G is equal to $h_a/3$, we get

$$GB+GC \geq \sqrt{a^2+4\left(\frac{1}{3}h_a\right)^2}.$$

Also, we have $BG=\frac{2}{3}m_b, CG=\frac{2}{3}m_c$. Hence, inequality (16) can be obtained immediately from the above inequality. Note that equality in (17) holds if and only if $b=c$. Then we easily determine that the equality condition of (16) is the same as that of (17). This completes the proof. □

Next, we prove Theorem 1.

Proof. We first prove the left inequality of (7), i.e.,

$$h_a(m_b+m_c-h_a) \geq h_bh_c, \tag{18}$$

which is equivalent to

$$h_a^2(m_b+m_c)^2 \geq (h_a^2+h_bh_c)^2.$$

By Lemma 5, it is enough to prove that

$$h_a^2(9a^2+4h_a^2)-4(h_a^2+h_bh_c)^2 \geq 0.$$

We denote by S the area of triangle ABC in the sequel. Using the formula $h_a=2S/a$, one sees that the above inequality is equivalent to

$$A_0 \equiv (bc)^2(9a^4+16S^2)-16S^2(a^2+bc)^2 \geq 0. \tag{19}$$

Applying Heron's formula

$$S=\sqrt{s(s-a)(s-b)(s-c)}, \tag{20}$$

where $s=(a+b+c)/2$, we easily obtain

$$A_0=a^2A_1, \tag{21}$$

where

$$A_1 = a^6 - 2(b^2 - bc + c^2)a^4 + (b^4 - 4b^3c + 7b^2c^2 - 4bc^3 + c^4)a^2 + 2(b-c)^2(b+c)^2bc.$$

Thus, we have to prove $A_1 \geq 0$. But A_1 can be rewritten as

$$A_1 = 2bc(a+b+c)(b+c-a)(b-c)^2 + a^2(a^2 - b^2 - c^2 + bc)^2, \tag{22}$$

which shows $A_1 \geq 0$. Hence inequality (18) is proved. Furthermore, it is easily seen that equality in (18) holds if and only if $a = b = c$.

We now prove the right inequality of (7), i.e.,

$$h_b h_c + w_a h_a \geq h_a(h_b + h_c). \tag{23}$$

Let I be the incenter of the triangle ABC. It is clear that $w_a \geq AI + r$, and then

$$w_a \geq r + \frac{r}{\sin\frac{A}{2}}. \tag{24}$$

Therefore, to prove inequality (23) it is sufficient to prove that

$$h_b h_c + h_a r + \frac{r h_a}{\sin\frac{A}{2}} \geq h_a(h_b + h_c).$$

Applying $r = S/s$ and $h_a = 2S/a$ etc., one sees that the above inequality is equivalent to

$$\frac{2}{bc} + \frac{1}{sa} + \frac{1}{sa\sin\frac{A}{2}} \geq \frac{2}{a}\left(\frac{1}{b} + \frac{1}{c}\right).$$

Using $s = (a+b+c)/2$ and simplifying gives the equivalent inequality

$$\sin\frac{A}{2} \leq \frac{bc}{b^2 + c^2 + bc - a^2}. \tag{25}$$

In view of the well known formula

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \tag{26}$$

for proving (25) we need to show

$$b^3c^3 - (b^2 + c^2 + bc - a^2)^2(s-b)(s-c) \geq 0.$$

Using once more $s = (a+b+c)/2$ and factoring gives

$$\frac{1}{4}(a+b+c)(b+c-a)(b^2 + c^2 - bc - a^2)^2 \geq 0,$$

which clearly is true. Thus, inequalities (25) and (23) are proved. Also, it is easy to deduce that the equality in (23) occurs only when $a = b = c$. This completes the proof of Theorem 1. □

3. Proof of Theorem 2

We first give the following snappy double inequality.

Lemma 6. *In any triangle ABC, we have*

$$m_a w_a \geq r_b r_c \geq w_a^2. \tag{27}$$

Equality holds in either inequality of (27) *if and only if* $b = c$.

Proof. We recall (see [8] or [7] p. 223) that for any triangle ABC it holds

$$\frac{m_a}{w_a} \geq \frac{(b+c)^2}{4bc}, \tag{28}$$

with equality if and only if $b = c$. Also, it is well known that the angle-bisector w_a is given by

$$w_a = \frac{2}{b+c}\sqrt{s(s-a)bc}. \tag{29}$$

Multiplying both sides of (28) by w_a^2 and using (29) and the following known identity

$$r_b r_c = s(s-a), \tag{30}$$

we obtain

$$m_a w_a \geq r_b r_c. \tag{31}$$

which is just the first inequality of (27). The second inequality of (27) can be rapidly obtained from another known inequality (see [1]), viz.,

$$w_a^2 \leq s(s-a), \tag{32}$$

and identity (30). Moreover, we easily determine the equality condition of (27). This completes the proof of Lemma 6. □

Remark 1. Inequality (28) is equivalent to (31) and the following two inequalities:

$$4m_a \sin\frac{A}{2} \geq h_b + h_c, \tag{33}$$

$$m_a \geq \frac{1}{2}(b+c)\cos\frac{A}{2}. \tag{34}$$

The later can be obtained from the identity

$$m_a = \frac{1}{2}\sqrt{(b+c)\cos^2\frac{A}{2} + (b-c)^2\sin^2\frac{A}{2}}. \tag{35}$$

We are now in the position to prove Theorem 2.

Proof. We first prove inequality (8), which has the following two analogues:

$$h_b(m_b + r_b - w_b) \geq h_c h_a, \tag{36}$$

$$h_c(m_c + r_c - w_c) \geq h_a h_b. \tag{37}$$

Without loss of generality we may suppose that $a \geq b \geq c$. Clearly, in order to prove inequality (8) is valid for any triangle we must prove that all inequalities (8), (36) and (37) hold under the hypothesis that $a \geq b \geq c$.

By $h_a = 2S/a$ and $r_a = S/(s-a)$, we get

$$h_a r_a - h_b h_c = \frac{4(a-b)(a-c)S^2}{abc(b+c-a)}. \tag{38}$$

Hence, we have $h_a r_a \geq h_b h_c, h_b r_b \leq h_c h_a$ and $h_c r_c \geq h_a h_b$ under the hypothesis $a \geq b \geq c$. Noting that $m_a \geq w_a$ and $m_c \geq w_c$, we deduce that inequalities (8) and (37) hold.

It remains to prove inequality (36) under the hypothesis $a \geq b \geq c$. According to Lemma 6, it is sufficient to prove

$$h_b \left(\frac{r_c r_a}{w_b} + r_b - w_b \right) \geq h_c h_a,$$

i.e.,

$$h_b (r_c r_a - w_b^2) \geq w_b (h_c h_a - h_b r_b). \tag{39}$$

By Lemma 6, we see that the left hand side of (39) is non-negative. Also, the above inequality $h_b r_b \leq h_c h_a$ shows that the right hand side of (39) is non-negative, too. Thus, for proving inequality (39) we need to prove that

$$B_0 \equiv h_b^2 (r_c r_a - w_b^2)^2 - w_b^2 (h_c h_a - h_b r_b)^2 \geq 0. \tag{40}$$

With the help of software Maple, using the following known formulas

$$h_a = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}, \tag{41}$$

$$r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}}, \tag{42}$$

and Heron's formula, we easily obtain the identity

$$B_0 = \frac{(a+b+c)^2 S^2}{4cab^2(c+a)^4} B_1, \tag{43}$$

where

$$\begin{aligned} B_1 = & -4(c+a)^2 b^6 + 8(c+a)^3 b^5 - 24(c+a)^2 cab^4 - 8(c^2 - 3ca \\ & + a^2)(c+a)^3 b^3 + (4c^6 + 17c^5 a - 12c^4 a^2 - 34c^3 a^3 \\ & - 12c^2 a^4 + 17ca^5 + 4a^6) b^2 - 2ca(c+a)(5c^2 + 6ca \\ & + 5a^2)(c-a)^2 cab + (c-a)^2 (c+a)^4. \end{aligned}$$

Consequently, we have to prove that $B_1 \geq 0$ holds under the hypothesis $a \geq b \geq c$.

Putting $s-a=x, s-b=y$ and $s-c=z$, then we get $a=y+z, b=z+x$ and $c=x+y$ $(x,y,z>0)$. Substituting $a=y+z$ etc. into B_1 and reorganizing gives

$$\begin{aligned}B_1 = &-64zxy^6+64(z+x)zxy^5+(4z^4+32z^3x-8z^2x^2+32zx^3\\&+4x^4)y^4+4(z+x)(z^4-12z^3x-10z^2x^2-12zx^3+x^4)y^3\\&-4(-z+3x)(-3z+x)(z+x)^2zxy^2+32(z+x)(z^2+x^2)z^2x^2y\\&-16(z+x)^2z^3x^3. \qquad (44)\end{aligned}$$

Let us denote by $B_1(x,y,z)$ the polynomial B_1, i.e., $B_1=B_1(x,y,z)$. From the hypothesis $a\ge b\ge c$ we have $x\le y\le z$, thus we may set $y=x+m, z=x+m+n$ $(m\ge 0, n\ge 0)$. Substituting them into B_1 and reorganizing gives

$$\begin{aligned}&B_1(x,x+m,x+m+n)\\=&\,16(m^2+6mn+n^2)(m-n)^2x^4+8(m-n)(7m^4+36m^3n+2m^2n^2\\&-12mn^3-n^4)x^3+4(18m^5+77m^4n-4m^3n^2-50m^2n^3\\&+2mn^4+5n^5)mx^2+8(m+n)(5m^3+17m^2n+3mn^2-n^3)m^3x\\&+4(2m+n)(m+n)^4m^3. \qquad (45)\end{aligned}$$

Therefore, we only need to prove that $B_1(x,x+m,x+m+n)\ge 0$ holds for the positive number x, the non-negative numbers m and n. If $m>n$, the inequality clearly holds. If $n\ge m$, we may set $n=m+t$ $(t\ge 0)$. Substituting this into $B_1(x,x+m,x+m+n)$, we can obtain the identity

$$B_1(x,x+m,x+2m+t)=B_2t^3+16m^2B_3, \qquad (46)$$

where

$$\begin{aligned}B_2 =&\,(8x^3+20x^2m+4m^3)t^2+(16x^4+128x^3m+108x^2m^2-8xm^3\\&+44m^4)t+16(8x^4+20x^3m+2x^2m^2-xm^3+12m^4)m,\\B_3 =&\,(8x^4-23x^2m^2+10xm^3+26m^4)t^2-4m^2(x+m)(4x^2-xm\\&-7m^2)t+12m^4(x+m)^2.\end{aligned}$$

As $m\ge 0, n\ge 0$, it is easy to show that

$$108x^2m^2-8xm^3+44m^4=4m^2(27x^2-2xm+11m^2)>0,$$
$$2x^2m^2-xm^3+12m^4=m^2(2x^2-xm+12m^2)>0.$$

So we have $B_2\ge 0$. Note that B_3 is a quadratic function in t and $8x^4-23x^2m^2+26m^4>0$. Also, we easily compute its discriminant F_t as

$$F_t=-16m^4(8x^4+8mx^3-14m^2x^2+16m^3x+29m^4)(x+m)^2.$$

Note that $8x^4-14m^2x^2+29m^4>0$, hence $F_t\le 0$, and then inequality $B_3\ge 0$ holds. Since $B_2\ge 0$ and $B_3\ge 0$, inequality $B_1(x,x+m,x+2m+t)\ge 0$ follows from (46). Hence, we have proved that for $x>0, m\ge 0, n\ge 0$

inequality $B_1(x, x+m, x+m+n) \geq 0$ holds. We also proved that $B_1 \geq 0$ holds under the hypothesis $a \geq b \geq c$. This completes the proof of inequality (8). Moreover, it is easy to determine that the equality in (8) holds if and only if $a = b = c$.

Next, we prove inequality (9).

By the inequality $m_a w_a \geq r_b r_c$ given in Lemma 6, to prove inequality (9) it suffices to show that

$$r_b r_c - w_a^2 \geq w_b w_c - w_a r_a, \tag{47}$$

which has the following two analogues:

$$r_c r_a - w_b^2 \geq w_c w_a - w_b r_b, \tag{48}$$

$$r_a r_b - w_c^2 \geq w_a w_b - w_c r_c. \tag{49}$$

Suppose that $a \geq b \geq c$. In this setting, we need to prove that all three inequalities (47)–(49) hold. Using the previous formulas (29) and (42), we easily obtain

$$(w_a r_a)^2 - (w_b w_c)^2 = \frac{bc(a-b)(a-c)(c+a-b)(a+b-c)(a^2+3ab+3ac+b)(a+b+c)^2}{4(b+c)^2(c+a)^2(a+b)^2}.$$

Thus, one can see that if $a \geq b \geq c$ then $w_a r_a \geq w_b w_c, w_b r_b \leq w_c w_a$ and $w_c r_c \geq w_a w_b$ hold. Then, we conclude by Lemma 6 that both inequalities (47) and (49) are true. Thus, it remains to prove inequality (48). Noting that the values of both sides of (48) are non-negative, because we have $r_c r_a \geq w_b^2$ (which follows from Lemma 6) and $w_c w_a \geq w_b r_b$ we only need to prove

$$(r_c r_a - w_b^2)^2 - (w_c w_a - w_b r_b)^2 \geq 0,$$

i.e.,

$$C_0 \equiv (r_c r_a)^2 + w_b^4 + 2w_a w_b w_c r_b - 2r_c r_a w_b^2 - (w_c w_a)^2 - (w_b r_b)^2 \geq 0. \tag{50}$$

Applying (29) and Heron's formula, we get

$$w_a w_b w_c = \frac{8abcsS}{(b+c)(c+a)(a+b)}. \tag{51}$$

Substituting this into C_0 and using (29) and (42), with the help of software Maple we obtain the identity

$$C_0 = \frac{(a+b+c)^2}{16(a+b)^2(b+c)^2(c+a)^4} C_1, \tag{52}$$

where

$$
\begin{aligned}
C_1 = {} & (c^4 - 8c^3a - 2c^2a^2 - 8ca^3 + a^4)b^6 + 8(c+a)^3cab^5 \\
& - (2c^6 - 6c^5a + 30c^4a^2 + 44c^3a^3 + 30c^2a^4 - 6ca^5 + 2a^6)b^4 \\
& - 2(c+a)(5c^4 - 8c^3a - 10c^2a^2 - 8ca^3 + 5a^4)cab^3 \\
& + (c^8 + 2c^7a + 17c^6a^2 - 10c^5a^3 - 36c^4a^4 - 10c^3a^5 \\
& + 17c^2a^6 + 2ca^7 + a^8)b^2 + 2(c+a)(c^4 - 5c^3a - 8c^2a^2 \\
& - 5ca^3 + a^4)(c-a)^2cab + (c-a)^2(c+a)^4c^2a^2.
\end{aligned}
$$

Consequently, we have to prove $C_1 \geq 0$ under the hypothesis $a \geq b \geq c$. If we set $s - a = x, s - b = y$ and $s - c = z$, then $a = y + z, b = z + x$, and $c = x + y$ $(x, y, z > 0)$. Substituting $a = y + z$ etc. into C_1 and reorganizing gives

$$
\begin{aligned}
C_1 = {} & -64zxy^8 + (4z^4 + 96z^3x + 56z^2x^2 + 96zx^3 + 4x^4)y^6 \\
& + 8(z+x)(3z^2 - 4zx + 3x^2)(z-x)^2y^5 + (52z^6 - 144z^5x \\
& - 132z^4x^2 + 64z^3x^3 - 132z^2x^4 - 144zx^5 + 52x^6)y^4 \\
& + 8(z+x)(6z^4 + z^3x - 18z^2x^2 + zx^3 + 6x^4)(z-x)^2y^3 \\
& + 4(4z^6 - 4z^5x - 15z^4x^2 + 46z^3x^3 - 15z^2x^4 - 4zx^5 \\
& + 4x^6)(z+x)^2y^2 + 16(z+x)(z-x)^2z^3x^3y \\
& - 16(z+x)^2z^4x^4.
\end{aligned} \tag{53}
$$

From the hypothesis $a \geq b \geq c$ we have $x \leq y \leq z$. Thus, it remains to prove that $C_1 \geq 0$ holds for $0 < x \leq y \leq z$. Let $C_1 = C_1(x, y, z)$. We may let $y = x + m, z = x + m + n$ $(m \geq 0, n \geq 0)$. Then, we need to prove

$$C_1(x, x+m, x+m+n) \geq 0. \tag{54}$$

Making use of software Maple and expanding $C_1(x, x+m, x+m+n)$, we find that all the terms are non-negative. So, inequality (54) holds. This completes the proof of inequality (9). Moreover, the above proof clearly shows that equality in (9) holds if and only if $a = b = c$. Theorem 2 is proved. □

4. PROOF OF THEOREM 3

In the proofs of both Theorem 3 and Theorem 4 we will use the following lemma.

Lemma 7. *In any triangle ABC the following inequality holds*

$$m_b m_c \geq w_a r_a, \tag{55}$$

with equality if and only if triangle ABC is equilateral.

Proof. By the first inequality of (27) and the fact that $m_a \geq w_a$, we obtain

$$m_a^2 \geq r_b r_c, \tag{56}$$

which was used in my recent paper [6]. Using two inequalities similar to (56) and the second inequality of (27), we have

$$m_b m_c \geq \sqrt{r_c r_a} \cdot \sqrt{r_a r_b} = r_a \sqrt{r_b r_c} \geq r_a w_a,$$

which proves (55). It is easily seen that equality in (55) holds if and only if $a = b = c$. This completes the proof of Lemma 7. □

Remark 2. Inequality (55) can be extended to

$$m_b m_c \geq w_a r_a \geq h_b h_c \geq h_a(h_b + h_c - w_a), \tag{57}$$

in which the second inequality could be proved by using the previous inequality (24), the third inequality is just the second inequality of (7).

Next, we prove Theorem 3.

Proof. We first prove inequality (10), that is

$$m_b m_c + w_a h_a \geq h_a(m_b + m_c). \tag{58}$$

Squaring both sides gives

$$(m_b m_c)^2 + (w_a h_a)^2 + 2 m_b m_c w_a h_a \geq h_a^2 (m_b + m_c)^2.$$

By Lemma 7, it is sufficient to prove that

$$(m_b m_c)^2 + (w_a h_a)^2 + 2 r_a h_a w_a^2 \geq h_a^2 (m_b + m_c)^2. \tag{59}$$

Using Cauchy's inequality, we have

$$(m_b + m_c)^2 \leq (b m_b^2 + c m_c^2)\left(\frac{1}{b} + \frac{1}{c}\right). \tag{60}$$

Thus to prove (59) we need to show that

$$D_0 \equiv bc\left[(m_b m_c)^2 + h_a(h_a + 2r_a)w_a^2\right] - (b + c)(b m_b^2 + c m_c^2)h_a^2 \geq 0. \tag{61}$$

With the help of software Maple, using the previous formulas (29), (41), (42) and the well-known median formula

$$4m_a^2 = 2b^2 + 2c^2 - a^2, \tag{62}$$

we easily obtain

$$D_0 = \frac{D_1}{16a^2(b + c)^2}, \tag{63}$$

where

$$\begin{aligned}D_1 &= (2b^4+12b^3c+16b^2c^2+12bc^3+2c^4)a^6-8(b+c)b^2c^2a^5\\&\quad-(5b^6+15b^5c+15b^4c^2+26b^3c^3+15b^2c^4+15bc^5+5c^6)a^4\\&\quad+16(b+c)(b^2+c^2)b^2c^2a^3+(4b^6-4b^4c^2+b^3c^3\\&\quad-4b^2c^4+4c^6)(b+c)^2a^2-8(b-c)^2(b+c)^3b^2c^2a\\&\quad-(b^2+bc+c^2)(b-c)^4(b+c)^4.\end{aligned}$$

Consequently, we have to prove $D_1 \geq 0$. Putting $s-a=x, s-b=y, s-c=z$, then we have $a=y+z, b=z+x, c=x+y$. Substituting the later three relations into D_1 and reorganizing gives

$$\begin{aligned}D_1 &= 4(y+z)^2x^8+16(y+z)(y^2+18yz+z^2)x^7+(9y^4+816y^3z\\&\quad+846y^2z^2+816yz^3+9z^4)x^6-(y+z)(29y^4-688y^3z+358y^2z^2\\&\quad-688yz^3+29z^4)x^5-(29y^6-9y^5z+971y^4z^2+1762y^3z^3\\&\quad+971y^2z^4-9yz^5+29z^6)x^4+(y+z)(9y^6-152y^5z-921y^4z^2\\&\quad-752y^3z^3-921y^2z^4-152yz^5+9z^6)x^3+(16y^6-13y^5z\\&\quad-213y^4z^2+404y^3z^3-213y^2z^4-13yz^5+16z^6)(y+z)^2x^2\\&\quad+(4y^6+24y^5z+31y^4z^2+282y^3z^3+31y^2z^4+24yz^5\\&\quad+4z^6)(y+z)^3x+(y+2z)^2(2y+z)^2(y+z)^4yz.\end{aligned}\tag{64}$$

Consequently, we need to prove $D_1 \geq 0$ for the positive real numbers x, y, z. We set $D_1 = D_1(x, y, z)$. Since $D_1(x, y, z)$ is symmetric with respect to y and z we may suppose that $y \geq z$ and let $y = z + m$ $(m \geq 0)$. With the help of Maple, we easily obtain

$$\begin{aligned}&D_1(x,z+m,z)\\&=4(z+x)m^9+8(3z+x)(3z+2x)m^8+(569z^3+547z^2x\\&\quad+147zx^2+9x^3)m^7+(2591z^4+2600z^3x+358z^2x^2\\&\quad-80zx^3-29x^4)m^6+(7491z^5+8635z^4x-78z^3x^2\\&\quad-1742z^2x^3-165zx^4-29x^5)m^5+(14257z^6+19778z^5x\\&\quad-1353z^4x^2-8868z^3x^3-1361z^2x^4+514zx^5+9x^6)m^4\\&\quad+4(z+x)(4464z^6+2977z^5x-3416z^4x^2-1994z^3x^3\\&\quad+460z^2x^4+209zx^5+4x^6)m^3+4(3546z^6-166z^5x\\&\quad-3435z^4x^2-168z^3x^3+664z^2x^4+86zx^5+x^6)(z+x)^2m^2\\&\quad+16(z-x)(405z^4+90z^3x-196z^2x^2-58zx^3\\&\quad-x^4)(z+x)^3zm+16(81z^2+38zx+x^2)(z-x)^2(z+x)^4z^2.\end{aligned}\tag{65}$$

Thus, we have to prove $D_1(x, z+m, z) \geq 0$ for positive numbers x, z and non-negative number m. If $z > x$, then it is easy to see that $D_1(x, z+m, z) \geq 0$ holds. If $z \leq x$, we may let $x = z + n$ ($n \geq 0$) and substitute it into $D_1(x, z+m, z)$. Then, we can obtain the identity

$$\begin{aligned} D_1(z+n, z+m, z) & \\ &= m_8 z^8 + m_7 z^7 + m_6 z^6 + m_5 z^5 + m_4 z^4 \\ &\quad + m_3 z^3 + m_2 z^2 + m_1 z + m_0, \end{aligned} \tag{66}$$

where

$$\begin{aligned} m_8 &= 8448m^2 - 30720mn + 30720n^2, \\ m_7 &= 21632m^3 - 62720m^2n + 15360mn^2 + 71680n^3, \\ m_6 &= 22976m^4 - 44608m^3n - 54336m^2n^2 + 117248mn^3 + 66816n^4, \\ m_5 &= 14112m^5 - 12352m^4n - 63616m^3n^2 + 50816m^2n^3 \\ &\quad + 122368mn^4 + 31232n^5, \\ m_4 &= 5440m^6 + 2448m^5n - 30848m^4n^2 - 1824m^3n^3 + 75312m^2n^4 \\ &\quad + 55808mn^5 + 7424n^6, \\ m_3 &= 1272m^7 + 2960m^6n - 6584m^5n^2 - 8992m^4n^3 + 20584m^3n^4 \\ &\quad + 32688m^2n^5 + 12160mn^6 + 768n^7, \\ m_2 &= 4(m+n)(40m^7 + 177m^6n - 191m^5n^2 - 482m^4n^3 + 818m^3n^4 \\ &\quad + 1213m^2n^5 + 268mn^6 + 4n^7), \\ m_1 &= 2m(4m^8 + 52m^7n + 87m^6n^2 - 98m^5n^3 - 155m^4n^4 + 284m^3n^5 \\ &\quad + 482m^2n^6 + 192mn^7 + 8n^8), \\ m_0 &= (m+n)(m-n)^2(2m+n)^2(m+2n)^2m^2n. \end{aligned}$$

Clearly, in order to prove $D_1(z+n, z+m, z) \geq 0$ we only need to show that the coefficients $m_8, m_7, \ldots, m_0$ are all non-negative. As $m \geq 0$ and $n \geq 0$, it is clear that $m_0 \geq 0$. Also, it is easy to show that $m_8 \geq 0$ and $m_7 \geq 0$. In addition, we can rewrite m_6, m_5, m_4 and m_3 as follows:

$$\begin{aligned} m_6 &= m(22976m + 47296n)(m-2n)^2 + 64(671m^2 - 1124mn \\ &\quad + 1044n^2)n^2, \end{aligned} \tag{67}$$

$$\begin{aligned} m_5 &= m(m-2n)^2(14112m^2 + 44096mn + 56320n^2) \\ &\quad + 128(779m^2 - 804mn + 244n^2)n^3, \end{aligned} \tag{68}$$

$$\begin{aligned} m_4 &= m(5440m + 24208n)(m-n)^4 + 16(464n + 1975m)n^5 \\ &\quad + 16(2084m^2 - 7832mn + 10419n^2)m^2n^2, \end{aligned} \tag{69}$$

$$m_3 = m(636m + 3706n)(2m + n)(m - n)^4 + 2(7135m^2 - 18684mn + 14634n^2)m^3n^2 + 2(19732m^2 + 4227mn + 384n^2)n^5. \tag{70}$$

Note that $671m^2 - 1124mn + 1044n^2 \geq 0$ etc. One sees that m_6, m_5, m_4 and m_3 are non-negative. Furthermore, it is easy to verify the following two identities:

$$\begin{aligned} &2(40m^7 + 177m^6n - 191m^5n^2 - 482m^4n^3 + 818m^3n^4 \\ &+ 1213m^2n^5 + 268mn^6 + 4n^7) \\ &= m(2m + n)(40m + 317n)(m - n)^4 \\ &\quad + (3020m^2 + 219mn + 8n^2)n^5 \\ &\quad + (1517m^2 - 3420mn + 2350n^2)m^3n^2, \end{aligned} \tag{71}$$

$$\begin{aligned} &4m^8 + 52m^7n + 87m^6n^2 - 98m^5n^3 - 155m^4n^4 + 284m^3n^5 \\ &+ 482m^2n^6 + 192mn^7 + 8n^8 \\ &= m(2m^2 + 33mn + 151n^2)(m - n)^4(2m + n) \\ &\quad + (751m^2 + 41mn + 8n^2)n^6 \\ &\quad + (699m^2 - 1293mn + 650n^2)m^3n^3. \end{aligned} \tag{72}$$

We can easily show that both right hand sides of the above two identities are non-negative. So, we have $m_2 \geq 0$ and $m_1 \geq 0$. And $D_1(z + n, z + m, z) \geq 0$ follows from identity (66). We thus proved that inequality $D_1(x, z+m, z) \geq 0$ is valid for non-negative numbers m, n and positive number x. This completes the proof of inequality (58).

Next, we prove inequality (11), that is

$$m_b m_c + w_a h_a \geq h_a(m_a + r_a). \tag{73}$$

Squaring both sides gives

$$(m_b m_c)^2 + (w_a h_a)^2 + 2m_b m_c w_a h_a \geq h_a^2(m_a + r_a)^2.$$

By Lemma 7, to prove the above inequality we need to show

$$(m_b m_c)^2 + (w_a h_a)^2 + 2r_a h_a w_a^2 \geq h_a^2(m_a + r_a)^2,$$

or equivalently

$$(m_b m_c)^2 + (w_a h_a)^2 + 2r_a h_a w_a^2 - h_a^2(m_a^2 + r_a^2) \geq 2m_a r_a h_a^2. \tag{74}$$

By the simplest arithmetic mean–geometric mean inequality we have

$$h_a + \frac{m_a^2}{h_a} \geq 2m_a. \tag{75}$$

Thus we only need to prove

$$E_0 \equiv (m_b m_c)^2 + (w_a h_a)^2 + 2r_a h_a w_a^2 - h_a^2(m_a^2 + r_a^2) - r_a h_a^3 - r_a h_a m_a^2 \geq 0. \tag{76}$$

Making use of formulas (29), (41), (42), and (62), with the help of software we obtain

$$E_0 = \frac{E_1}{16a^3(b+c)^2}, \tag{77}$$

where

$$\begin{aligned}E_1 = {}& 4(b^2+bc+c^2)a^7 - 8(b+c)bca^6 + (b^2+6bc+c^2)(b-c)^2a^5 \\ & - 2(b+c)(b-c)^4a^4 - (b^2+bc-c^2)(b^2-bc-c^2)(b+c)^2a^3 \\ & + (3b^2-2bc+3c^2)(b-c)^2(b+c)^3a^2 - (b-c)^4(b+c)^5.\end{aligned}$$

To prove inequality (76) we need to show $E_1 \ge 0$. Putting $s-a=x, s-b=y, s-c=z$, we easily show that $E_1 \ge 0$ is equivalent to

$$\begin{aligned}E_1 \equiv {}& 4(y+z)^3x^6 + (12y^4+176y^3z-184y^2z^2+176yz^3+12z^4)x^5 \\ & - (y+z)(3y^4-380y^3z+514y^2z^2-380yz^3+3z^4)x^4 \\ & - 2(13y^2-38yz+13z^2)(y^2-6yz+z^2)(y+z)^2x^3 \\ & - 3(y^4-10y^3z+62y^2z^2-10yz^3+z^4)(y+z)^3x^2 \\ & + 2(6y^4+5y^3z-16y^2z^2+5yz^3+6z^4)(y+z)^4x \\ & + (4y^4+12y^3z-15y^2z^2+12yz^3+4z^4)(y+z)^5 \ge 0. \end{aligned} \tag{78}$$

Through analysis, we find the following identity which can be verified by expanding:

$$\begin{aligned}4E_1 = {}& t_1 + t_2x(x-y)^2(x-z)^2 + t_3xyz(y+z-2x)^2 \\ & + yz(a_0x^2+b_0x+c_0),\end{aligned} \tag{79}$$

where

$$\begin{aligned}t_1 &= 4y^5(33x^2+20xy+4y^2)(x-y)^2 + 4z^5(33x^2+20zx+4z^2)(x-z)^2, \\ t_2 &= 16(y+z)^3x + 80(y^4+z^4) + 32yz(26y^2-17yz+26z^2), \\ t_3 &= (y+z)(805y^2-859yz+805z^2)x + 3(y^2+yz+z^2)(y-z)^2 \\ &\quad + 2(351y^4-242y^2z^2+351z^4), \\ a_0 &= (y+z)(2259y^4+381y^3z-4208y^2z^2+381yz^3+2259z^4), \\ b_0 &= -473y^6-1167y^5z-1087y^4z^2+1166y^3z^3-1087y^2z^4 \\ &\quad -1167yz^5-473z^6, \\ c_0 &= 4(y+z)(32y^6+53y^5z+44y^4z^2+10y^3z^3+44y^2z^4+53yz^5+32z^6).\end{aligned}$$

As $x, y, z > 0$, we have $t_1 > 0$. Also, it is easy to show that $t_2 > 0$ and $t_3 > 0$. Clearly, to prove $E_1 \ge 0$ it remains to prove

$$a_0x^2 + b_0x + c_0 \ge 0. \tag{80}$$

The left hand side is a quadratic function in x. Note that $a_0 > 0$ and the discriminant F_x is given by

$$\begin{aligned}F_x =& b_0^2 - 4a_0c_0 \\ =& -(932879y^{10} + 5185696y^9z + 12185266y^8z^2 + 16635726y^7z^3 \\ &+ 16037951y^6z^4 + 15621924y^5z^5 + 16037951z^6y^4 + 16635726z^7y^3 \\ &+ 12185266z^8y^2 + 5185696yz^9 + 932879z^{10})(y-z)^2.\end{aligned}$$

It is clear $F_x \le 0$, thus inequality (80) holds. This completes the proof of inequality (11). Moreover, it is easy to determine that the equality in (11) holds if and only if $a = b = c$. Theorem 3 is proved. □

5. PROOF OF THEOREM 4

Proof. We first prove inequality (12), that is

$$m_ar_a + w_ah_a \ge h_a(m_b + m_c). \tag{81}$$

Squaring both sides gives

$$(m_ar_a)^2 + (w_ah_a)^2 + 2m_aw_ar_ah_a \ge h_a^2(m_b + m_c)^2.$$

By Lemma 7, we only need to prove that

$$(m_ar_a)^2 + (w_ah_a)^2 + 2r_br_cr_ah_a \ge h_a^2(m_b + m_c)^2,$$

which, by the previous inequality (60) follows from

$$F_0 \equiv bc\left[(m_ar_a)^2 + (w_ah_a)^2 + 2r_br_cr_ah_a\right] - (b+c)(bm_b^2 + cm_c^2)h_a^2 \ge 0. \tag{82}$$

Applying the previous formulas (29), (41), (42), and (62), we can obtain the identity

$$F_0 = \frac{(c+a-b)(a+b-c)(a+b+c)}{16(b+c-a)a^2(b+c)^2}F_1, \tag{83}$$

where

$$\begin{aligned}F_1 =& -(2b^4 + 7b^3c + 14b^2c^2 + 7bc^3 + 2c^4)a^4 + 2(b+c)(2b^4 + 7b^3c \\ &+ 14b^2c^2 + 7bc^3 + 2c^4)a^3 - (b^2 + 6bc + c^2)(b^2 + 3bc \\ &+ c^2)(b+c)^2a^2 - 2(b^4 - 2b^3c - 2b^2c^2 - 2bc^3 \\ &+ c^4)(b+c)^3a + (b^2 + bc + c^2)(b-c)^2(b+c)^4.\end{aligned}$$

Thus, we need to show $F_1 \ge 0$. After analyzing, we obtain the nice identity

$$F_1 = x(x^2 - yz)^2F_2 + (y-z)^2F_3, \tag{84}$$

where $x = s - a, y = s - b, z = s - c$ and

$$\begin{aligned} F_2 =& 64(y+z)x^2 + (112y^2 + 32yz + 112z^2)x + 64y^3 + 64z^3, \\ F_3 =& 20(y^2 + 6yz + z^2)x^4 + 12(y+z)(y^2 + 10yz + z^2)x^3 \\ &+ (5y^4 + 40y^3z + 22y^2z^2 + 40yz^3 + 5z^4)x^2 \\ &+ (y+z)(y^2 + 14yz + z^2)(y-z)^2x + (y+z)^4yz. \end{aligned}$$

As $x, y, z > 0$, we have $F_1 \geq 0$. This completes the proof of inequality (81). Also, the above proof immediately implies that equality in (81) holds if and only if $a = b = c$.

Finally, we prove inequality (13), i.e.,

$$m_a r_a + w_a h_a \geq h_a(m_a + r_a), \tag{85}$$

which is equivalent to

$$(m_a r_a)^2 + (w_a h_a)^2 + 2m_a w_a r_a h_a \geq h_a^2(m_a^2 + r_a^2 + 2r_a m_a).$$

According to the first inequality $m_a w_a \geq r_b r_c$ of (27) and the inequality

$$m_a \leq \frac{1}{4}(r_b + r_c) + \frac{m_a^2}{r_b + r_c}, \tag{86}$$

which follows from the simplest arithmetic mean–geometric mean inequality, we only need to prove

$$\begin{aligned} &(m_a r_a)^2 + (w_a h_a)^2 + 2r_b r_c r_a h_a \\ &\geq h_a^2 \left[m_a^2 + r_a^2 + \frac{1}{2} r_a(r_b + r_c) + \frac{2r_a}{r_b + r_c} m_a^2 \right], \end{aligned}$$

i.e.,

$$\begin{aligned} G_0 \equiv& 2(r_b + r_c)\left[(m_a r_a)^2 + (w_a h_a)^2 + 2r_b r_c r_a h_a\right] \\ &- h_a^2\left[2(r_b + r_c)(m_a^2 + r_a^2) + r_a(r_b + r_c)^2 + 4r_a m_a^2\right]. \end{aligned} \tag{87}$$

Making use of the previous formulas (29), (41), (42), and (62), we obtain the factorization

$$G_0 = \frac{(a+b+c)(b-c)^2 S}{2(b+c-a)a^2(b+c)^2} G_1, \tag{88}$$

where

$$\begin{aligned} G_1 =& a^5 - 2(b+c)a^4 + (b+c)^2a^3 + 2(b+c)^3a^2 - (3b^2 + 2bc \\ &+ 3c^2)(b+c)^2a + 2(b^2 + c^2)(b+c)^3. \end{aligned}$$

Thus, to prove inequality (85) we need to prove the strict inequality $G_1 > 0$. But it is easy to get the following identity:

$$\begin{aligned} G_1 =& 32x^5 + 48(y+z)x^4 + 8(5y^2 + 6yz + 5z^2)x^3 + 4(y+z)(7y^2 + 6yz \\ &+ 7z^2)x^2 + 8(y^2 + yz + z^2)(y+z)^2x + (y+z)^5, \end{aligned} \tag{89}$$

where $x = s - a, y = s - b, z = s - c$. So we have $G_1 > 0$ and inequality (13) is proved. From Lemma 7 and identity (88), we immediately deduce that equality in (13) holds if and only if $b = c$. This completes the proof of Theorem 4. □

6. Several conjectures

In this section, we shall propose several conjectures as open problems.

Conjecture 1. *In any triangle ABC the following double inequality holds:*

$$h_a(m_a + r_a - h_a) \geq w_a r_a \geq h_a(m_b + m_c - m_a). \tag{90}$$

Conjecture 2. *In any triangle ABC the following two inequalities hold:*

$$w_a(m_b + m_c - m_a) \geq h_b h_c, \tag{91}$$

$$h_a(m_b + m_c - w_a) \geq h_b h_c. \tag{92}$$

Remark 3. In the acute triangle ABC we have $w_a \geq k_a$ etc., thus by inequality (6) we know that inequality (91) holds for the acute triangle.

Conjecture 3. *In any triangle ABC the following two inequalities hold:*

$$w_b w_c \geq h_a(w_b + w_c - w_a), \tag{93}$$

$$w_b w_c \geq h_a(m_b + m_c - m_a). \tag{94}$$

Conjecture 4. *In any triangle ABC the following two inequalities hold:*

$$m_b m_c \geq w_a(m_b + m_c - m_a), \tag{95}$$

$$m_a r_a \geq w_a(m_b + m_c - m_a). \tag{96}$$

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Refined asymptotic expansions for some recurrent sequences

Dumitru Popa[1)]

Abstract. Let $a \in \mathbb{R}$ and $f : (a, \infty) \to \mathbb{R}$ be a continuous function such that $f(x) > x$ for all $x > a$, and $(x_n)_{n \geq 1}$ the sequence defined by $x_1 > a$ and $x_{n+1} = f(x_n)$ for all $n \geq 1$. We prove that if there exist $b_0, b_1, b_2 \in \mathbb{R}$, $b_0 \neq 0$, such that $\lim_{x \to \infty} x^2 \left(f(x) - x - b_0 - \frac{b_1}{x} \right) = b_2$, then there exists $C := \lim_{n \to \infty} \left(x_n - b_0 n - \frac{b_1}{b_0} \cdot \ln n \right) \in \mathbb{R}$. Moreover,
$x_n = b_0 n + \frac{b_1}{b_0} \cdot \ln n + C + \frac{b_1^2}{b_0^3} \cdot \frac{\ln n}{n} - \left(\frac{b_1}{2b_0} + \frac{b_2 - b_1 C}{b_0^2} - \frac{b_1^2}{b_0^3} \right) \cdot \frac{1}{n} + o\left(\frac{1}{n}\right)$.
Many and various concrete examples are given.

Keywords: Recurrent sequences, asymptotic expansion of a function, asymptotic expansion of a sequence, Cesàro lemma.

MSC: 35C20, 11B37, 40A05, 40A25.

1. Introduction and a preliminary result

In the recent paper [5] we have proved that there is a connection between the asymptotic expansions of functions and the asymptotic expansions of some recurrent sequences; several similar results can be found in [1, 6]. We recall Theorem 2 from the paper [5].

Theorem 1. *Let $a \in \mathbb{R}$ and $f : (a, \infty) \to \mathbb{R}$ be a continuous function such that $f(x) > x$ for all $x > a$. Let $(x_n)_{n \geq 1}$ be the sequence defined by $x_1 > a$ and $x_{n+1} = f(x_n)$ for all $n \geq 1$. If there exist $b_0, b_1, b_2 \in \mathbb{R}$, $b_0 \neq 0$, such that $\lim_{x \to \infty} x^2 \left(f(x) - x - b_0 - \frac{b_1}{x} \right) = b_2$, then there exists $C \in \mathbb{R}$ such that $x_n = b_0 n + \frac{b_1}{b_0} \cdot \ln n + C + \frac{b_1^2}{b_0^3} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right)$.*

A look at the statement of Theorem 1 reveals the absence of b_2 in the conclusion. The main purpose of this paper is to complete Theorem 1, see Theorem 3. All our notation and notions are standard, see [3]. We need the following well-known evaluation. We include one of its possible proofs.

Proposition 2. *The following evaluation holds*

$$\frac{\ln(n+1)}{n+1} - \frac{\ln n}{n} = -\frac{\ln n}{n^2} + \frac{1}{n^2} + o\left(\frac{1}{n^2}\right).$$

Proof. From $\ln(n+1) = \ln n + \ln\left(1 + \frac{1}{n}\right) = \ln n + \frac{1}{n} + o\left(\frac{1}{n}\right)$ we deduce $\frac{\ln(n+1)}{n+1} = \frac{\ln n}{n+1} + \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$. From $\frac{1}{n+1} = \frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^3} + o\left(\frac{1}{n^3}\right)$ we deduce

$$\frac{\ln n}{n+1} = \frac{\ln n}{n} - \frac{\ln n}{n^2} + \frac{\ln n}{n^3} + o\left(\frac{\ln n}{n^3}\right) = \frac{\ln n}{n} - \frac{\ln n}{n^2} + o\left(\frac{1}{n^2}\right)$$

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and the proof is finished. □

2. The main result

Theorem 3. *Let $a \in \mathbb{R}$ and $f : (a, \infty) \to \mathbb{R}$ be a continuous function such that $f(x) > x$ for all $x > a$. Let $(x_n)_{n \geq 1}$ be the sequence defined by $x_1 > a$ and $x_{n+1} = f(x_n)$ for all $n \geq 1$. If there exist $b_0, b_1, b_2 \in \mathbb{R}$, $b_0 \neq 0$, such that $\lim\limits_{x \to \infty} x^2 \left(f(x) - x - b_0 - \frac{b_1}{x} \right) = b_2$, then there exists $C := \lim\limits_{n \to \infty} \left(x_n - b_0 n - \frac{b_1}{b_0} \cdot \ln n \right) \in \mathbb{R}$. Moreover,*

$$x_n = b_0 n + \frac{b_1}{b_0} \cdot \ln n + C + \frac{b_1^2}{b_0^3} \cdot \frac{\ln n}{n} - \left(\frac{b_1}{2b_0} + \frac{b_2 - b_1 C}{b_0^2} - \frac{b_1^2}{b_0^3} \right) \cdot \frac{1}{n} + o\left(\frac{1}{n} \right).$$

Proof. For all $n \geq 1$ let us put $z_n = x_n - b_0 n - \frac{b_1}{b_0} \cdot \ln n - C - \frac{b_1^2}{b_0^3} \cdot \frac{\ln n}{n}$. Using $x_{n+1} = f(x_n)$, $\ln\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)$ and Proposition 2, we get

$$\begin{aligned}
z_{n+1} - z_n &= x_{n+1} - x_n - b_0 - \frac{b_1}{b_0} \ln\left(1 + \frac{1}{n}\right) - \frac{b_1^2}{b_0^3} \left(\frac{\ln(n+1)}{n+1} - \frac{\ln n}{n} \right) \\
&= f(x_n) - x_n - b_0 - \frac{b_1}{b_0} \left(\frac{1}{n} - \frac{1}{2n^2} \right) - \frac{b_1^2}{b_0^3} \left(-\frac{\ln n}{n^2} + \frac{1}{n^2} \right) + o\left(\frac{1}{n^2} \right) \\
&= \left(f(x_n) - x_n - b_0 - \frac{b_1}{x_n} \right) + \frac{b_1}{x_n} - \frac{b_1}{b_0} \left(\frac{1}{n} - \frac{1}{2n^2} \right) \\
&\quad - \frac{b_1^2}{b_0^3} \left(-\frac{\ln n}{n^2} + \frac{1}{n^2} \right) + o\left(\frac{1}{n^2} \right). \qquad (1)
\end{aligned}$$

Since $\lim\limits_{n \to \infty} x_n = \infty$, the hypothesis $\lim\limits_{x \to \infty} x^2 \left(f(x) - x - b_0 - \frac{b_1}{x} \right) = b_2$ implies that $\lim\limits_{n \to \infty} x_n^2 \left(f(x_n) - x_n - b_0 - \frac{b_1}{x_n} \right) = b_2$. Passing to the limit in the equality

$$n^2 \left(f(x_n) - x_n - b_0 - \frac{b_1}{x} \right) = x_n^2 \left(f(x_n) - x_n - b_0 - \frac{b_1}{x} \right) \cdot \frac{n^2}{x_n^2}$$

and remembering that $\lim\limits_{n \to \infty} \frac{x_n}{n} = b_0$, we get

$$\lim_{n \to \infty} n^2 \left(f(x_n) - x_n - b_0 - \frac{b_1}{x_n} \right) = \frac{b_2}{b_0^2}$$

or equivalently

$$f(x_n) - x_n - b_0 - \frac{b_1}{x_n} = \frac{b_2}{b_0^2} \cdot \frac{1}{n^2} + o\left(\frac{1}{n^2} \right). \qquad (2)$$

From (1) and (2) we deduce that

$$z_{n+1}-z_n=\frac{b_1}{x_n}-\frac{b_1}{b_0}\cdot\frac{1}{n}+\frac{b_1^2}{b_0^3}\cdot\frac{\ln n}{n^2}+\left(\frac{b_2}{b_0^2}+\frac{b_1}{2b_0}-\frac{b_1^2}{b_0^3}\right)\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right). \quad (3)$$

From $C:=\lim\limits_{n\to\infty}\left(x_n-b_0n-\frac{b_1}{b_0}\cdot\ln n\right)$, that is $x_n=b_0n+\frac{b_1}{b_0}\cdot\ln n+C+o\left(1\right)$, we get $\frac{1}{x_n}=\frac{1}{b_0n}\cdot\frac{1}{1+\frac{b_1}{b_0^2}\cdot\frac{\ln n}{n}+\frac{C}{b_0}\cdot\frac{1}{n}+o\left(\frac{1}{n}\right)}$. But

$$\frac{1}{1+\frac{b_1}{b_0^2}\cdot\frac{\ln n}{n}+\frac{C}{b_0}\cdot\frac{1}{n}+o\left(\frac{1}{n}\right)}=1-\frac{b_1}{b_0^2}\cdot\frac{\ln n}{n}-\frac{C}{b_0}\cdot\frac{1}{n}+o\left(\frac{1}{n}\right)$$

and hence

$$\frac{b_1}{x_n}=\frac{b_1}{b_0n}-\frac{b_1^2}{b_0^3}\cdot\frac{\ln n}{n^2}-\frac{b_1C}{b_0^2}\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right). \quad (4)$$

Relations (3) and (4) give us that

$$z_{n+1}-z_n=\left(\frac{b_1}{2b_0}+\frac{b_2-b_1C}{b_0^2}-\frac{b_1^2}{b_0^3}\right)\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right),$$

that is

$$\lim_{n\to\infty}\frac{z_{n+1}-z_n}{\frac{1}{n^2}}=\frac{b_1}{2b_0}+\frac{b_2-b_1C}{b_0^2}-\frac{b_1^2}{b_0^3}. \quad (5)$$

Since by Theorem 1, $\lim\limits_{n\to\infty}z_n=0$ from the Cesàro lemma in the case $\left[\frac{0}{0}\right]$, see [2], relation (5) gives us that $\lim\limits_{n\to\infty}\frac{z_n}{\frac{1}{n}}=-\left(\frac{b_1}{2b_0}+\frac{b_2-b_1C}{b_0^2}-\frac{b_1^2}{b_0^3}\right)$, that is, $z_n=-\left(\frac{b_1}{2b_0}+\frac{b_2-b_1C}{b_0^2}-\frac{b_1^2}{b_0^3}\right)\cdot\frac{1}{n}+o\left(\frac{1}{n}\right)$ and the proof is finished. □

3. Some applications

We apply in the sequel Theorem 3 to refine asymptotic evaluations of some recurrent sequences from the paper [5].

Theorem 4. *Let $\varphi:[0,\infty)\to\mathbb{R}$ be a twice differentiable function such that $\varphi\left(x\right)>0$ for all $x\geq 0$. Let $\left(x_n\right)_{n\geq 1}$ be the sequence defined by $x_1>0$ and $x_{n+1}=x_n+\varphi\left(\frac{1}{x_n}\right)$ for all $n\geq 1$. Then there exists $C\in\mathbb{R}$ such that*

$$\begin{aligned} x_n &= \varphi\left(0\right)n+\frac{\varphi'\left(0\right)}{\varphi\left(0\right)}\cdot\ln n+C+\frac{\left[\varphi'\left(0\right)\right]^2}{\left[\varphi\left(0\right)\right]^3}\cdot\frac{\ln n}{n} \\ &\quad -\left(\frac{\varphi'\left(0\right)}{2\varphi\left(0\right)}+\frac{\varphi''\left(0\right)-2C\varphi'\left(0\right)}{2\left[\varphi\left(0\right)\right]^2}-\frac{\left[\varphi'\left(0\right)\right]^2}{\left[\varphi\left(0\right)\right]^3}\right)\cdot\frac{1}{n}+o\left(\frac{1}{n}\right). \end{aligned}$$

Proof. Let $f:(0,\infty)\to\mathbb{R}$, $f(x)=x+\varphi\left(\frac{1}{x}\right)$. In this case $\varphi(0)=b_0>0$, $\varphi'(0)=b_1$, $\frac{\varphi''(0)}{2}=b_2$, and $\frac{b_1}{2b_0}+\frac{b_2-b_1C}{b_0^2}-\frac{b_1^2}{b_0^3}=\frac{\varphi'(0)}{2\varphi(0)}+\frac{\varphi''(0)-2C\varphi'(0)}{2[\varphi(0)]^2}-\frac{[\varphi'(0)]^2}{[\varphi(0)]^3}$. We apply now Theorem 3. □

Theorem 5. *Let $\varphi:[0,\infty)\to\mathbb{R}$ be a three times differentiable function such that $\varphi(x)>1$ for all $x>0$, $\varphi(0)=1$, $\varphi'(0)\neq 0$. Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_1>0$ and $x_{n+1}=x_n\varphi\left(\frac{1}{x_n}\right)$ for all $n\geq 1$. Then there exists $C\in\mathbb{R}$ such that*

$$\begin{aligned} x_n &= \varphi'(0)\,n+\frac{\varphi''(0)}{2\varphi'(0)}\cdot\ln n+C+\frac{[\varphi''(0)]^2}{4[\varphi'(0)]^3}\cdot\frac{\ln n}{n}\\ &\quad-\left(\frac{\varphi''(0)}{2\varphi'(0)}+\frac{\varphi'''(0)-3C\varphi''(0)}{6[\varphi'(0)]^2}-\frac{[\varphi''(0)]^2}{4[\varphi'(0)]^3}\right)\cdot\frac{1}{n}+o\left(\frac{1}{n}\right). \end{aligned}$$

Proof. Let $f:(0,\infty)\to\mathbb{R}$, $f(x)=x\varphi\left(\frac{1}{x}\right)$. We have $\varphi'(0)=b_0$, $\frac{\varphi''(0)}{2}=b_1$, $\frac{\varphi'''(0)}{6}=b_2$, and $\frac{b_1}{2b_0}+\frac{b_2-b_1C}{b_0^2}-\frac{b_1^2}{b_0^3}=\frac{\varphi''(0)}{2\varphi'(0)}+\frac{\varphi'''(0)-3C\varphi''(0)}{6[\varphi'(0)]^2}-\frac{[\varphi''(0)]^2}{4[\varphi'(0)]^3}$. From Theorem 3 we deduce the evaluation from the statement. □

Proposition 6. (i) *If $\alpha_n^3=o\left(\frac{1}{n^2}\right)$, then for all $r\in\mathbb{R}\setminus\{0\}$, $(1+\alpha_n)^r=1+r\alpha_n+\frac{r(r-1)}{2}\alpha_n^2+o\left(\frac{1}{n^2}\right)$.*
(ii) *Let a, b, c, $d\in\mathbb{R}$. Then for all $r\in\mathbb{R}\setminus\{0\}$*

$$\left(1+a\cdot\frac{\ln n}{n}+b\cdot\frac{1}{n}+c\cdot\frac{\ln n}{n^2}+d\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)\right)^r=1+ra\cdot\frac{\ln n}{n}+rb\cdot\frac{1}{n}+$$
$$\frac{r(r-1)a^2}{2}\cdot\frac{\ln^2 n}{n^2}+[abr(r-1)+rc]\cdot\frac{\ln n}{n^2}+\left[\frac{r(r-1)b^2}{2}+rd\right]\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right).$$

Proof. (i) We will use the well-known evaluation $(1+x)^r=1+rx+\frac{r(r-1)}{2}x^2+\frac{r(r-1)(r-2)}{6}x^3+o\left(x^3\right)$ as $x\to 0$. Since $\alpha_n=o\left(\frac{1}{n^{2/3}}\right)$, $\lim\limits_{n\to\infty}\alpha_n=0$, and hence $(1+\alpha_n)^r=1+r\alpha_n+\frac{r(r-1)}{2}\alpha_n^2+\frac{r(r-1)(r-2)}{6}\alpha_n^3+o\left(\alpha_n^3\right)$. Now from $\alpha_n^3=o\left(\frac{1}{n^2}\right)$ we get $(1+\alpha_n)^r=1+r\alpha_n+\frac{r(r-1)}{2}\alpha_n^2+o\left(\frac{1}{n^2}\right)$.

(ii) From (i) it follows that

$$\begin{aligned} &\left(1+a\cdot\frac{\ln n}{n}+b\cdot\frac{1}{n}+c\cdot\frac{\ln n}{n^2}+d\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)\right)^r\\ &=1+r\left(a\cdot\frac{\ln n}{n}+b\cdot\frac{1}{n}+c\cdot\frac{\ln n}{n^2}+d\cdot\frac{1}{n^2}\right)\\ &+\frac{r(r-1)}{2}\left(a\cdot\frac{\ln n}{n}+b\cdot\frac{1}{n}+c\cdot\frac{\ln n}{n^2}+d\cdot\frac{1}{n^2}\right)^2+o\left(\frac{1}{n^2}\right). \end{aligned}\tag{6}$$

Also we have

$$\left(a\cdot\frac{\ln n}{n}+b\cdot\frac{1}{n}+c\cdot\frac{\ln n}{n^2}+d\cdot\frac{1}{n^2}\right)^2=\left(a\cdot\frac{\ln n}{n}+b\cdot\frac{1}{n}\right)^2+o\left(\frac{1}{n^2}\right)$$

$$=a^2\cdot\frac{\ln^2 n}{n^2}+2ab\cdot\frac{\ln n}{n^2}+b^2\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right). \tag{7}$$

Replacing (7) in (6), we get the stated evaluation. □

Proposition 7. *Let $\alpha,\ \beta,\ \gamma,\ \delta,\ C\in\mathbb{R},\ \alpha\neq 0,$ and $r\in\mathbb{R}\setminus\{0\}$. If $a_n=\alpha n+\beta\ln n+C+\gamma\cdot\frac{\ln n}{n}+\delta\cdot\frac{1}{n}+o\left(\frac{1}{n}\right)$, then*

$$a_n^r=\alpha^r n^r+\frac{r\beta}{\alpha^{1-r}}\cdot\frac{\ln n}{n^{1-r}}+\frac{rC}{\alpha^{1-r}}\cdot\frac{1}{n^{1-r}}+\frac{r\left(r-1\right)\beta^2}{2\alpha^{2-r}}\cdot\frac{\ln^2 n}{n^{2-r}}$$
$$+\frac{r\left(r-1\right)\beta C+r\alpha\gamma}{\alpha^{2-r}}\cdot\frac{\ln n}{n^{2-r}}+\frac{r\left(r-1\right)C^2+2r\alpha\delta}{2\alpha^{2-r}}\cdot\frac{1}{n^{2-r}}+o\left(\frac{1}{n^{2-r}}\right).$$

Proof. From Proposition 6 we have

$$\begin{aligned}
a_n^r &= \alpha^r n^r\left(1+\frac{\beta}{\alpha}\cdot\frac{\ln n}{n}+\frac{C}{\alpha}\cdot\frac{1}{n}+\frac{\gamma}{\alpha}\cdot\frac{\ln n}{n^2}+\frac{\delta}{\alpha}\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)\right)^r\\
&= \alpha^r n^r\left(1+\frac{r\beta}{\alpha}\cdot\frac{\ln n}{n}+\frac{rC}{\alpha}\cdot\frac{1}{n}+\frac{r\left(r-1\right)\beta^2}{2\alpha^2}\cdot\frac{\ln^2 n}{n^2}\right.\\
&\quad+\left[\frac{r\left(r-1\right)\beta C}{\alpha^2}+\frac{r\gamma}{\alpha}\right]\cdot\frac{\ln n}{n^2}+\left[\frac{r\left(r-1\right)C^2}{2\alpha^2}+\frac{r\delta}{\alpha}\right]\cdot\frac{1}{n^2}\\
&\quad\left.+\,o\left(\frac{1}{n^2}\right)\right)\\
&= \alpha^r n^r\left(1+\frac{r\beta}{\alpha}\cdot\frac{\ln n}{n}+\frac{rC}{\alpha}\cdot\frac{1}{n}+\frac{r\left(r-1\right)\beta^2}{2\alpha^2}\cdot\frac{\ln^2 n}{n^2}\right.\\
&\quad\left.+\frac{r\left(r-1\right)\beta C+r\alpha\gamma}{\alpha^2}\cdot\frac{\ln n}{n^2}+\frac{r\left(r-1\right)C^2+2r\alpha\delta}{2\alpha^2}\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)\right).
\end{aligned}$$

By calculations we get the stated evaluation. □

In the next result we refine the evaluation from the exercise 2.8 in [4].

Corollary 8. *Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_1>0$ and $x_{n+1}=x_n+\frac{1}{x_n}$ for all $n\geq 1$. Then there exists $K\in\mathbb{R}$ such that*

$$\begin{aligned}
x_n &= \sqrt{2}\sqrt{n}+\frac{1}{4\sqrt{2}}\cdot\frac{\ln n}{\sqrt{n}}+K\cdot\frac{1}{\sqrt{n}}-\frac{1}{64\sqrt{2}}\cdot\frac{\ln^2 n}{n\sqrt{n}}\\
&\quad-\frac{2K\sqrt{2}-1}{16\sqrt{2}}\cdot\frac{\ln n}{n\sqrt{n}}-\frac{\left(2K\sqrt{2}-1\right)^2}{16\sqrt{2}}\cdot\frac{1}{n\sqrt{n}}+o\left(\frac{1}{n\sqrt{n}}\right).
\end{aligned}$$

Proof. Squaring the recurrence relation, we obtain $x_{n+1}^2 = x_n^2 + 2 + \frac{1}{x_n^2}$, $\forall n \geq 1$. For all $n \geq 1$ we denote $x_n^2 = a_n$ and thus $a_{n+1} = a_n + 2 + \frac{1}{a_n} = a_n + \varphi\left(\frac{1}{a_n}\right)$, where $\varphi(x) = 2 + x$. From Theorem 4 there exists $C \in \mathbb{R}$ such that

$$\begin{aligned} a_n &= \varphi(0)\, n + \frac{\varphi'(0)}{\varphi(0)} \cdot \ln n + C + \frac{[\varphi'(0)]^2}{[\varphi(0)]^3} \cdot \frac{\ln n}{n} \\ &\quad - \left(\frac{\varphi'(0)}{2\varphi(0)} + \frac{\varphi''(0) - 2C\varphi'(0)}{2[\varphi(0)]^2} - \frac{[\varphi'(0)]^2}{[\varphi(0)]^3} \right) \cdot \frac{1}{n} + o\left(\frac{1}{n}\right). \end{aligned}$$

or, $x_n^2 = 2n + \frac{1}{2} \cdot \ln n + C + \frac{1}{8} \cdot \frac{\ln n}{n} + \frac{2C-1}{8} \cdot \frac{1}{n} + o\left(\frac{1}{n}\right)$. By Proposition 7

$$\begin{aligned} x_n &= \left(2n + \frac{1}{2} \cdot \ln n + C + \frac{1}{8} \cdot \frac{\ln n}{n} + \frac{2C-1}{8} \cdot \frac{1}{n} + o\left(\frac{1}{n}\right) \right)^{1/2} \\ &= \sqrt{\alpha}\sqrt{n} + \frac{\beta}{2\sqrt{\alpha}} \cdot \frac{\ln n}{\sqrt{n}} + \frac{C}{2\sqrt{\alpha}} \cdot \frac{1}{\sqrt{n}} - \frac{\beta^2}{8\alpha\sqrt{\alpha}} \cdot \frac{\ln^2 n}{n\sqrt{n}} \\ &\quad + \frac{2\alpha\gamma - \beta C}{4\alpha\sqrt{\alpha}} \cdot \frac{\ln n}{n\sqrt{n}} + \frac{4\alpha\delta - C^2}{8\alpha\sqrt{\alpha}} \cdot \frac{1}{n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right), \end{aligned}$$

where $\alpha = 2$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{8}$, $\delta = \frac{2C-1}{8}$. Hence

$$\begin{aligned} x_n &= \sqrt{2}\sqrt{n} + \frac{1}{4\sqrt{2}} \cdot \frac{\ln n}{\sqrt{n}} + \frac{C}{2\sqrt{2}} \cdot \frac{1}{\sqrt{n}} - \frac{1}{64\sqrt{2}} \cdot \frac{\ln^2 n}{n\sqrt{n}} \\ &\quad - \frac{C-1}{16\sqrt{2}} \cdot \frac{\ln n}{n\sqrt{n}} - \frac{(C-1)^2}{16\sqrt{2}} \cdot \frac{1}{n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right). \end{aligned}$$

If we denote $\frac{C}{2\sqrt{2}} = K$, we get the stated evaluation. □

Proposition 9. *Let $A > 0$ and $g : [0, A) \to \mathbb{R}$ be a function with the property that there exist a_1, a_2, $a_3 \in \mathbb{R}$ such that $g(x) = x + a_1x^2 + a_2x^3 + a_3x^4 + o(x^4)$ as $x \to 0$, $x > 0$. Then*

$$\frac{1}{g(x)} = \frac{1}{x} - a_1 + \left(a_1^2 - a_2\right)x - \left(a_1^3 - 2a_1a_2 + a_3\right)x^2 + o\left(x^2\right) \quad \textit{as } x \to 0, x > 0.$$

In particular,

$$\frac{1}{g\left(\frac{1}{x}\right)} = x - a_1 + \frac{a_1^2 - a_2}{x} - \frac{a_1^3 - 2a_1a_2 + a_3}{x^2} + o\left(\frac{1}{x^2}\right) \quad \textit{as } x \to \infty.$$

Proof. By direct calculus $\lim\limits_{x \to 0, x > 0} \frac{\frac{1}{g(x)} - \frac{1}{x} + a_1 - \left(a_1^2 - a_2\right)x}{x^2} = -\left(a_1^3 - 2a_1a_2 + a_3\right)$. A different proof can be found in [5, Proposition 16]. □

Theorem 10. *Let $A > 0$, $g : [0, A) \to [0, \infty)$ be a continuous function such that $0 < g(x) < x$ for all $0 < x < A$. Let $(x_n)_{n \geq 1}$ be the sequence defined by*

$x_1 \in (0, A)$ and $x_{n+1} = g(x_n)$ for all $n \geq 1$. If there exist the real numbers a_1, a_2, a_3, $a_1 \neq 0$ such that $g(x) = x + a_1 x^2 + a_2 x^3 + a_3 x^4 + o(x^4)$ as $x \to 0$, $x > 0$, then there exists $K \in \mathbb{R}$ such that

$$\begin{aligned} x_n &= -\frac{1}{a_1} \cdot \frac{1}{n} + \frac{a_1^2 - a_2}{a_1^3} \cdot \frac{\ln n}{n^2} + K \cdot \frac{1}{n^2} - \frac{\left(a_1^2 - a_2\right)^2}{a_1^5} \cdot \frac{\ln^2 n}{n^3} \\ &\quad + \frac{\left(a_1^2 - a_2\right)\left(a_1^2 - a_2 - 2a_1^3 K\right)}{a_1^5} \cdot \frac{\ln n}{n^3} \\ &\quad - \frac{2a_1^4 K^2 - 2a_1^3\left(a_1^2 - a_2\right)K + a_1^4 - a_1^2 a_2 + 2a_1 a_3 - 2a_2^2}{2a_1^5} \cdot \frac{1}{n^3} + o\left(\frac{1}{n^3}\right). \end{aligned}$$

Proof. Let $f : \left(\frac{1}{A}, \infty\right) \to \mathbb{R}$, $f(x) = \frac{1}{g\left(\frac{1}{x}\right)}$. Then f is continuous and from $0 < g(x) < x$, $\forall 0 < x < A$, it follows that $f(x) > x$, $\forall x > \frac{1}{A}$. For all $n \geq 1$ we define $v_n = \frac{1}{x_n}$. Then $v_1 > \frac{1}{A}$ and the recurrence relation becomes $v_{n+1} = f(v_n)$ for all $n \geq 1$. From Proposition 9 it follows that $f(x) = x + b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + o\left(\frac{1}{x^2}\right)$ as $x \to \infty$, where $b_0 = -a_1$, $b_1 = a_1^2 - a_2$, $b_2 = -a_1^3 + 2a_1 a_2 - a_3$. From Theorem 3 we have

$$v_n = \alpha n + \beta \ln n + C + \gamma \frac{\ln n}{n} + \delta \frac{1}{n} + o\left(\frac{1}{n}\right),$$

where $\alpha = b_0$, $\beta = \frac{b_1}{b_0}$, $\gamma = \frac{b_1^2}{b_0^3}$, $\delta = -\left(\frac{b_1}{2b_0} + \frac{b_2 - b_1 C}{b_0^2} - \frac{b_1^2}{b_0^3}\right)$. By Proposition 7

$$\begin{aligned} x_n &= \frac{1}{v_n} = \frac{1}{\alpha n} - \frac{\beta}{\alpha^2} \cdot \frac{\ln n}{n^2} - \frac{C}{\alpha^2} \cdot \frac{1}{n^2} + \frac{\beta^2}{\alpha^3} \cdot \frac{\ln^2 n}{n^3} + \frac{2\beta C - \gamma\alpha}{\alpha^3} \cdot \frac{\ln n}{n^3} \\ &\quad + \frac{C^2 - \alpha\delta}{\alpha^3} \cdot \frac{1}{n^3} + o\left(\frac{1}{n^3}\right). \end{aligned}$$

Since $\alpha = -a_1$, $\beta = -\dfrac{a_1^2 - a_2}{a_1}$, $\gamma = -\dfrac{\left(a_1^2 - a_2\right)^2}{a_1^3}$, and

$$\delta = \frac{a_1^2 - a_2}{2a_1} + \frac{\left(a_1^3 - 2a_1 a_2 + a_3\right) + \left(a_1^2 - a_2\right)C}{a_1^2} - \frac{\left(a_1^2 - a_2\right)^2}{a_1^3},$$

we obtain

$$\begin{aligned} \frac{2\beta C - \gamma\alpha}{\alpha^3} &= \frac{\left(a_1^2 - a_2\right)\left(a_1^2 - a_2 + 2a_1 C\right)}{a_1^5}, \\ \frac{C^2 - \alpha\delta}{\alpha^3} &= -\frac{C^2 + \frac{a_1^2 - a_2}{2} + \frac{\left(a_1^3 - 2a_1 a_2 + a_3\right) + \left(a_1^2 - a_2\right)C}{a_1} - \frac{\left(a_1^2 - a_2\right)^2}{a_1^2}}{a_1^3} \\ &= -\frac{2a_1^2 C^2 + 2a_1\left(a_1^2 - a_2\right)C + a_1^4 - a_1^2 a_2 + 2a_1 a_3 - 2a_2^2}{2a_1^5}. \end{aligned}$$

Hence

$$x_n = -\frac{1}{a_1}\cdot\frac{1}{n}+\frac{a_1^2-a_2}{a_1^3}\cdot\frac{\ln n}{n^2}-\frac{C}{a_1^2}\cdot\frac{1}{n^2}-\frac{\left(a_1^2-a_2\right)^2}{a_1^5}\cdot\frac{\ln^2 n}{n^3}$$
$$+\frac{\left(a_1^2-a_2\right)\left(a_1^2-a_2+2a_1C\right)}{a_1^5}\cdot\frac{\ln n}{n^3}$$
$$-\frac{2a_1^2C^2+2a_1\left(a_1^2-a_2\right)C+a_1^4-a_1^2a_2+2a_1a_3-2a_2^2}{2a_1^5}\cdot\frac{1}{n^3}$$
$$+o\left(\frac{1}{n^3}\right).$$

If we denote $K=-\dfrac{C}{a_1^2}$, we get the stated evaluation. □

In the next result we refine the evaluation from Exercise 2.3 in [4].

Corollary 11. *Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_1>0$ and $x_{n+1}=\ln(1+x_n)$ for all $n\geq 1$. Then there exists $K\in\mathbb{R}$ such that*

$$x_n=2\cdot\frac{1}{n}+\frac{2}{3}\cdot\frac{\ln n}{n^2}+K\cdot\frac{1}{n^2}+\frac{2}{9}\cdot\frac{\ln^2 n}{n^3}+\frac{2\left(3K-1\right)}{9}\cdot\frac{\ln n}{n^3}$$
$$-\frac{18K^2-3K+1}{144}\cdot\frac{1}{n^3}+o\left(\frac{1}{n^3}\right).$$

Proof. As is well known, $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+o\left(x^4\right)$ as $x\to 0$, i.e., $a_1=-\frac{1}{2}$, $a_2=\frac{1}{3}$, $a_3=-\frac{1}{4}$. By Theorem 10 we get

$$x_n = -\frac{1}{a_1}\cdot\frac{1}{n}+\frac{a_1^2-a_2}{a_1^3}\cdot\frac{\ln n}{n^2}+K\cdot\frac{1}{n^2}-\frac{\left(a_1^2-a_2\right)^2}{a_1^5}\cdot\frac{\ln^2 n}{n^3}$$
$$+\frac{\left(a_1^2-a_2\right)\left(a_1^2-a_2-2a_1^3K\right)}{a_1^5}\cdot\frac{\ln n}{n^3}$$

$$-\frac{2a_1^4K^2-2a_1^3\left(a_1^2-a_2\right)K+a_1^4-a_1^2a_2+2a_1a_3-2a_2^2}{2a_1^5}\cdot\frac{1}{n^3}+o\left(\frac{1}{n^3}\right).$$

We have $a_1^2-a_2=-\frac{1}{12}$, $\dfrac{a_1^2-a_2}{a_1^3}=\frac{2}{3}$, $\dfrac{(a_1^2-a_2)^2}{a_1^5}=-\frac{2}{9}$, $a_1^4-a_1^2a_2+2a_1a_3-2a_2^2=\frac{1}{144}$, $\left(a_1^2-a_2\right)\left(a_1^2-a_2-2a_1^3K\right)=-\dfrac{3K-1}{144}$, $2a_1^4K^2-2a_1^3\left(a_1^2-a_2\right)K+a_1^4-a_1^2a_2+2a_1a_3-2a_2^2=\dfrac{18K^2-3K+1}{144}$, and therefore we get the stated evaluation. □

In the end we refine the asymptotic evaluation for the logistic map.

Corollary 12. *Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_1 \in (0,1)$ and $x_{n+1} = x_n - x_n^2$ for all $n \geq 1$. Then there exists $K \in \mathbb{R}$ such that*

$$x_n = \frac{1}{n} - \frac{\ln n}{n^2} + K \cdot \frac{1}{n^2} + \frac{\ln^2 n}{n^3} - (2K+1) \cdot \frac{\ln n}{n^3} + \frac{2K^2+2K+1}{2} \cdot \frac{1}{n^3} + o\left(\frac{1}{n^3}\right).$$

Proof. We apply Theorem 10 for the function $g : [0,1) \to \mathbb{R}$, $g(x) = x - x^2$ with $a_1 = -1$, $a_2 = a_3 = 0$. □

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17th South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2023

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Abstract. The 17th South Eastern European Mathematical Olympiad for University Students (SEEMOUS 2023) took place on March 7–12, 2023, in Struga, North Macedonia. We present the competition problems and their solutions, as given by the authors, or in the line of the official ones. Alternative solutions provided by members of the jury or by contestants are also included.

Keywords: Trace, commutator, Jacobson's Lemma, normal matrix, Schur's Triangularization Theorem, Riemann integral, integral convergence, differentiable function, sequences of real numbers, series of real numbers.

MSC: Primary 15A21; Secondary 40A05, 26A24, 26A42.

The Mathematical Society of South-Eastern Europe (MASSEE) organized in 2023 the 17th South Eastern European Mathematical Competition for University Students with International Participation (SEEMOUS 2023), which is addressed to students in the first or second year of undergraduate studies, from universities in countries that are members of MASSEE or from invited countries that are not affiliated to MASSEE.

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This year's competition was hosted between March 7–12, 2023, in Struga by the Union of Mathematicians of Macedonia and the Faculty of Natural Sciences and Mathematics, Ss. Cyril and Methodius University in Skopje. The number of students that participated in the contest was 85, representing 24 universities from Albania, Bulgaria, Greece, North Macedonia, Romania, and Turkmenistan. The jury awarded 10 gold medals, 21 silver medals and 30 bronze medals. The student Mario–Cosmin Drăguţ from University of Bucharest, Romania, was the absolute winner of the competition, being the only contestant that obtained full marks to all four problems.

We present the problems from the contest and their solutions as given by the corresponding authors, together with alternative solutions provided by members of the jury or by the contestants.

Problem 1. *Prove that if A and B are $n \times n$ square matrices with complex entries satisfying*

$$A = AB - BA + A^2B - 2ABA + BA^2 + A^2BA - ABA^2,$$

then $\det(A) = 0$.

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Authors' solution. Let k be an arbitrary positive integer. By multiplying the previous equality with A^{k-1}, we obtain

$$A^k = A^kB - A^{k-1}BA + A^{k+1}B - 2A^kBA + A^{k-1}BA^2 + A^{k+1}BA - A^kBA^2.$$

Taking the trace and considering that $\operatorname{Tr}(MN) = \operatorname{Tr}(NM)$, for all $M, N \in \mathcal{M}_n(\mathbb{C})$, we deduce

$$\begin{aligned}\operatorname{Tr}\left(A^k\right) =\ & \operatorname{Tr}\left(A^kB\right) - \operatorname{Tr}\left(\left(A^{k-1}B\right)A\right) + \operatorname{Tr}\left(A^{k+1}B\right) - 2\operatorname{Tr}\left(\left(A^kB\right)A\right)\\ & + \operatorname{Tr}\left(\left(A^{k-1}B\right)A^2\right) + \operatorname{Tr}\left(\left(A^{k+1}B\right)A\right) - \operatorname{Tr}\left(\left(A^kB\right)A^2\right)\\ =\ & 0.\end{aligned}$$

Therefore, we proved that $\operatorname{Tr}\left(A^k\right) = 0$ for any positive integer k, which implies (see [3], p. 144) that the matrix A is nilpotent, so $\det(A) = 0$.

Alternative solution. Suppose on the contrary that $\det(A) \neq 0$. By multiplying on the left the equality from the statement with A^{-1}, we obtain

$$I_n = B - A^{-1}BA + AB - 2BA + A^{-1}BA^2 + ABA - BA^2.$$

As in the previous solution, taking the trace and considering that $\operatorname{Tr}(MN) = \operatorname{Tr}(NM)$, for all $M, N \in \mathcal{M}_n(\mathbb{C})$, we deduce

$$\begin{aligned}\operatorname{Tr}(I_n) &= \operatorname{Tr}(B) - \operatorname{Tr}\left(\left(A^{-1}B\right)A\right) + \operatorname{Tr}(AB) - 2\operatorname{Tr}(BA)\\ &\quad + \operatorname{Tr}\left(A^{-1}\left(BA^2\right)\right) + \operatorname{Tr}(A(BA)) - \operatorname{Tr}\left(BA^2\right)\\ &= 0,\end{aligned}$$

which is a contradiction. Hence $\det(A) = 0$.

Second alternative solution. *The following solution was given by Carol–Luca Gasan and Horia Mercan, from University Politehnica of Bucharest (contestants).*

Let us consider the linear operator $\Delta_A : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$, with $\Delta_A(X) = AX - XA$. With this notation, the equality in the statement can be rewritten

$$A = \Delta_A(B) + \Delta_A(AB) - \Delta_A(BA) + \Delta_A(ABA),$$

so $A = \Delta_A(B + AB - BA + ABA)$. As an obvious consequence, the matrix A commute with the matrix $\Delta_A(B + AB - BA + ABA)$. Therefore, considering Jacobson's Lemma (see [1], p. 126), we obtain that the matrix $\Delta_A(B + AB - BA + ABA)$ is nilpotent, that is, the matrix A is nilpotent, and the conclusion follows immediately.

Problem 2. *For the sequence*

$$S_n = \frac{1}{\sqrt{n^2+1^2}} + \frac{1}{\sqrt{n^2+2^2}} + \cdots + \frac{1}{\sqrt{n^2+n^2}},$$

find

$$\lim_{n\to\infty} n\left(n\left(\ln\left(1+\sqrt{2}\right) - S_n\right) - \frac{1}{2\sqrt{2}\left(1+\sqrt{2}\right)}\right).$$

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Author's solution. Observe that, if f is a C^2 function on $[0,1]$, by Taylor formula, we can write

$$\int_0^1 f(x)\,\mathrm{d}x = \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,\mathrm{d}x = \sum_{k=1}^{n}\left(F\left(\frac{k}{n}\right) - F\left(\frac{k-1}{n}\right)\right)$$
$$= \sum_{k=1}^{n}\left(\frac{1}{n} f\left(\frac{k-1}{n}\right) + \frac{1}{2n^2} f'\left(\frac{k-1}{n}\right) + \frac{1}{6n^3} f''(c_{k,n})\right),$$

where F is an antiderivative of f and $c_{k,n} \in \left(\frac{k-1}{n}, \frac{k}{n}\right)$. Moreover,

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k-1}{n}\right) - \frac{f(0) - f(1)}{n},$$

hence

$$n\left(\int_0^1 f(x)\,\mathrm{d}x - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right)$$
$$= \sum_{k=1}^{n}\left(\frac{1}{2n} f'\left(\frac{k-1}{n}\right) + \frac{1}{6n^2} f''(c_{k,n})\right) + (f(0) - f(1)). \tag{1}$$

Furthermore, also by Taylor formula,

$$\begin{aligned} f(1)-f(0) &= \sum_{k=1}^{n}\left(f\left(\frac{k}{n}\right)-f\left(\frac{k-1}{n}\right)\right) \\ &= \sum_{k=1}^{n}\left(\frac{1}{n}f'\left(\frac{k-1}{n}\right)+\frac{1}{2n^2}f''(d_{k,n})\right), \qquad (2) \end{aligned}$$

with $d_{k,n}\in\left(\frac{k-1}{n},\frac{k}{n}\right)$. By combining relations (1) and (2), we obtain

$$\begin{aligned} & n\left(n\left(\int_0^1 f(x)\,\mathrm{d}x-\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\right)-\frac{f(0)-f(1)}{2}\right) \\ &= n\left(\sum_{k=1}^{n}\left(\frac{1}{2n}f'\left(\frac{k-1}{n}\right)+\frac{1}{6n^3}f''(c_{k,n})\right)-\frac{f(1)-f(0)}{2}\right) \\ &= n\left(\sum_{k=1}^{n}\left(\frac{1}{6n^2}f''(c_{k,n})-\frac{1}{4n^2}f''(d_{k,n})\right)\right) \\ &= \frac{1}{6}\cdot\frac{1}{n}\sum_{k=1}^{n}f''(c_{k,n})-\frac{1}{4}\cdot\frac{1}{n}\sum_{k=1}^{n}f''(d_{k,n}), \end{aligned}$$

which converges to

$$\frac{1}{6}\int_0^1 f''(x)\,\mathrm{d}x-\frac{1}{4}\int_0^1 f''(x)\,\mathrm{d}x=-\frac{1}{12}\left(f'(1)-f'(0)\right).$$

In the particular case of $f(x)=\frac{1}{\sqrt{1+x^2}}$, we obtain that

$$\begin{aligned} \int_0^1 f(x)\,\mathrm{d}x &= \ln\left(1+\sqrt{2}\right), \\ S_n &= \frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right), \\ \frac{f(0)-f(1)}{2} &= \frac{1}{2\sqrt{2}\left(\sqrt{2}+1\right)}, \end{aligned}$$

i.e., the requested limit can be obtained by the procedure described before. Therefore, the result is

$$-\frac{1}{12}\left(f'(1)-f'(0)\right)=\frac{1}{24\sqrt{2}}.$$

Remark. This problem is closely related to a well-known result (see [2], p. 7), stating that if f is a C^1 function on $[0,1]$, then

$$\lim_{n\to\infty} n\left(\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)-\int_0^1 f(x)\,\mathrm{d}x\right)=\frac{f(1)-f(0)}{2},$$

whose proof uses the Taylor expansion on each interval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ for an antiderivative of f. So, the idea and the path of the proof described before should naturally have been visible, and for this reason the problem was considered to be of easy to medium difficulty. The presented proof follows the author's official solution, but tries to better emphasize the natural way to solve the problem.

Problem 3. *Prove that: if A is $n \times n$ square matrix with complex entries such that $A + A^* = A^2 A^*$, then $A = A^*$. (For any matrix M, denote by $M^* = \overline{M}^t$, the conjugate transpose of M.)*

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Authors' solution. We will show first that the matrix A is normal, i.e., $AA^* = A^*A$. The equality $A + A^* = A^2 A^*$ leads to

$$A = \left(A^2 - I_n\right) A^*, \tag{3}$$

wherefrom $A \pm I_n = (A - I_n)(A + I_n) A^* \pm I_n$, so

$$(A - I_n)\left[(A + I_n) A^* - I_n\right] = I_n \quad \text{and} \quad (A + I_n)\left[I_n - (A - I_n) A^*\right] = I_n,$$

relations that imply that the matrices $A - I_n$ and $A + I_n$ are invertible. Then $A^2 - I_n$ is also invertible and by (3) it follows that $A^* = \left(A^2 - I_n\right)^{-1} A$. On the other hand, using the Cayley–Hamilton theorem, we get that $\left(A^2 - I_n\right)^{-1}$ is a polynomial of $A^2 - I_n$, hence a polynomial of A, so $AA^* = A^*A$.

Since the matrix A is normal, it is unitarily diagonalizable, that is, there exists an unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ and a diagonal matrix $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ such that $A = UDU^*$. Then $A^* = U\overline{D}U^*$, which, due to the equality in the statement, leads to $D + \overline{D} = D^2 \overline{D}$, from where we find that $\lambda_i + \overline{\lambda_i} = \lambda_i^2 \overline{\lambda_i}$, for all $i \in \overline{1,n}$. Then $2\operatorname{Re}(\lambda_i) = \lambda_i |\lambda_i|^2$, so $\lambda_i \in \mathbb{R}$, for all $i \in \overline{1,n}$, hence $D = \overline{D}$ and the conclusion follows immediately.

Alternative solution. *The following solution was given by George–Vlad Manolache and Horia Mercan, from University Politehnica of Bucharest (contestants).*

By multiplying on the right with A^*, the equality in the statement implies that $AA^* + (A^*)^2 = A^2 (A^*)^2$, so the conjugate transposes of the matrices in the two members of the previous equality are also equal, that is, $AA^* + A^2 = A^2 (A^*)^2$, hence $A^2 = \left(A^2\right)^*$. By multiplying on the left with A^*, the equality in the statement implies that $A^*A + (A^*)^2 = A^*A^2A^*$ and, considering that $A^2 = \left(A^2\right)^*$, we obtain that $A^*A + (A^*)^2 = (A^*)^4 = A^2 (A^*)^2 = AA^* + (A^*)^2$, so $A^*A = AA^*$, which means that the matrix A is normal.

As in the previous solution, there exists an unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ and a diagonal matrix $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ such that $A = UDU^*$. From

$A^2 = \left(A^2\right)^*$ we obtain that $D^2 = \left(D^2\right)^*$, that is $\lambda_i^2 = \overline{\lambda_i^2}$, so $\lambda_i^2 \in \mathbb{R}$, for all $i \in \overline{1,n}$.

Considering that $A^2 = \left(A^2\right)^*$, the equality in the statement also implies that $A + A^* = (A^*)^3$, so $(A + A^*)^* = A^3$, that is $A + A^* = A^3$, which means that $A^3 = \left(A^3\right)^*$. Thus $D^3 = \left(D^3\right)^*$, that is $\lambda_i^3 = \overline{\lambda_i^3}$, so $\lambda_i^3 \in \mathbb{R}$, for all $i \in \overline{1,n}$.

Taking into account those established in the last two paragraphs, we obtain that $\lambda_i \in \mathbb{R}$, for all $i \in \overline{1,n}$, and, as in the previous solution, the conclusion follows immediately.

Second alternative solution. *The following solution of the fact that the matrix A is normal was given by Carol–Luca Gasan, from University Politehnica of Bucharest (contestant).*

It can be shown (for example, as in the previous solution) that $A^2 = (A^*)^2$. According to Schur's Triangularization Theorem (see [1], p. 101), there exists an unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ and an upper triangular matrix $T \in \mathcal{M}_n(\mathbb{C})$ such that $A = UTU^*$. It immediately follows that $T^2 = (T^*)^2$. On the other hand, the equality in the statement implies that $T + T^* = T^2T^*$, so $T = (T^*)^3 - T^*$, which means that the matrix T is normal. Since T is an upper triangular matrix, it follows (see [1], p. 132) that T is a diagonal matrix and the conclusion is obvious.

Third alternative solution. *The following solution of the fact that the matrix A is normal was given by Alexandru–Constantin Ariton, from University Politehnica of Bucharest (contestant).*

The equality in the statement implies that $(A + A^*)^* = \left(A^2A^*\right)^*$, so $A + A^* = A\,(A^*)^2$. It follows that $A^2A^* = A\,(A^*)^2$.

By multiplying on the right with A, the equality in the statement implies that $A^2 + A^*A = A^2A^*A$. By multiplying on the left with A, the equality in the statement implies that $A^2 + AA^* = A^3A^*$. Subtracting the last equality from the previous one, we obtain that $A^*A - AA^* = A^2\,(A^*A - AA^*)$.

Suppose on the contrary that there exists $\lambda \neq 0$, eigenvalue of the matrix $A^*A - AA^*$, and $\mathbf{x} \neq \mathbf{0}$, a corresponding eigenvector. That is, $(A^*A - AA^*)\,\mathbf{x} = \lambda\mathbf{x}$, wherefrom we obtain that $A^2\,(A^*A - AA^*)\,\mathbf{x} = \lambda\mathbf{x}$, so $\lambda A^2\mathbf{x} = \lambda\mathbf{x}$. Hence $A^2\mathbf{x} = \mathbf{x}$, which means that $\omega = 1$ is an eigenvalue of the matrix A^2, so $\omega = 1$ is also an eigenvalue of the matrix $(A^*)^2$. Hence there exists $\mathbf{y} \neq \mathbf{0}$ such that $(A^*)^2\,\mathbf{y} = \mathbf{y}$.

On the other hand, the equality $A + A^* = A\,(A^*)^2$ implies that $A\mathbf{y} + A^*\mathbf{y} = A\,(A^*)^2\,\mathbf{y}$. Then $A\mathbf{y} + A^*\mathbf{y} = A\mathbf{y}$, so $A^*\mathbf{y} = \mathbf{0}$. As an immediate consequence, we obtain that $\mathbf{y} = (A^*)^2\,\mathbf{y} = \mathbf{0}$, which is a contradiction. Hence, all the eigenvalues of the self-adjoint matrix $A^*A - AA^*$ are equal to zero, which means that $A^*A - AA^* = O_n$, that is, the matrix A is normal.

Problem 4. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous, strictly decreasing function such that $f([0,1])\subset[0,1]$.*

(a) *For all $n\in\mathbb{N}^*$, prove that there exists a unique $a_n\in(0,1)$, solution of the equation*

$$f(x)=x^n. \tag{4}$$

Moreover, if (a_n) is the sequence defined as above, prove that $\lim_{n\to\infty} a_n=1$.

(b) *Suppose f is C^1 with $f(1)=0$ and $f'(1)<0$. For any $x\in\mathbb{R}$, we define*

$$F(x)=\int_x^1 f(t)\ \mathrm{d}t.$$

Study the convergence of the series $\sum_{n=1}^{\infty}(F(a_n))^{\alpha}$, with $\alpha\in\mathbb{R}$.

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Author's solution. (a) Consider the continuous function $g:[0,1]\to\mathbb{R}$ given by $g(x)=f(x)-x^n$, and observe that $g(0)=f(0)>0$, and $g(1)=f(1)-1<0$. It follows the existence of $a_n\in(0,1)$ such that $g(a_n)=0$. For uniqueness, observe that if would exist two solutions of the equation (4), say $a_n<b_n$, we would obtain

$$f(a_n)>f(b_n)\Leftrightarrow a_n^n>b_n^n\Leftrightarrow a_n>b_n,$$

a contradiction.

We prove that the sequence (a_n) is strictly increasing. If it would exist $n\in\mathbb{N}^*$ such that $a_n\geq a_{n+1}$, we would obtain that

$$f(a_n)\leq f(a_{n+1})\Leftrightarrow a_n^n\leq a_{n+1}^{n+1}<a_{n+1}^n,$$

since f is strictly decreasing and $a_{n+1}\in(0,1)$. It follows that $a_n<a_{n+1}$, a contradiction. Hence, (a_n) is strictly increasing and bounded above by 1, so it converges to $\ell\in(0,1]$. Suppose, by contradiction, that $\ell<1$. Since $f(a_n)=a_n^n$ for any n, using the continuity of f it follows that $f(\ell)=0$ for $\ell<1$, contradicting the fact that f is strictly decreasing with $f(1)\geq 0$. Hence, $\lim_{n\to\infty} a_n=1$.

(b) Observe that F is well-defined, of class C^2, with $F(1)=0$, $F'(x)=-f(x)\Rightarrow F'(1)=0$, $F''(x)=-f'(x)\Rightarrow F''(1)>0$. Moreover, notice that $F(x)>0$ on $[0,1)$. Using the Taylor formula on the interval $[a_n,1]$, it follows that for any n, there exist $c_n,d_n\in(a_n,1)$ such that

$$\begin{aligned} F(a_n)&=F(1)+F'(1)(a_n-1)+\frac{F''(c_n)}{2}(a_n-1)^2=\frac{F''(c_n)}{2}(a_n-1)^2,\\ f(a_n)&=f(1)+f'(d_n)(a_n-1)=f'(d_n)(a_n-1). \end{aligned} \tag{5}$$

Hence, since $c_n \to 1$ and F is C^2, we obtain

$$\lim_{n\to\infty} \frac{(1-a_n)^2}{F(a_n)} = \frac{2}{F''(1)} \in (0,\infty),$$

so, due to the comparison test,

$$\sum_{n=1}^{\infty} (F(a_n))^\alpha \sim \sum_{n=1}^{\infty} (1-a_n)^{2\alpha}.$$

But

$$\begin{aligned}\lim_{n\to\infty} n(1-a_n) &= -\lim_{n\to\infty} n \cdot \frac{(a_n-1)}{\ln(1+(a_n-1))} \cdot \ln a_n \\ &= -\lim_{n\to\infty} \ln a_n^n = -\lim_{n\to\infty} \ln f(a_n) = -\ln\left(\lim_{n\to\infty} f(a_n)\right) = \infty.\end{aligned}$$

It follows that $\sum_{n=1}^{\infty}(1-a_n)$ diverges and, furthermore, $\sum_{n=1}^{\infty}(1-a_n)^{2\alpha}$ diverges for any $2\alpha \le 1$.

Remark that

$$\frac{n(1-a_n)}{-\ln f(a_n)} = -\frac{(a_n-1)}{\ln(1+(a_n-1))} \cdot \frac{n\ln a_n}{-\ln f(a_n)} = \frac{(a_n-1)}{\ln(1+(a_n-1))} \to 1.$$

Next, consider arbitrary $\gamma \in (0,1)$. Using (5) and the fact that $d_n \to 1$, we obtain

$$\begin{aligned}\lim_{n\to\infty} n^\gamma(1-a_n) &= \lim_{n\to\infty} [n(1-a_n)]^\gamma \cdot (1-a_n)^{1-\gamma} \\ &= \lim_{n\to\infty} [n(1-a_n)]^\gamma \cdot \left[\frac{f(a_n)}{-f'(d_n)}\right]^{1-\gamma} \\ &= \frac{1}{(-f'(1))^{1-\gamma}} \cdot \lim_{n\to\infty} [-\ln f(a_n)]^\gamma \cdot \left[e^{\ln f(a_n)}\right]^{1-\gamma}.\end{aligned}$$

Observe that

$$-\ln f(a_n) \to \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{x^\gamma}{e^{(1-\gamma)x}} = 0,$$

hence $\lim_{n\to\infty} n^\gamma(1-a_n) = 0$. So, if $\alpha > \frac{1}{2}$, we obtain that there exists $\varepsilon > 0$ such that $2\alpha > 1+\varepsilon$, hence for $\gamma := \frac{1+\varepsilon}{2\alpha} < 1$, we get

$$\lim_{n\to\infty} n^{2\alpha\gamma}(1-a_n)^{2\alpha} = \lim_{n\to\infty} n^{(1+\varepsilon)}(1-a_n)^{2\alpha} = 0.$$

Using the comparison test, it follows that the series $\sum_{n=1}^{\infty}(1-a_n)^{2\alpha}$ converges.

In conclusion, the series $\sum_{n=1}^{\infty}(F(a_n))^\alpha$ converges iff $\alpha > \frac{1}{2}$.

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MATHEMATICAL NOTES

A series involving a product of three consecutive harmonic numbers

OVIDIU FURDUI[1], ALINA SÎNTĂMĂRIAN[2]

Abstract. In this paper we calculate the following remarkable series $\sum_{n=1}^{\infty} \frac{H_n H_{n+1} H_{n+2}}{n(n+1)(n+2)}$, where H_n denotes the nth harmonic number. We prove that this series equals a rational linear combination of $\zeta(2)$, $\zeta(3)$, and $\zeta(4)$.

Keywords: Harmonic series, product of harmonic numbers, Euler's series, Goldbach's series, Riemann zeta function values.

MSC: 40A05, 40C99

1. INTRODUCTION AND THE MAIN RESULT

In this paper we calculate the remarkable series $\sum\limits_{n=1}^{\infty} \frac{H_n H_{n+1} H_{n+2}}{n(n+1)(n+2)}$, where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the nth harmonic number. This paper is motivated by the following series formula (see [1]) involving the product of two consecutive harmonic numbers

$$\sum_{n=1}^{\infty} \frac{H_n H_{n+1}}{n(n+1)} = \zeta(2) + 2\zeta(3). \tag{1}$$

We extend this result by calculating the corresponding series involving the product of consecutive harmonic numbers H_n, H_{n+1} and H_{n+2}.

The main result of this paper, which answers to an open problem posed in [3, pag. 119], is the following theorem.

Theorem 1. *The following identity holds*

$$\sum_{n=1}^{\infty} \frac{H_n H_{n+1} H_{n+2}}{n(n+1)(n+2)} = -\frac{1}{2}\zeta(2) + \frac{5}{4}\zeta(3) + \frac{5}{8}\zeta(4).$$

Before we prove the theorem we compile the following lemma.

Lemma 2. ***An Euler and a Goldbach series***

The following identities hold:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$;

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(b) $\displaystyle\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4}\zeta(4).$

Proof. A proof of the lemma can be found in [3, pp. 238–240]. The two parts of the lemma also appear as problems 3.55 and 3.58 in [2]. We mention that the series in part (a) is due to Euler and the sum in part (b) is due to Goldbach, and thus we correct the statement of problem 3.58 in [2], where the problem is wrongly attributed to Klamkin. □

Proof. (Proof of Theorem 1.) Let $x_n = \frac{H_n H_{n+1} H_{n+2}}{n(n+1)(n+2)}$. We calculate the series $\sum_{n=1}^{\infty} x_n$ by showing that it telescopes. We have

$$\begin{aligned}
\frac{H_n H_{n+1} H_{n+2}}{n(n+1)(n+2)} &= \frac{H_n H_{n+1} H_{n+2}}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)\\
&= \frac{1}{2}\left[\frac{H_n H_{n+1} H_{n+2}}{n(n+1)} - \frac{H_{n+1} H_{n+2}}{(n+1)(n+2)}\left(H_{n+3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)\right]\\
&= \frac{1}{2}\left(\frac{H_n H_{n+1} H_{n+2}}{n(n+1)} - \frac{H_{n+1} H_{n+2} H_{n+3}}{(n+1)(n+2)}\right)\\
&\quad + \frac{1}{2}\left(\frac{H_{n+1} H_{n+2}}{(n+1)^2(n+2)} + \frac{H_{n+1} H_{n+2}}{(n+1)(n+2)^2} + \frac{H_{n+1} H_{n+2}}{(n+1)(n+2)(n+3)}\right).
\end{aligned}$$

We calculate

$$\begin{aligned}
&\frac{H_{n+1} H_{n+2}}{(n+1)^2(n+2)} + \frac{H_{n+1} H_{n+2}}{(n+1)(n+2)^2}\\
&= H_{n+1} H_{n+2}\left(\frac{1}{(n+1)^2} - \frac{1}{(n+1)(n+2)}\right)\\
&\quad + H_{n+1} H_{n+2}\left(\frac{1}{(n+1)(n+2)} - \frac{1}{(n+2)^2}\right)\\
&= \frac{H_{n+1} H_{n+2}}{(n+1)^2} - \frac{H_{n+1} H_{n+2}}{(n+2)^2}\\
&= \frac{H_{n+1}\left(H_{n+1} + \frac{1}{n+2}\right)}{(n+1)^2} - \frac{\left(H_{n+2} - \frac{1}{n+2}\right) H_{n+2}}{(n+2)^2}\\
&= \frac{H_{n+1}^2}{(n+1)^2} - \frac{H_{n+2}^2}{(n+2)^2} + \frac{H_{n+1}}{(n+1)^2(n+2)} + \frac{H_{n+2}}{(n+2)^3}\\
&= \frac{H_{n+1}^2}{(n+1)^2} - \frac{H_{n+2}^2}{(n+2)^2} + \frac{H_{n+2}}{(n+2)^3} + H_{n+1}\left(\frac{1}{(n+1)^2} - \frac{1}{(n+1)(n+2)}\right)\\
&= \frac{H_{n+1}^2}{(n+1)^2} - \frac{H_{n+2}^2}{(n+2)^2} + \frac{H_{n+2}}{(n+2)^3} + \frac{H_{n+1}}{(n+1)^2} - H_{n+1}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)
\end{aligned}$$

$$= \frac{H_{n+1}^2}{(n+1)^2} - \frac{H_{n+2}^2}{(n+2)^2} + \frac{H_{n+2}}{(n+2)^3} + \frac{H_{n+1}}{(n+1)^2} - \left(\frac{H_{n+1}}{n+1} - \frac{H_{n+2} - \frac{1}{n+2}}{n+2}\right)$$

$$= \frac{H_{n+1}^2}{(n+1)^2} - \frac{H_{n+2}^2}{(n+2)^2} + \frac{H_{n+2}}{(n+2)^3} + \frac{H_{n+1}}{(n+1)^2} - \left(\frac{H_{n+1}}{n+1} - \frac{H_{n+2}}{n+2}\right) - \frac{1}{(n+2)^2}.$$

Now we calculate

$$\frac{H_{n+1}H_{n+2}}{(n+1)(n+2)(n+3)} = \frac{H_{n+1}H_{n+2}}{2}\left(\frac{1}{(n+1)(n+2)} - \frac{1}{(n+2)(n+3)}\right)$$

$$= \frac{1}{2}\left[\frac{H_{n+1}H_{n+2}}{(n+1)(n+2)} - \frac{H_{n+2}\left(H_{n+3} - \frac{1}{n+2} - \frac{1}{n+3}\right)}{(n+2)(n+3)}\right]$$

$$= \frac{1}{2}\left(\frac{H_{n+1}H_{n+2}}{(n+1)(n+2)} - \frac{H_{n+2}H_{n+3}}{(n+2)(n+3)} + \frac{H_{n+2}}{(n+2)^2(n+3)} + \frac{H_{n+2}}{(n+2)(n+3)^2}\right). \tag{2}$$

We have

$$\frac{H_{n+2}}{(n+2)^2(n+3)} + \frac{H_{n+2}}{(n+2)(n+3)^2} = H_{n+2}\left(\frac{1}{(n+2)^2} - \frac{1}{(n+2)(n+3)}\right) + H_{n+2}\left(\frac{1}{(n+2)(n+3)} - \frac{1}{(n+3)^2}\right)$$

$$= \frac{H_{n+2}}{(n+2)^2} - \frac{H_{n+2}}{(n+3)^2}$$

$$= \frac{H_{n+2}}{(n+2)^2} - \frac{H_{n+3}}{(n+3)^2} + \frac{1}{(n+3)^3}. \tag{3}$$

Combining (2) and (3) we have that

$$\frac{H_{n+1}H_{n+2}}{(n+1)(n+2)(n+3)} = \frac{1}{2}\left[\frac{H_{n+1}H_{n+2}}{(n+1)(n+2)} - \frac{H_{n+2}H_{n+3}}{(n+2)(n+3)} + \frac{H_{n+2}}{(n+2)^2} - \frac{H_{n+3}}{(n+3)^2} + \frac{1}{(n+3)^3}\right].$$

It follows that

$$\begin{aligned} x_n &= \frac{1}{2}\left(\frac{H_nH_{n+1}H_{n+2}}{n(n+1)} - \frac{H_{n+1}H_{n+2}H_{n+3}}{(n+1)(n+2)}\right) \\ &+ \frac{1}{4}\left(\frac{H_{n+1}H_{n+2}}{(n+1)(n+2)} - \frac{H_{n+2}H_{n+3}}{(n+2)(n+3)}\right) \\ &+ \frac{1}{4}\left(\frac{H_{n+2}}{(n+2)^2} - \frac{H_{n+3}}{(n+3)^2}\right) + \frac{1}{4(n+3)^3} \\ &+ \frac{1}{2}\left(\frac{H_{n+1}^2}{(n+1)^2} - \frac{H_{n+2}^2}{(n+2)^2}\right) + \frac{H_{n+1}}{2(n+1)^2} + \frac{H_{n+2}}{2(n+2)^3} \\ &- \frac{1}{2}\left(\frac{H_{n+1}}{n+1} - \frac{H_{n+2}}{n+2}\right) - \frac{1}{2(n+2)^2}. \end{aligned}$$

This implies, based on Lemma 2, that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_nH_{n+1}H_{n+2}}{n(n+1)(n+2)} &= \frac{1}{2}\cdot\frac{H_1H_2H_3}{2} + \frac{1}{4}\cdot\frac{H_2H_3}{6} + \frac{1}{4}\cdot\frac{H_3}{9} \\ &+ \frac{1}{4}\left(\zeta(3) - 1 - \frac{1}{2^3} - \frac{1}{3^3}\right) + \frac{1}{2}\cdot\frac{H_2^2}{4} \\ &+ \frac{1}{2}\left(2\zeta(3) - \frac{H_1}{1^2}\right) + \frac{1}{2}\left(\frac{5}{4}\zeta(4) - \frac{H_1}{1^3} - \frac{H_2}{2^3}\right) \\ &- \frac{1}{2}\cdot\frac{H_2}{2} - \frac{1}{2}\left(\zeta(2) - 1 - \frac{1}{2^2}\right) \\ &= -\frac{1}{2}\zeta(2) + \frac{5}{4}\zeta(3) + \frac{5}{8}\zeta(4), \end{aligned}$$

and the theorem is proved. □

Motivated by formula (1) and by Theorem 1 we conjecture that, if $k \geq 1$ is an integer, then

$$\sum_{n=1}^{\infty} \frac{H_nH_{n+1}\cdots H_{n+k}}{n(n+1)\cdots(n+k)} = a_{k,2}\zeta(2) + a_{k,3}\zeta(3) + \cdots + a_{k,k+2}\zeta(k+2),$$

where $a_{k,i}$, $i = 2, \ldots, k+2$, are rational numbers.

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PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of November 2023**.

PROPOSED PROBLEMS

536. Let p be a prime number, $\mathbb{F}_p$ the field with p elements, and $n \geq 1$ an integer. If $f \in \mathbb{F}_p[X]$ is the polynomial $X^p - X \in \mathbb{F}_p[X]$ composed with itself n times, determine the splitting field of f over $\mathbb{F}_p$.

Proposed by Tudor Păișanu, École Polytechnique, Paris, France.

537. Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be such that $A^2 = A$ and $B^2 = B$.

Prove that

$$\operatorname{Im}(AB - BA) = \operatorname{Im}(A + B - I_n) \cap \operatorname{Im}(A - B),$$

where $\operatorname{Im} M = \{MX \mid X \in \mathcal{M}_{n,1}(\mathbb{C})\}$ for every $M \in \mathcal{M}_n(\mathbb{C})$.

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania.

538. Let $n \geq 1$ be an integer and let $a_1, \ldots, a_{2n} \in \mathbb{Z}$ be pairwise distinct. Prove that

$$\sum_{i=1}^{2n} a_i^2 + \left(\sum_{i=1}^{2n} a_i\right)^2 \geq \frac{n(n+1)(2n+1)}{3}.$$

When do we have equality?

Proposed by Leonard Giugiuc, Traian National College, Drobeta-Turnu Severin, Romania.

539. Let $n \geq 1$ be an integer and let $X = \{1, \ldots, n\}$. We denote by F_X the set of all functions $f : X \to X$ and by S_X the symmetric group on X, i.e. the set of all permutations on X. If $f, g \in F_X$, we say that f and g are conjugate and we write $f \sim g$ if there is $\sigma \in S_X$ such that $g = \sigma f \sigma^{-1}$.

Let M_X be the set of all $f \in F_X$ such that for every $\emptyset \neq Y \subseteq X$ with $f(Y) \subseteq Y$ we have $f(Y) = f(X)$.

(i) Prove that if $f \in M_X$ and $g \sim f$ then $g \in M_X$.

(ii) Prove that $|M_X/\sim| = \frac{1}{n} \sum_{d|n} \phi(d)\, 2^{n/d} - 1$.

Proposed by Constantin-Nicolae Beli, IMAR, Bucharest, Romania.

540. For any matrix M, denote $M^* = \overline{M}^t$ the transpose conjugate of M. Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be such that $A^*B = O_n$. Prove that

$$\operatorname{rank}(A^*A + B^*B) \leq \operatorname{rank}(AA^* + BB^*).$$

Proposed by Mihai Opincariu, Brad, Romania, and Vasile Pop, Technical University of Cluj-Napoca, Romania.

541. Calculate

$$\sum_{n=1}^{\infty} \frac{H_n H_{n+1}}{(2n+1)(2n+3)},$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the nth harmonic number.

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Romania.

542. Let $n \geq 2$ be an integer and let $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x_1, \ldots, x_n) = 0$ if $(x_1, \ldots, x_n) = (0, \ldots, 0)$ and

$$f(x_1, \ldots, x_n) = \frac{\sqrt[5]{x_1^4 \cdots x_n^4}}{\sqrt[3]{x_1^2 \cdots x_n^2} + (x_2 - x_1)^2 + (x_3 - x_1)^2 + \cdots + (x_n - x_1)^2}$$

otherwise.

Prove that:

(i) f is continuous at $(0, \ldots, 0)$.

(ii) f is Fréchet differentiable at $(0, \ldots, 0)$ if and only if $n \geq 8$.

Proposed by Dumitru Popa, Department of Mathematics, Ovidius University of Constanța, Romania.

SOLUTIONS

521. Prove that
$$\lim_{p\to\infty}\sum_{i,j=0}^{p}\frac{(-4a^2)^{i+j}}{2i+2j+1}\binom{p+i}{p-i}\binom{p+j}{p-j}=\frac{1}{4a}\ln\left(\frac{1+a}{1-a}\right)\qquad \forall a\in(0,1).$$

Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.

Solution by the author. We denote our sum by $S_p(a)$. Then
$$\begin{aligned}S_p(a)&=\frac{1}{a}\int_0^a\sum_{i,j=0}^{p}(-4x^2)^{i+j}\binom{p+i}{p-i}\binom{p+j}{p-j}\mathrm{d}x\\&=\frac{1}{a}\int_0^a\left(\sum_{i=0}^{p}(-4x^2)^i\binom{p+i}{p-i}\right)^2\mathrm{d}x.\end{aligned}$$

We make the change of index $k=p-i$ and we get
$$\sum_{i=0}^{p}(-4x^2)^i\binom{p+i}{p-i}=\sum_{i=0}^{p}(-1)^{p-k}\binom{2p-k}{k}(2x)^{2p-2k}=(-1)^pU_{2p},$$
where U_n is the Chebyshev polynomial of the second kind, which is defined by the relation $U_n(\cos\theta)\sin\theta=\sin(n+1)\theta$ for all $\theta\in\mathbb{R}$, and writes explicitly as
$$U_n(x)=\sum_{k=0}^{[n/2]}(-1)^k\binom{n-k}{k}(2x)^{n-2k}.$$

Hence $S_p(a)=\frac{1}{a}\int_0^a U_{2p}^2\mathrm{d}x$. We put $I_n=\int_0^a U_n^2(x)\mathrm{d}x$ and make the substitution $x=\cos u$. Since $\sin(n+1)u=\sin uU_n(\cos u)$, we get
$$I_n=\int_{\pi/2}^{\theta}U_n^2(\cos u)(-\sin u)\mathrm{d}u=\int_{\theta}^{\pi/2}\frac{\sin^2(n+1)u}{\sin u}\mathrm{d}u,$$
where $\theta=\arccos u$. We have $I_0=\int_\theta^{\pi/2}\sin u\mathrm{d}u=\cos u$ and for $n\geq 1$ it holds
$$\begin{aligned}\sin^2(n+1)u-\sin^2 nu&=\frac{1}{2}(1-\cos 2(n+1)u)-\frac{1}{2}(1-\cos 2nu)\\&=\sin u\sin(2n+1)u,\end{aligned}$$
so we obtain
$$\begin{aligned}I_n-I_{n-1}&=\int_\theta^{\pi/2}\frac{\sin^2(n+1)u-\sin^2 nu}{\sin u}\mathrm{d}u=\int_\theta^{\pi/2}\sin(2n+1)u\mathrm{d}u\\&=\frac{\cos(2n+1)\theta}{2n+1}.\end{aligned}$$

It follows that

$$I_n = I_0 + (I_1 - I_0) + \cdots + (I_n - I_{n-1}) = \cos\theta + \frac{\cos 3\theta}{3} + \cdots + \frac{\cos(2n+1)\theta}{2n+1}$$
$$= \cos\theta + \frac{\cos 2\theta}{2} + \cdots + \frac{\cos(2n+1)\theta}{2n+1} - \frac{\cos 2\theta}{2} - \frac{\cos 4\theta}{4} - \cdots - \frac{\cos 2n\theta}{2n}.$$

By using the formula $\sum_{k\geq 1} \frac{\cos k\alpha}{k} = -\ln\left(2\sin\frac{\alpha}{2}\right)$, we get

$$\lim_{n\to\infty} I_n = -\ln\left(2\sin\frac{\theta}{2}\right) + \frac{1}{2}\ln(2\sin\theta) = -\frac{1}{2}\ln\left(4\sin^2\frac{\theta}{2}\right) + \frac{1}{4}\ln(4\sin^2\theta)$$
$$= -\frac{1}{2}\ln(2(1-\cos\theta)) + \frac{1}{4}\ln(4(1-\cos^2\theta)) = \frac{1}{4}\ln\left(\frac{4(1-\cos^2\theta)}{4(1-\cos\theta)^2}\right)$$
$$= \frac{1}{4}\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right) = \frac{1}{4}\ln\left(\frac{1+a}{1-a}\right).$$

It follows that

$$\lim_{p\to\infty} S_p(a) = \lim_{p\to\infty}\frac{1}{a}I_{2p} = \frac{1}{4a}\ln\left(\frac{1+a}{1-a}\right).$$

Editor's note. One may write U_n^2 in terms of the first Chebyshev polynomials of the first kind as $U_n^2(x) = \frac{1-T_{2n+2}(x)}{2(1-x^2)}$. This follows from the fact that

$$U_n^2(\cos\theta) = \sin^2((n+1)\theta)/\sin^2\theta = (1-\cos(2n+2)\theta)/2\sin^2\theta$$
$$= \frac{1-T_{2n+2}(\cos\theta)}{2(1-\cos^2\theta)}.$$

We get

$$S_p(a) = \int_0^a U_{2p}^2(x)\mathrm{d}x = \frac{1}{a}\int_0^a \frac{1}{2(1-x^2)}\mathrm{d}x - \frac{1}{a}\int_0^a \frac{T_{4p+2}(x)}{2(1-x^2)}\mathrm{d}x$$
$$= \frac{1}{4a}\ln\left(\frac{1+a}{1-a}\right) - \frac{1}{a}\int_0^a \frac{T_{4p+2}(x)}{2(1-x^2)}\mathrm{d}x.$$

To complete the proof, we show that $\lim_{n\to\infty}\int_0^a \frac{T_n(x)}{1-x^2}\mathrm{d}x = 0$. We have

$$\int_0^a \frac{T_n(x)}{1-x^2}\mathrm{d}x = \int_0^a \frac{\cos(n\arccos x)}{1-x^2}\mathrm{d}x.$$

We use integration by parts. Let $f(x) = \sin(n\arccos x)$. Then $f'(x) = -\frac{n}{\sqrt{1-x^2}}\cos(n\arccos x)$, so $\frac{\cos(n\arccos x)}{1-x^2} = f'(x)g(x)$, with $g(x) = -\frac{1}{n\sqrt{1-x^2}}$.

Since $g'(x) = \frac{4}{n}x\sqrt{1-x^2}$, we get

$$\int_0^a \frac{\cos(n\arccos x)}{1-x^2}\mathrm{d}x = -\left.\frac{\sin(n\arccos x)}{n\sqrt{1-x^2}}\right|_0^a - \int_0^a \frac{4}{n}x\sqrt{1-x^2}\sin(n\arccos x)\mathrm{d}x.$$

To conclude the proof, we show that if a_n and b_n are the two terms in the right side of the above formula, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. We have

$$|a_n| \leq \left|\frac{\sin(n\arccos x)}{n\sqrt{1-a^2}}\right| + \left|\frac{\sin(n\pi/2)}{n}\right| \leq \frac{1}{n\sqrt{1-a^2}} + \frac{1}{n},$$

which proves that $\lim_{n\to\infty} a_n = 0$, and

$$|b_n| \leq \int_0^a \left|\frac{4}{n}x\sqrt{1-x^2}\sin(n\arccos x)\right|\mathrm{d}x \leq \int_0^a \frac{4}{n}x\sqrt{1-x^2}\,\mathrm{d}x = \frac{4}{n}x\int_0^a \sqrt{1-x^2}\mathrm{d}x,$$

whence the conclusion that $\lim_{n\to\infty} a_n = 0$.

We encounter essentially this approach in a solution we received from G. C. Greubel, from Newport News, VA, USA. Instead of Chebyshev polynomials, he uses the Fibonacci and Lucas polynomials F_n and L_n, which are related to the Chebyshev polynomials of the first and second kind by $F_n(x) = i^{n-1}U_{n-1}(-ix/2)$ and $L_n(x) = 2i^nT_n(-ix/2)$.

He starts by writing

$$S_p = \frac{1}{2a}\int_0^{2a}\left(\sum_{j=0}^{p}\binom{p+j}{p-j}(ix)^{2j}\right)^2\mathrm{d}x = \frac{1}{2a}\int_0^{2a}F_{2p+1}^2(ix)\mathrm{d}x.$$

Then he uses the formula $F_n^2(x) = \dfrac{L_{2n}(x) - 2(-1)^n}{x^2+4}$ to get $F_{2p+1}^2(ix) = \dfrac{L_{4p+2}-2}{x^2-4}$. (Note that L_{4p+2} is an even function.) This gives

$$S_p = \frac{1}{2a}\int_0^{2a}\frac{L_{4p+2}(it)}{x^2-4}\mathrm{d}x - \frac{1}{a}\int_0^{2a}\frac{\mathrm{d}x}{x^2-4} = \frac{1}{2a}\int_0^{2a}\frac{L_{4p+2}(it)}{x^2-4}\mathrm{d}x + \frac{1}{4a}\log\left(\frac{1+a}{1-a}\right).$$

From here he noticed that $L_{4p+2}(ix) = -T_{2p+2}(x/2)$, but he failed to provide a correct proof of the fact that

$$\lim_{p\to\infty}\frac{1}{2a}\int_0^{2a}\frac{L_{4p+2}(ix)}{x^2-4}\mathrm{d}x = \lim_{p\to\infty} -\frac{1}{2a}\int_0^{2a}\frac{T_{4p+2}(x/2)}{x^2-4}\mathrm{d}x = 0.$$

Hence his proof is incomplete.

522. Evaluate the integral

$$\int_0^\infty \frac{\log\left(1+x^{10}\right)}{1+x^2}\,\mathrm{d}x.$$

Proposed by Seán M. Stewart, King Abdullah University of Science and Technology (KAUST), Saudi Arabia.

Solution by the author. Denote the integral to be found by I. We claim $I = \pi\log(10+4\sqrt{5})$.

Consider the parametric integral

$$J(a) = \int_0^\infty \frac{\log(a+x^{10})}{1+x^2}\,\mathrm{d}x, \quad a \geqslant 0.$$

Observe that

$$J(0) = 10\int_0^\infty \frac{\log(x)}{1+x^2}\,\mathrm{d}x = 0,$$

a result that can be readily seen on enforcing a substitution of $x \mapsto 1/x$, which proves that $J(0) = -J(0)$. The value for the desired integral we seek is $J(1)$. Differentiating with respect to the parameter a we have

$$J'(a) = \int_0^\infty \frac{\mathrm{d}x}{(1+x^2)(a+x^{10})},$$

or

$$J'(a) = \frac{1}{a-1}\int_0^\infty \frac{\mathrm{d}x}{1+x^2} - \frac{1}{a-1}\int_0^\infty \frac{x^8-x^6+x^4-x^2+1}{a+x^{10}}\,\mathrm{d}x,$$

after a partial fraction decomposition has been applied to the integrand. The first of the integrals appearing to the right of the equality is elementary. It has a value equal to $\pi/2$. The second integral can be written as a linear combination of $\int_0^\infty \frac{x^n}{a+x^{10}}\,\mathrm{d}x$ for $n = 8,6,4,2,0$. These integrals can be written in terms of the beta function,

$$\mathrm{B}(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,\mathrm{d}t, \quad x,y>0,$$

which satisfies $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$, where $\Gamma(\cdot)$ denotes the gamma function.

We make the substitution $ay = x^{10}$, i.e. $x = (ay)^{1/10}$, and we get

$$\int_0^\infty \frac{x^n}{a+x^{10}}\mathrm{d}x = \int_0^\infty \frac{a^{n/10}y^{n/10}}{a+ay}\,a^{1/10}y^{-9/10}\mathrm{d}y = a^{\frac{n-9}{10}}\int_0^\infty \frac{y^{-\frac{n-9}{10}}}{1+y}\mathrm{d}y$$
$$= a^{\frac{n-9}{10}}\mathrm{B}\left(\frac{n+1}{10},\frac{9-n}{10}\right) = \frac{\pi a^{\frac{n-9}{10}}}{\sin\frac{(n+1)\pi}{10}}.$$

Here we used the fact that $\Gamma(1)=1$ and the Euler's reflection formula to get $\mathrm{B}\left(z,1-z\right) = \Gamma(z)\Gamma(1-z)/\Gamma(1) = \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$.

By plugging $n = 8, 4, 2, 0$ in the formula above, we get

$$J'(a) = \frac{\pi/2}{a-1} + \frac{\pi}{10(a-1)} \left(\frac{a^{-1/10}}{\sin 9\pi/10} - \frac{a^{-3/10}}{\sin 7\pi/10} + \frac{a^{-1/2}}{\sin \pi/2} - \frac{a^{-7/10}}{\sin 3\pi/10} + \frac{a^{-9/10}}{\sin \pi/10} \right).$$

But

$$\frac{1}{\sin \pi/10} = \frac{1}{\sin 9\pi/10} = \sqrt{5}+1 =: \alpha, \quad \frac{1}{\sin 3\pi/10} = \frac{1}{\sin 7\pi/10} = \sqrt{5}-1 =: \beta,$$

so

$$J'(a) = \frac{\pi}{10(a-1)} (5 - \alpha a^{-1/10} + \beta a^{-3/10} - a^{-1/2} + \beta a^{-7/10} - \alpha a^{-9/10}).$$

We make the substitution $a = u^{10}$ in the integral $J(1) = \int_0^1 J'(a)\,\mathrm{d}a$ and we get

$$J(1) = \pi \int_0^1 \frac{5u^9 - \alpha u^8 + \beta u^6 - u^4 + \beta u^2 - \alpha}{u^{10}-1}\,\mathrm{d}x. \tag{4}$$

Over $\mathbb{R}$ the denominator $u^{10}-1$ decomposes into the factors $u-1$, $u+1$ and $u^2 - 2\cos\theta u + 1$, with $\theta = \pi/5$, $2\pi/5$, $3\pi/5$ and $4\pi/5$. But the cosines of these numbers are $\frac{\pm\sqrt{5}\pm 1}{4}$, i.e. $\pm\alpha/4$ and $\pm\beta/4$. Hence we have the factors $u^2 \pm \frac{\alpha}{2}u + 1 = \frac{1}{2}(2u^2 \pm \alpha u + 2)$ and $u^2 \pm \frac{\beta}{2}u + 1 = \frac{1}{2}(2u^2 \pm \beta u + 2)$. It turns out that in the fraction above the factors $u-1$, $\frac{1}{2}(2u^2 - \alpha u + 2)$ and $\frac{1}{2}(2u^2 - \beta u + 2)$ simplify and we get the following partial fraction decomposition

$$\frac{5u^9 - \alpha u^8 + \beta u^6 - u^4 + \beta u^2 - \alpha}{u^{10}-1} = \frac{1}{u+1} + \frac{4u+\alpha}{2u^2 + \alpha u + 2} + \frac{4u+\beta}{2x^2 + \beta u + 2}.$$

Then our integral becomes

$$\begin{aligned} J(1) &= \pi \int_0^1 \frac{\mathrm{d}u}{u+1} + \pi \int_0^1 \frac{4u+\alpha}{2u^2+\alpha u+2}\,\mathrm{d}u + \pi \int_0^1 \frac{4u+\beta}{2u^2+\beta u+2}\,\mathrm{d}u \\ &= \pi \left[\log(u+1) + \log\left(2u^2 + \alpha u + 2\right) + \log\left(2u^2 + \beta u + 2\right)\right]_0^1 \\ &= \pi \log\left(\frac{(5+\sqrt{5})(3+\sqrt{5})}{2}\right) = \pi \log\left(10 + 4\sqrt{5}\right), \end{aligned}$$

as announced. □

Solution by Constantin-Nicolae Beli, IMAR, Bucharest. We denote by I our integral. Since the real map $x \mapsto \frac{\log(1+x^{10})}{1+x^2}$ is even, we have

$$\int_{-\infty}^{\infty} \frac{\log(1+x^{10})}{1+x^2}\,\mathrm{d}x = 2I.$$

To evaluate this integral we will use complex contour integration.

We have $1+x^{10}=\prod_{\lambda\in A}(x-\lambda)$, where $A\subseteq\mathbb{C}$ is the set of roots of the polynomial $X^{10}+1$, $A=\{e^{k\pi i/10}\mid k \text{ odd}\}$. We write $A=A_+\cup A_-$, where A_+ and A_- are the roots of $X^{10}+1$ in the upper and lower half-plane in $\mathbb{C}$, respectively. We have

$$A_\pm=\{e^{i\theta}\mid \theta=\pm\pi/10,\pm 3\pi/10,\pm 5\pi/10,\pm 7\pi/10,\pm 9\pi/10\}.$$

(Here the $\pm$ sign from $A_\pm$ is the same as the $\pm$ signs for the values of θ.)

Note that $\lambda\mapsto\bar\lambda$ is a bijection from A_- to A_+ so for every $x\in\mathbb{R}$ we have

$$1+x^{10}=|1+x^{10}|=\left|\prod_{\lambda\in A}(x-\lambda)\right|=\left|\prod_{\lambda\in A_-}(x-\lambda)(x-\bar\lambda)\right|=\prod_{\lambda\in A_-}|x-\lambda|^2.$$

(We have $|x-\lambda|=|x-\bar\lambda|$.) Hence $\log(1+x^{10})=2\sum_{\lambda\in A_-}\log|x-\lambda|$.

We now denote by $\log:\mathbb{C}\setminus(-\infty,0]\to\mathbb{C}$ the principal branch of the logarithm. If $z\in\mathbb{C}\setminus(-\infty,0]$ then z writes uniquely as $z=re^{i\theta}$, with $r>0$ and $\theta\in(-\pi/2,\pi/2)$ and $\log(z)$ is defined as $\log(z)=\log(r)+\theta i$. Moreover θ will be called the argument of z and denoted $\arg(z)=\theta$. Note that $\Re\log(z)=\log(r)=\log|z|$.

We now define the function

$$f(z)=\frac{2}{1+z^2}\sum_{\lambda\in A_-}\log(z-\lambda).$$

Note that each function $\log(z-\lambda)$ is defined everywhere except on $(-\infty,0]+\lambda$, which is the horizontal half-line to the left of λ. In particular, $\log(z-\lambda)$ is defined in the half-plane $\{z\in\mathbb{C}\mid \operatorname{Im}z>\operatorname{Im}\lambda\}$. Since the imaginary parts of the elements $\lambda\in A_-$ are $-\sin\pi/10,-\sin 3\pi/10,\ldots,-\sin 9\pi/10$ and the largest of these imaginary parts is $-\sin\pi/10$, the function f is defined on the half-plane $\{z\in\mathbb{C}\mid \operatorname{Im}z>-\sin\pi/10\}$, which contains the upper half-plane $\{z\in\mathbb{C}\mid \operatorname{Im}z\geq 0\}$.

Now for $z\in\mathbb{R}$ we have

$$\begin{aligned}\Re f(z)&=\frac{2}{1+z^2}\sum_{\lambda\in A_-}\Re\log(z-\lambda)\\&=\frac{2}{1+z^2}\sum_{\lambda\in A_-}\log|z-\lambda|\\&=\frac{1}{1+z^2}\log(1+z^{10}).\end{aligned}$$

It follows that

$$2I=\int_{-\infty}^{\infty}\Re f(z)\mathrm{d}z=\Re\int_{-\infty}^{\infty}f(z)\mathrm{d}z.$$

Let $R\gg 0$. We consider the closed contour made of the interval $[-R,R]$ and the semicircle C_R of diameter $[-R,R]$ in the upper half-plane. The only

singularity of f in this domain is at i and this singularity is simple. We have
$\operatorname{Res}_{z=i}(f) = \lim_{z\to i}(z-i)f(z) = \left(\frac{2}{z+i}\sum_{\lambda\in A_-}\log(z-\lambda)\right)\Big|_{z=i} = \frac{1}{i}\sum_{\lambda\in A_-}\log(i-\lambda).$

It follows that

$$\int_{-R}^{R} f(z)\mathrm{d}z + \int_{C_R} f(z)\mathrm{d}z = 2\pi i \operatorname{Res}_{z=i}(f) = 2\pi \sum_{\lambda\in A_-}\log(i-\lambda).$$

When we take the real parts in both sides, since $\Re\log(z) = \log|z|$, we get

$$\Re\int_{-R}^{R} f(z)\mathrm{d}z + \Re\int_{C_R} f(z)\mathrm{d}z = 2\pi \sum_{\lambda\in A_-}\log|i-\lambda|.$$

We now take limits when $R\to\infty$. Let $R \gg 0$. If $z\in C_R$, then for every $\lambda\in A_-$ we have $|z|=R$ and $|\lambda|=1$, so $1\le R-1\le |z-\lambda|\le R+1$. Hence $0\le \log|z-\lambda|\le \log(R+1)$ and so $|\Re\log(z-\lambda)| = |\log|z-\lambda|| \le \log(R+1)$. We also have $|\operatorname{Im}\log(z-\lambda)|\le\pi$, so $|\log(z-\lambda)|\le\log(R+1)+\pi$. Since $|A_-|=5$, we get

$$\Big|\sum_{\lambda\in A_-}\log(z-\lambda)\Big| \le \sum_{\lambda\in A_-}|\log(z-\lambda)| \le 5(\log(R+1)+\pi).$$

We also have $|1+z^2|\ge|z^2|-1 = R^2-1$, so $\frac{2}{1+z^2}\le\frac{2}{R^2-1}$. It follows that $|f(z)|\le\frac{2}{R^2-1}\cdot 5(\log(R+1)+\pi) = \frac{10(\log(R+1)+\pi)}{R^2-1}$ $\forall z\in C_R$. Since the length of C_R is πR, this implies that $|\int_{C_R} f(z)\mathrm{d}z| \le \frac{10\pi R(\log(R+1)+\pi)}{R^2-1}$. It follows that $\lim_{R\to\infty}\int_{C_R} f(z)\mathrm{d}z = 0$, which implies that $\lim_{R\to\infty}\Re\int_{C_R} f(z)\mathrm{d}z = 0$.

We also have $\lim_{R\to\infty}\Re\int_{-R}^{R} f(z)\mathrm{d}z = \Re\int_{-\infty}^{\infty} f(z)\mathrm{d}z = 2I$. Thus

$$2I + 0 = 2\pi\sum_{\lambda\in A_-}\log|i-\lambda|, \qquad \text{whence} \qquad I = \pi\sum_{\lambda\in A_-}\log|i-\lambda|.$$

To evaluate $|i-\lambda|$, we note that for every $\alpha,\beta\in\mathbb{R}$ we have

$$\begin{aligned}|e^{\alpha i}-e^{\beta i}| &= |e^{(\alpha+\beta)i/2}|\cdot|e^{(\alpha-\beta)i/2}-e^{-(\alpha-\beta)i/2}| = 1\cdot|2i\sin(\alpha-\beta)/2| \\ &= 2|\sin(\alpha-\beta)/2|.\end{aligned}$$

In our case $|i-\lambda| = |e^{\pi i/2}-e^{\theta}| = |2\sin(\pi/4-\theta/2)|$, for $\lambda = -\pi/10,\ldots,-9\pi/10$. We get

$$\begin{aligned}\sum_{\lambda\in A_-}\log|i-\lambda| &= \log\prod_{\lambda\in A_-}|i-\lambda| \\ &= \log|2^5\sin(3\pi/10)\sin(2\pi/5)\sin(\pi/2)\sin(3\pi/5)\sin(7\pi/10)|.\end{aligned}$$

But $\sin(\pi/2)=1$, $\sin(3\pi/10)\sin(7\pi/10) = \sin^2(3\pi/10) = \frac{1}{2}(1-\cos(3\pi/5)) = \frac{1}{8}(3+\sqrt{5})$ and $\sin(2\pi/5)\sin(3\pi/5) = \sin^2(2\pi/5) = \frac{1}{2}(1-\cos(4\pi/5)) = \frac{1}{8}(5+$

$\sqrt{5}$). (Here we used the formulas $\cos(3\pi/5) = -\cos(2\pi/5) = -\frac{-1+\sqrt{5}}{4}$ and $\cos(4\pi/5) = \frac{-1-\sqrt{5}}{4}$.) We get

$$\sum_{\lambda \in A_-} \log|i-\lambda| = \log\left(2^5 \cdot \frac{1}{8}(3+\sqrt{5}) \cdot \frac{1}{8}(5+\sqrt{5})\right) = \log\left(\frac{1}{2}(20+8\sqrt{5})\right)$$
$$= \log(10+4\sqrt{5}),$$

which implies $I = \pi \log(10+4\sqrt{5})$. □

523. Prove that

$$\operatorname{rank}(A - ABA) - \operatorname{rank}(B - BAB) = \operatorname{rank}(A) - \operatorname{rank}(B)$$

for all $A, B \in \mathcal{M}_n(\mathbb{C})$.

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania and Mihai Opincariu, Avram Iancu National College, Brad, Romania.

Solutions by the authors. **First solution.** This solution is based on the observation that

$$\operatorname{Ker}(A) \subseteq \operatorname{Im}(I_n - BA)$$

for all $A, B \in \mathcal{M}_n(\mathbb{C})$. (If $x \in \operatorname{Ker}(A)$, then $x = x - BAx = (I_n - BA)x$, so $x \in \operatorname{Im}(I_n - BA)$.)

It follows that in the Sylvester inequality for A and $I_n - BA$ we have equality, i.e.

$$n - \operatorname{rank}(A) = \operatorname{rank}(I_n - BA) - \operatorname{rank}(A(I_n - BA))$$
$$= \operatorname{rank}(I_n - BA) - \operatorname{rank}(A - ABA)\,.$$

By symmetry,

$$n - \operatorname{rank}(B) = \operatorname{rank}(I_n - AB) - \operatorname{rank}(B - BAB)\,.$$

Since $\operatorname{rank}(I_n - BA) = \operatorname{rank}(I_n - AB)$ for all $A, B \in \mathcal{M}_n(\mathbb{C})$, the required identity follows by subtracting termwise the previous two equalities. □

Second solution. Consider the block matrices

$$M = \begin{bmatrix} A - ABA & O_n \\ O_n & B \end{bmatrix}, \quad N = \begin{bmatrix} A & O_n \\ O_n & B - BAB \end{bmatrix}, \text{ and } \quad K = \begin{bmatrix} A & AB \\ BA & B \end{bmatrix}.$$

Since

$$\operatorname{rank}(M) = \operatorname{rank}(A - ABA) + \operatorname{rank}(B)$$

and

$$\operatorname{rank}(N) = \operatorname{rank}(B - BAB) + \operatorname{rank}(A)\,,$$

the required identity is equivalent to $\operatorname{rank}(M) = \operatorname{rank}(N)$, which follows from the representation $L_1MR_1 = L_2NR_2 = K$, where

$$L_1 = \begin{bmatrix} I_n & A \\ O_n & I_n \end{bmatrix}, \quad R_1 = \begin{bmatrix} I_n & O_n \\ A & I_n \end{bmatrix}, \quad L_2 = \begin{bmatrix} I_n & O_n \\ B & I_n \end{bmatrix}, \quad R_2 = \begin{bmatrix} I_n & B \\ O_n & I_n \end{bmatrix}$$

are all non-singular matrices. □

524. Determine all bijective and differentiable functions $f : \mathbb{R} \to \mathbb{R}$, with $f'(x) \neq 0\ \forall x \in \mathbb{R}$, satisfying

$$f(x) + f\left(\frac{1}{f'(x)} - x\right) = 1, \quad \text{for every } x \in \mathbb{R}.$$

Proposed by Mircea Rus, Technical University of Cluj-Napoca, Romania.

Solution by the author. Let $g : \mathbb{R} \to \mathbb{R}$, $g(x) = f(x) - \frac{1}{2}$. Then g is bijective and differentiable, with $g' = f'$ and

$$g\left(\frac{1}{g'(x)} - x\right) = -g(x), \quad \text{for every } x \in \mathbb{R}. \tag{1}$$

Let $h : \mathbb{R} \to \mathbb{R}$ be the inverse function of g. For every $y \in \mathbb{R}$ if we take $x = h(y)$ in (1) we get

$$g\left(h'(y) - h(y)\right) = -y, \quad \text{for every } y \in \mathbb{R}, \tag{2}$$

as h is differentiable, with $h'(y) = \dfrac{1}{g'(h(y))} = \dfrac{1}{g'(x)}$. We apply h to (2) and we get $h'(y) - h(y) = h(-y)$, so

$$h'(y) = h(y) + h(-y), \quad \text{for every } y \in \mathbb{R}, \tag{3}$$

from which we also get

$$h'(-y) = h(-y) + h(y) = h'(y), \quad \text{for every } y \in \mathbb{R}.$$

Hence,

$$(h(y) + h(-y))' = h'(y) - h'(-y) = 0, \quad \text{for every } y \in \mathbb{R},$$

and so $h(y) + h(-y)$ is a constant.

Then, by (3), we get that h' is a constant, i.e. $h(y) = ay + b$, with $a, b \in \mathbb{R}$ and $a \neq 0$ (h is assumed bijective). After plugging $h(y) = ay + b$ into (3) we get $a = 2b$, so that $h(y) = ay + \frac{a}{2}\ \forall y \in \mathbb{R}$ and so $g(x) = \frac{1}{a}x - \frac{1}{2}$, $\forall x \in \mathbb{R}$. Consequently, $f(x) = \frac{x}{a}$, i.e. the functions f satisfying the required properties are of the form

$$f_c : \mathbb{R} \to \mathbb{R}, \quad f(x) = cx, \quad \text{for every } x \in \mathbb{R},$$

where $c \in \mathbb{R}^*$. One verifies that each such function f_c satisfies the required properties. □

Editor's note. We received a similar solution from Leonard Giugiuc, except that he doesn't consider the map $g(x) = f(x) - 1/2$ and its inverse $h = g^{-1}$, but he takes $h = f^{-1}$. Then the relation from the hypothesis applied to $x = h(y)$ yields $f(h'(y) - h(y)) = 1 - y$ which, after applying h, gives $h'(y) = h(y) + h(1-y)$. After substituting $y \mapsto 1-y$ one gets $(h(y) + h(1-y))' = h'(y) - h'(1-y) = 0$, so $h'(y) = h(y) + h(1-y)$ is a constant. It follows that h is affine and so is its inverse f. If $f(x) = ax + b$, then, after plugging this into the original relation, one gets $f(x) = ax$, with $a \in \mathbb{R}^*$.

525. Let $n \geq 4$ be an integer and $a_1, \ldots, a_n$ be positive real numbers.

(i) Prove that if $\dfrac{1}{a_1} + \cdots + \dfrac{1}{a_n} \leq 1$, then

$$2 \sum_{1 \leq i < j \leq n} a_i a_j - n^3(n-1) \geq 2(n-1)^2(a_1 + \cdots + a_n - n^2).$$

(ii) Prove that if $\dfrac{1}{a_1} + \cdots + \dfrac{1}{a_n} \geq 1$ and $k = \dfrac{n^2 - n + 1}{(n-1)^3}$, then

$$a_1^2 + \cdots + a_n^2 - n^3 \geq k \left(2 \sum_{1 \leq i < j \leq n} a_i a_j - n^3(n-1) \right).$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploieşti, Romania and Leonard Giugiuc, Traian National College, Drobeta-Turnu Severin, Romania.

Solution by the authors. (i) Let $a_1, \ldots, a_n > 0$, with $\sum_{i=1}^n 1/a_i \leq 1$. Put $p = \sum_{i=1}^n a_i$ and $q = \sum_{i<j} a_i a_j$. These relations are equivalent to $\sum_{i=1}^n a_i = p$ and $\sum_{i=1}^n a_i^2 = p^2 - 2q$. By The Equal Variables Theorem applied to the map $\theta \mapsto 1/\theta$, i.e. Corollary 1.4 in [1], of all $c_1, \ldots, c_n$ with $\sum_{i=1}^n c_i = p$ and $\sum_{i<j} c_i c_j = q$, i.e. with $\sum_{i=1}^n c_i = p$ and $\sum_{i=1}^n c_i^2 = p^2 - 2q$, the minimum of $\sum_{i=1}^n 1/c_i$ is reached for $(c_1, \ldots, c_n) = (x, \ldots, x, y)$ for some x, y with $0 < x \leq y$. In this case we have $p = \sum_{i=1}^n c_i = (n-1)x + y$ and $q = \sum_{i<j} c_i c_j = \dfrac{(n-1)(n-2)}{2} x^2 + (n-1)xy$. Hence the relation we want to prove writes as

$$(n-1)(n-2)x^2 + 2(n-1)xy - n^3(n-1) \geq 2(n-1)^2\big((n-1)x + y - n^2\big),$$

that is,

$$(n-2)x^2 + 2xy - n^3 \geq 2(n-1)\big((n-1)x + y - n^2\big),$$

or equivalently

$$2(x-n+1)y+(n-2)x^2-2(n-1)^2x+n^2(n-2)\geq 0.$$

The minimality property implies that $(n-1)/x+1/y=\sum_{i=1}^n 1/c_i \leq \sum_{i=1}^n 1/a_n = 1$. Then $(n-1)/x<(n-1)/x+1/y\leq 1$, so $x>n-1$, i.e. $x-n+1>0$. Also $(n-1)/x+1/y\leq 1$ implies that $y\geq \dfrac{x}{x-n+1}$. Since also $y\geq x$, we have

$$y\geq \max\left\{\frac{x}{x-n+1},x\right\}=\begin{cases}\frac{x}{x-n+1} & \text{if} \quad n-1<x\leq n,\\ x & \text{if} \quad x>n.\end{cases}$$

When $n-1<x\leq n$, the inequality we want to prove follows from $y\geq\frac{x}{x-n+1}$ and

$$2(x-n+1)\frac{x}{x-n+1}+(n-2)x^2-2(n-1)^2x+n^2(n-2)\geq 0,$$

which is equivalent to $(n-2)(x-n)^2\geq 0$. The equality holds when $x=n$ and $y=\frac{x}{x-n+1}=n$.

When $x>n$, the inequality we want to prove follows from $y\geq x$ and

$$2(x-n+1)x+(n-2)x^2-2(n-1)^2x+n^2(n-2)\geq 0,$$

which is equivalent to $n(x-n+2)(x-n)\geq 0$. Since $x>n$, we have strict inequality, so the only case of equality is the one from the first case, i.e. for $(a_1,\dots,a_n)=(n,\dots,n)$.

(ii) We keep the notations p and q from (i). We first prove the following statement

Lemma 1. One of the following holds.

(a) There are unique $0<y\leq x$ such that $(n-1)x+y=p$ and $(n-1)(n-2)x^2+2(n-1)xy=2q$.

(b) There are $b_1\geq\cdots\geq b_{n-1}\geq 0$ such that $\sum_{i=1}^n b_i=p$ and $\sum_{i<j}b_ib_j=q$.

Proof. Our lemma states that there are $0\leq c_1\leq\cdots\leq c_n$ with $\sum_{i=1}^n c_i=p$ and $\sum_{i<j}c_ic_j=q$ such that $(c_1,\dots,c_n)$ is either of the form $(x,\dots,x,y)$, for some unique $0<y\leq x$, or the form $(b_1,\dots,b_{n-1},0)$, for some $b_1\geq\cdots\geq b_{n-1}\geq 0$.

Again the relations $\sum_{i=1}^n c_i=p$ and $\sum_{i<j}c_ic_j=q$ are equivalent to $\sum_{i=1}^n c_i=p$ and $\sum_{i=1}^n c_i^2=p^2-2q$.

For (a) let $s=p/n$, so that $p=ns$. Then the condition that $(n-1)x+y=p=ns$ with $x\geq y$ means that $x=s+t$ and $y=s-(n-1)t$ for some $t\geq 0$. The extra condition $y>0$ means $s>(n-1)t$. And the condition $(n-1)x^2+y^2=p^2-2q$ write as $(n-1)(s+t)^2+(s-(n-1)t)^2=p^2-2q$, i.e. $ns^2+n(n-1)t^2=p^2-2q$. But $ns=p$, so after multiplying by n, the equation becomes $p^2+n^2(n-1)t^2=np^2-2nq$, i.e. $n^2(n-1)t^2=(n-1)p^2-2nq$.

Note that the inequality $(\sum_{i=1}^n a_i)^2 \le n\sum_{i=1}^n a_i^2$ writes as $p^2 \le n(p^2-2q)$, i.e. $(n-1)p^2 \ge 2nq$. Thus there is a unique $t \ge 0$ with this property, namely

$$t = \sqrt{\frac{(n-1)p^2-2nq}{n^2(n-1)}}.$$

We still have to check the condition $p/n = s > (n-1)t$, i.e. $p > n(n-1)t$, which is equivalent to $p^2 \ge n^2(n-1)^2t^2 > (n-1)^2p^2 - 2n(n-1)q$, i.e.

$$2(n-1)q > (n-2)p^2.$$

For (b) we look for $(b_1,\dots,b_{n-1})$ of the type $(y,x,\dots,x)$, with $0 < x \le y$. Our conditions write as $(n-2)x+y = p$ and $(n-2)x^2+y^2 = p^2-2q$. We denote $u = p/(n-1)$, that is, $p = (n-1)u$. Then $(n-2)x+y = p$ with $x \le y$ is equivalent to $x = u-v$ and $y = u+(n-2)v$ for some $v \ge 0$. The extra condition $x > 0$ writes as $u > v$. By the reasoning from (a) applied with n,s,t replaced by $n-1,u,-v$, respectively, the second equation, $(n-2)x^2+y^2 = p^2-2q$, is equivalent to $(n-1)^2(n-2)v^2 = (n-2)p^2-2(n-1)q$. It has a solution iff $(n-2)p^2-2(n-1)q \ge 0$, i.e. $2(n-1)q \le (n-2)p^2$. Assuming this happens, we have $(n-1)^2(n-2)v^2 = (n-2)p^2-2(n-1)q < (n-2)p^2$, which is equivalent to $v < p/(n-1) = u$, so all conditions are fulfilled.

Hence if $2(n-1)q > (n-2)p^2$ then (a) holds and if $2(n-1)q \le (n-2)p^2$ then (b) holds. This concludes the proof of lemma. □

The relation we want to prove writes as $(p^2-2q)-n^3 \ge k\big(2q-n^3(n-1)\big)$ i.e. as

$$p^2-2q-n^3-k\big(2q-n^3(n-1)\big) = p^2-n^4-(k+1)\big(2q-n^3(n-1)\big) \ge 0.$$

Recall that in the case (a) of Lemma 1 we have $p^2-2q = (n-1)x^2+y^2$ and in the case (b) we have $p^2-2q = \sum_{i=1}^{n-1} b_i^2$. We now consider the two cases of Lemma 1.

(a) We have $p^2-n^4-(k+1)\big(2q-n^3(n-1)\big) = f(y)$, where f is a quadratic polynomial in the variable Y with coefficients depending on x, viz.

$$\begin{aligned} f(Y) &= (n-1)x^2+Y^2-n^3-k((n-1)(n-2)x^2+2(n-1)xY \\ &\qquad -n^3(n-1)) \\ &= Y^2-2k(n-1)xY+(n-1)\big(1-k(n-2)\big)x^2+\big(k(n-1)-1\big)n^3. \end{aligned}$$

Note that f is monic and the coefficient of Y is $-2k(n-1)x$. It follows that f is decreasing on $(-\infty, k(n-1)x]$. But $k(n-1) = \frac{n^2-n+1}{(n-1)^2} > 1$, so that $k(n-1)x > x$. Thus f is decreasing on $(-\infty, x]$. Since $y \le x$, we have

$$f(y) \ge f(x) = (n-kn(n-1))(x^2-n^2).$$

But $k(n-1) > 1$, so $n-kn(n-1) < 0$. Hence, if $x < n$, then $f(y) < 0$ and we are done.

Assume now that $x \geq n$. By The Equal Variables Theorem applied to the map $\theta \mapsto 1/\theta$, i.e. Corollary 1.4 in [1], of all $c_1, \ldots, c_n$ with $\sum_{i=1}^n c_i = p$ and $\sum_{i<j} c_i c_j = q$, the maximum of $\sum_{i=1}^n 1/c_i$ is reached for $(c_1, \ldots, c_i) = (x, \ldots, x, y)$. This maximality property implies that $(n-1)/x + 1/y \geq \sum_{i=1}^n 1/a_i \geq 1$, so $y \leq \frac{x}{x-n+1} < x$. (We have $x \geq n$.) Since f is decreasing on $(-\infty, x]$, we have that $f(y) \geq f\left(\frac{x}{x-n+1}\right)$, so it is enough to prove that $f(z) \geq 0$, where $z = \frac{x}{x-n+1}$.

Note that f can be rewritten as

$$((n-1)x+Y)^2 - n^4 - (k+1)((n-1)(n-2)x^2 + 2(n-1)xY - n^3(n-1)).$$

Thus, in order to get $f(z)$, we first compute

$$\begin{aligned}((n-1)x+z)^2 - n^4 &= ((n-1)x + z - n^2)((n-1)x + z + n^2)\\ &= \frac{(n-1)(x-n)^2}{x-n+1} \cdot \frac{(n-1)x^2 - 2nx - n^2(n-1)}{x-n+1}.\end{aligned}$$

With

$$xz - n^2 = \frac{x^2}{x-n+1} - n^2 = \frac{(x-n)(x-n(n-1))}{x-n+1},$$

we get

$$\begin{aligned}(n-2)x^2 + 2xz - n^3 &= (n-2)(x^2-n^2) + 2(xz - n^2)\\ &= (x-n)\left((n-2)(x+n) + 2\cdot\frac{x-n(n-1)}{x-n+1}\right)\\ &= (x-n)\cdot\frac{(n-2)x^2 + nx - n^2(n-1)}{x-n+1}\\ &= \frac{(x-n)^2((n-2)x + n(n-1))}{x-n+1}.\end{aligned}$$

Hence, since

$$(k+1)(n-1) = (n-1)\left(\frac{n^2-n+1}{(n-1)^3} + 1\right) = \frac{n(n^2-2n+2)}{(n-1)^2},$$

we have

$$\begin{aligned}f(z) &= \big((n-1)x+z\big)^2 - n^2 - (k+1)(n-1)\big((n-2)x^2 + 2xz - n^3\big)\\ &= \frac{(n-1)(x-n)^2\big((n-1)x^2 - 2nx - n^2(n-1)\big)}{(x-n+1)^2}\\ &\qquad - \frac{n(n^2-2n+2)}{(n-1)^2}\cdot\frac{(x-n)^2\big((n-2)x + n(n-1)\big)}{x-n+1}\\ &= \frac{(x-n)^2 g(x)}{(n-1)^2(x-n+1)^2},\end{aligned}$$

where

$$g(x) = x^2 - 2n(n-1)x + n^2(n-1)^2 = \big(x - n(n-1)\big)^2.$$

Thus $f(y) \geq f(z) = \frac{(x-n)^2(x-n(n-1))^2}{(n-1)^2(x-n+1)^2} \geq 0$. The equality holds iff $x = n$ or $n(n-1)$ and $y = z = \frac{x}{x-n+1}$, i.e. $y = n$ or $\frac{n}{n-1}$, respectively.

(b) We have $(n-1)\sum_{i=1}^{n-1} b_i^2 \geq (\sum_{i=1}^{n-1} b_i)^2$, i.e. $(n-1)(p^2-2q) \geq p^2$, which implies that $(n-2)p^2 \geq 2(n-1)q$, so $(n-2)(p^2-2q) \geq 2q$, i.e. $p^2 - 2q \geq \frac{2}{n-2}q$. But $(n-2)(n^2-n+1) = n^3 - 3n^2 + 3n - 2 < (n-1)^3$, so $\frac{1}{n-2} > \frac{n^2-n+1}{(n-1)^3} = k$. Hence $p^2 - 2q > 2kq$. We also have $(n-1)k = \frac{n^2-n+1}{(n-1)^2} > 1$, so $-n^3 > -kn^3(n-1)$. It follows that $p^2 - 2q - n^3 > k(2q - n^3(n-1))$ and we are done.

The equality holds for $(a_1, \ldots, a_n) = (n, \ldots, n)$ or $(n(n-1), \ldots, n(n-1), \dfrac{n}{n-1})$ and permutations. $\square$

References

[1] Vasile Cîrtoaje, The equal variable method, *J. Inequal. Pure Appl. Math.* **8** (2007), no. 1, art. 15, 21 pp. https://www.emis.de/journals/JIPAM/article828.html?sid=828

526. If $a, b \in \mathbb{C}$ and $k \geq 2$, then we define

$$\mathcal{F}_{a,b}^k = \{f : \mathbb{C} \to \mathbb{C} \mid f^{(k)}(z) = az + b\},$$

where $f^{(k)} = f \circ \cdots \circ f$, with k copies of f.

For given $a, b \in \mathbb{C}$ and $k \geq 2$ find necessary and sufficient condition such that a set $A \subseteq \mathbb{C}$ can be written as $A = \mathrm{Fix}(f)$ for some $f \in \mathcal{F}_{a,b}^k$.

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Solution by the authors. We prove the following.

1. If $a \neq 1$, then the only possible A is $A = \{\frac{b}{1-a}\}$.
2. If $a = 1$, $b \neq 0$, then the only possible A is $A = \emptyset$.
3. If $a = 1$, $b = 0$, then A can be any subset of $\mathbb{C}$ except a set of the form $A = \mathbb{C} \setminus X$, where $X \subset \mathbb{C}$ is a finite set with $|X| \notin p_1\mathbb{N} + \cdots + p_s\mathbb{N}$, where $p_1, \ldots, p_s$ are all prime divisors of k and $\mathbb{N} := \mathbb{Z}_{\geq 0}$.

In particular, if k is a power of a prime p, then the only sets A that do not qualify are $A = \mathbb{C} \setminus X$, where X is finite and $p \nmid |X|$.

And if k is not a power of a prime, i.e. if $s \geq 2$, then every $n \geq p_1p_2 - p_1 - p_2 + 1$ writes as $n = p_1n_1 + p_2n_2$ for some integers $n_1, n_2 \geq 0$, so $n \in p_1\mathbb{N} + p_2\mathbb{N} \subseteq p_1\mathbb{N} + \cdots + p_s\mathbb{N}$. Hence $S = \mathbb{N} \setminus (p_1\mathbb{N} + \cdots + p_s\mathbb{N})$ is a finite set contained in $\{1, 2, \ldots, p_1p_2 - p_1 - p_2\}$. Then the only sets A that don't qualify are $A = \mathbb{C} \setminus X$, where X is finite and $|X| \in S$.

To prove these statements, we use a couple of observations. Let $a, b \in \mathbb{C}$ be fixed. We denote by $g : \mathbb{C} \to \mathbb{C}$, the function given by $g(z) = az + b$. So $f \in \mathcal{F}^k_{a,b}$ means $f^{(k)} = g$.

First note that $\mathrm{Fix}(f) \subseteq \mathrm{Fix}(g)$. Indeed, if $f(z) = z$, then, by induction on i, we have $f^{(i)}(z) = z\ \forall i \geq 1$. In particular, $g(z) = f^{(k)}(z) = z$.

Next, we note that if $z \in \mathrm{Fix}(g)$, then $g(z) = z$, so $f(g(z)) = f(z)$. From $f \circ g = f \circ f^{(k)} = f^{(k+1)} = f^{(k)} \circ f = g \circ f$ it follows that $g(f(z)) = f(g(z)) = f(z)$ and so $f(z) \in \mathrm{Fix}(g)$. Hence $f(\mathrm{Fix}(g)) \subseteq \mathrm{Fix}(g)$.

We now prove our claims.

1. If $a \neq 1$, then $\mathrm{Fix}(g) = \{z_0\}$, where z_0 is the only solution of $az + b = z$, i.e. $z_0 = \frac{b}{1-a}$. So $\mathrm{Fix}(f) \subseteq \mathrm{Fix}(g) = \{z_0\}$. But we also have $f(z_0) \in f(\mathrm{Fix}(g)) \subseteq \mathrm{Fix}(g) = \{z_0\}$ and so $f(z_0) = z_0$, i.e. $z_0 \in \mathrm{Fix}(f)$. It follows that $\mathrm{Fix}(f) = \{z_0\}$, as claimed.

To complete the proof, we must show that $\mathcal{F}^k_{a,b} \neq 0$, i.e. that there are functions f with $f^{(k)} = g$. We look for affine functions $f(z) = \alpha z + \beta$. By induction on i, $f^{(i)}$ has the form $f^{(i)}(z) = \alpha^i z + \beta_i$ for some $\beta_i \in \mathbb{C}$. In particular, $f^{(k)}(z) = \alpha^k z + \beta_k$. In order that $f^{(k)}(z) = az + b$ we need that $\alpha^k = a$ and, in order that $f(z_0) = z_0$, we must have $f(z) = \alpha(z - z_0) + z_0$. Conversely, if f has this form, then $f^{(k)}(z) = az + \beta_k$ for some $\beta_k \in \mathbb{C}$ and z_0 is a fixed point of f, so it will be a fixed point of $f^{(k)}$, as well. Since $f^{(k)}(z) = az + \beta_k$, $g(z) = az + b$, and $f^{(k)}(z_0) = g(z_0) = z_0$, we must have $f^{(k)} = g$. Hence $\mathcal{F}_{a,b}$ has precisely k affine functions, namely $f(z) = \alpha(z - z_0) + z_0$ for each $\alpha \in \mathbb{C}$ with $\alpha \in \mathbb{C}$ with $\alpha^k = a$.

2. If $a = 1$, $b \neq 0$, i.e. $g(z) = z + b$, then $\mathrm{Fix}(g) = \emptyset$, so, since $\mathrm{Fix}(f) \subseteq \mathrm{Fix}(g)$, we must have $\mathrm{Fix}(f) = \emptyset$, as claimed.

To complete the proof, we must show that $\mathcal{F}^k_{1,b} \neq \emptyset$. An obvious element of $\mathcal{F}^k_{1,b}$ is $f(z) = z + \frac{b}{k}$. In fact, this is the only affine function from $\mathcal{F}^k_{1,b}$. Indeed, if $f(z) = \alpha z + \beta$ then, same as in the case 1, we have $f^{(k)}(z) = \alpha^k z + \beta_k$. Therefore, in order that $f^{(k)}(z) = z + b$, we must have $\alpha^k = 1$. Assuming that $\alpha^k = 1$, we have $f^{(k)}(z) = \alpha^k z + \beta_k = z + \beta_k$. If $\alpha \neq 1$, then $z_0 = \frac{\beta}{1-\alpha}$ is fixed point of f, so it will be a fixed point of $f^{(k)}$, as well. It follows that $z_0 = f^{(k)}(z_0) = z_0 + \beta_k$ and so $f^{(k)}(z) = z \neq g$. If $\alpha = 1$, then $f(z) = z + \beta$ and $f^{(k)}(z) = z + k\beta$. Thus $f^{(k)}(z) = g(z) = z + b$ if and only if $\beta = \frac{b}{k}$.

3. If $a = 1$ and $b = 0$, then $g(z) = z$. Let $A \subseteq \mathbb{C}$. We denote $X = \mathbb{C} \setminus A$. We have two cases.

(a) X *is infinite.* Then $|X| = k|X| = |\mathbb{Z}_k \times X|$. (See, e.g., [1].) Hence there is a bijection $\phi : \mathbb{Z}_k \times X \to X$. Then every element of X writes uniquely as $\phi(\bar{c}, x)$ for some $\bar{c} \in \mathbb{Z}_k$ and $x \in X$. (Here $\bar{c}$ denotes the class in $\mathbb{Z}_k$ of

$c \in \mathbb{Z}$.) Then we define the function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} z, & z \in \mathbb{C} \setminus X, \\ \phi(\bar{c} + \bar{1}, x), & z = \phi(\bar{c}, x) \text{ with } \bar{c} \in \mathbb{Z}_k,\ x \in X. \end{cases}$$

Obviously, $\mathrm{Fix}(f) = \mathbb{C} \setminus X = A$. If $z \in A$, then $f(z) = z$, so $f^{(k)}(z) = z$. If $z \in X$, then $z = \phi(\bar{c}, x)$ for some $\bar{c} \in \mathbb{Z}_k$ and $x \in X$. By induction on l, we have $f^{(l)}(z) = f^{(l)}(\phi(\bar{c}, x)) = \phi(\bar{c}+\bar{l}, x)$. In particular, $f^{(k)}(z) = \phi(\bar{c}+\bar{k}, x) = \phi(\bar{c}, x) = z$. Hence $f^{(k)}$ is the identity map, i.e. $f \in \mathcal{F}^k_{1,0}$.

(b) X *is finite,* $|X| = n$. If $f \in \mathcal{F}^k_{1,0}$, then $f^{(k)}$ is the identity map g, so f is invertible, with $f^{-1} = f^{(k-1)}$. Then f generates a group of invertible complex functions $(G, \circ)$. Since $f^{(k)}$ is the identity map, we necessarily have $|G| = \mathrm{ord}\, f \mid k$. We consider the action of G on $\mathbb{C}$ given by $f^{(i)} \cdot z = f^{(i)}(z)$. For every $z \in \mathbb{C}$ we denote by O_z its orbit with respect to the action of G. We have $|O_z| = |\{f^{(i)}(z) \mid i \in \mathbb{Z}\}| = 1$ iff $z \in \mathrm{Fix}(f) = A$. Hence $X = \mathbb{C} \setminus A$ is the union of all orbits O_z with $|O_z| \geq 2$. Since $|X| < \infty$, there are a finite number of such orbits, say, $O_{z_1}, \dots, O_{z_m}$. Then $n = |X| = \sum_{j=1}^m d_j$, with $d_j = |O_{z_j}|$. But we have $d_j \mid |G| \mid k$. Hence n writes as a sum of divisors of k which are > 1.

Conversely, assume that $|X| = n = d_1 + \cdots + d_m$, where $d_j \mid k$, $d_j > 1$. Then X writes as a disjoint union $X = X_1 \cup \cdots \cup X_m$ with $|X_k| = d_k$. We fix an index j, $1 \leq j \leq m$, and we index the elements of X_j by the set $\mathbb{Z}_{d_j}$, i.e. we put $X_j = \{z_{\bar{c}} \mid \bar{c} \in \mathbb{Z}_{d_j}\}$. We define $f_j : X_j \to X_j$ by $f_j(z_{\bar{c}}) = z_{\bar{c}+\bar{1}}$. Obviously, $f_j(z) \neq z\ \forall z \in X_j$. By induction on l we get $f^l(z_{\bar{c}}) = z_{\bar{c}+\bar{l}}$. In particular, $f^{(d_j)}(z_{\bar{c}}) = z_{\bar{c}+\bar{d}_j} = z_{\bar{c}}$. Hence $f^{(d_j)}$ is the identity map on X_j. Since $d_j \mid k$, this implies that $f_j^{(k)}$ is the identity map, i.e. $f_j^{(k)}(z) = z$ $\forall z \in X_j$.

We now define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} z, & z \in \mathbb{C} \setminus X, \\ f_j(z), & z \in X_j,\ 1 \leq j \leq m. \end{cases}$$

Since $f(z) \neq z$ for every $1 \leq j \leq m$ and every $z \in X_j$, we have $\mathrm{Fix}(f) = \mathbb{C} \setminus X = A$. If $z \in \mathbb{C} \setminus X$, then $f(z) = z$ and so $f^{(k)}(z) = z$. If $z \in X_j$ for some j, then $f^{(k)}(z) = f_j^{(k)}(z) = z$. Hence $f^{(k)}$ is the identity map and so $f \in \mathcal{F}^k_{1,0}$.

Let now M be the monoid $M = p_1\mathbb{N} + \cdots + p_s\mathbb{N}$, where $p_1, \dots, p_s$ are all primes divisors of k. If $n = d_1 + \cdots + d_m$, with $d_j \mid k$, $d_j > 1$, then for every j we have $p_t \mid d_j$ for some $1 \leq t \leq s$. It follows that $d_j \in p_t\mathbb{N} \subseteq M$ and so $n = d_1 + \cdots + d_m \in M$. Conversely, if $n \in M$, then n is a sum of copies of p_t, with $1 \leq t \leq s$, and for each t we have $p_t \mid k$ and $p_t > 1$.

Hence an equivalent necessary and sufficient condition is that $n \in p_1\mathbb{N} + \cdots + p_s\mathbb{N}$. □

References

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527. For every non-negative integer n, evaluate the integral

$$I_n = \int_0^\infty \frac{\log^n(x)\sin(x)}{x}\,\mathrm{d}x.$$

The answer may be given either in a closed form or recursively. Also it may include values of the zeta function at integers ≥ 2.

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Solution by the author. Denote the integral to be found by I_n. We claim for $n \in \mathbb{N}$

$$I_{n+1} = -\gamma I_n - \sum_{k=1}^{n} \frac{n!}{(n-k)!}\left\{(-1)^k\zeta(k+1) + \frac{|B_{k+1}|\pi^{k+1}}{(k+1)!}\right\} I_{n-k},$$

subject to the initial conditions of $I_0 = \frac{\pi}{2}$ and $I_1 = -\frac{\gamma\pi}{2}$. Here $\zeta(s)$ denotes the Riemann zeta function, B_n denotes the Bernoulli numbers, while γ denotes the Euler–Mascheroni constant.

Denote the integral to be evaluated by I_n. Thus

$$I_n = \int_0^\infty \frac{\log^n(x)\sin(x)}{x}\,\mathrm{d}x.$$

The value for the integral will be found using a technique that exploits an exponential generating function. Consider the exponential generating function $G(t) = \sum_{n=0}^{\infty} \frac{I_n t^n}{n!}$. We have

$$G(t) = \int_0^\infty \frac{\sin x}{x} \sum_{n=0}^{\infty} \frac{(t\log x)^n}{n!}\,\mathrm{d}x = \int_0^\infty x^{t-1}\sin x\,\mathrm{d}x = \Gamma(t)\sin\left(\frac{\pi t}{2}\right), \tag{5}$$

where the value for the integral corresponds to the well-known value for the Mellin transform of the sine function. Here $\Gamma(t)$ denotes the gamma function. Taking the logarithmic derivative of (5) with respect to t yields

$$\frac{G'(t)}{G(t)} = \psi(t) + \frac{\pi}{2}\cot\left(\frac{\pi t}{2}\right) = \psi(t+1) - \frac{1}{t} + \frac{\pi}{2}\cot\left(\frac{\pi t}{2}\right). \tag{6}$$

where $\psi(t)$ denotes the digamma function and we have made use of the functional relation for the digamma function of

$$\psi(t+1) = \psi(t) + \frac{1}{t}.$$

From the series expansion for the cotangent function

$$\cot(t) = \frac{1}{t} - \sum_{n=1}^{\infty} \frac{2^{n+1}|B_{n+1}|t^n}{(n+1)!}, \quad |t| < \pi,$$

and the Maclaurin series expansion for the digamma function

$$\psi(t+1) = -\gamma - \sum_{n=1}^{\infty} (-1)^n \zeta(n+1) t^n, \quad |t| < 1,$$

we may rewrite (6) as

$$\frac{G'(t)}{G(t)} = -\gamma - \sum_{n=1}^{\infty} (-1)^n \zeta(n+1) t^n - \sum_{n=1}^{\infty} \frac{|B_{n+1}|\pi^{n+1} t^n}{(n+1)!},$$

or

$$G'(t) = -\gamma \sum_{n=0}^{\infty} \frac{I_n t^n}{n!} - \sum_{n=0}^{\infty} \frac{I_n t^n}{n!} \cdot \sum_{n=1}^{\infty} (-1)^n \zeta(n+1) t^n - \sum_{n=0}^{\infty} \frac{I_n t^n}{n!} \cdot \sum_{n=1}^{\infty} \frac{|B_{n+1}|\pi^{n+1} t^n}{(n+1)!}.$$

Noting that

$$G'(t) = \sum_{n=1}^{\infty} \frac{I_n t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{I_{n+1} t^n}{n!},$$

adjusting all sums so they start at $n = 1$ we have

$$I_1 + \sum_{n=1}^{\infty} \frac{I_{n+1} t^n}{n!} = -\gamma I_0 - \gamma \sum_{n=1}^{\infty} \frac{I_n t^n}{n!} - \frac{1}{t} \sum_{n=1}^{\infty} \frac{I_{n-1} t^n}{(n-1)!} \cdot \sum_{n=1}^{\infty} (-1)^n \zeta(n+1) t^n - \frac{1}{t} \sum_{n=1}^{\infty} \frac{I_{n-1} t^n}{(n-1)!} \cdot \sum_{n=1}^{\infty} \frac{|B_{n+1}|\pi^{n+1} t^n}{(n+1)!}. \tag{7}$$

For the initial conditions, we have

$$I_0 = \int_0^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2} \quad \text{and} \quad I_1 = \int_0^{\infty} \frac{\log(x)\sin(x)}{x} \, dx = -\frac{\gamma\pi}{2}.$$

One way to find the value for I_1 is given in the Appendix. With these values, on employing the Cauchy product of power series allows one to rewrite (7) as

$$\sum_{n=1}^{\infty} I_{n+1}\frac{t^n}{n!} = -\sum_{n=1}^{\infty}\gamma I_n\frac{t^n}{n!} - \sum_{n=1}^{\infty}\sum_{k=1}^{n}\left\{\frac{n!}{(n-k)!}(-1)^k\zeta(k+1)I_{n-k}\right\}\frac{t^n}{n!}$$
$$-\sum_{n=1}^{\infty}\sum_{k=1}^{n}\left\{\frac{n!}{(n-k)!}\frac{|B_{k+1}|\pi^{k+1}I_{n-k}}{(k+1)!}\right\}\frac{t^n}{n!}.$$

Comparing coefficients of $t^n/n!$ on either side yields the recurrence relation of

$$I_{n+1} = -\gamma I_n - \sum_{k=1}^{n}\frac{n!}{(n-k)!}\left\{(-1)^k\zeta(k+1) + \frac{|B_{k+1}|\pi^{k+1}}{(k+1)!}\right\}I_{n-k},$$

subject to the initial conditions of $I_0 = \frac{\pi}{2}$ and $I_1 = -\frac{\gamma\pi}{2}$, as announced.

Appendix. The value for the initial condition I_1 will be found using the following property for the Laplace transform

$$\int_0^{\infty} f(x)g(x)\,\mathrm{d}x = \int_0^{\infty}\mathcal{L}\{f(x)\}(s)\cdot\mathcal{L}^{-1}\{g(x)\}(s)\mathrm{d}s, \tag{8}$$

provided both integrals exist.

We start by considering the integral

$$J = \int_0^{\infty}\frac{\log(x)}{1+x^2}\,\mathrm{d}x.$$

To evaluate this integral, enforcing a substitution of $x \mapsto \frac{1}{x}$ yields

$$J = -\int_0^{\infty}\frac{\log(x)}{1+x^2}\,\mathrm{d}x = -J,$$

from which $J = 0$. Now the Laplace transform for the logarithmic function is known (see, for example, [1, Entry **6.1**, p. 48]). Here

$$\mathcal{L}\{\log(x)\}(s) = -\frac{\log(s)+\gamma}{s},$$

so on applying (8) to the integral J we find

$$\int_0^{\infty}\frac{\log(x)}{1+x^2}\,\mathrm{d}x = \int_0^{\infty}\mathcal{L}\{\log(x)\}(s)\cdot\mathcal{L}^{-1}\left\{\frac{1}{1+x^2}\right\}(s)\mathrm{d}s$$
$$= -\int_0^{\infty}\frac{\log(s)+\gamma}{s}\cdot\sin(s)\mathrm{d}s,$$

whence

$$0 = -\int_0^{\infty}\frac{\log(s)\sin(s)}{s}\,\mathrm{d}s - \gamma\int_0^{\infty}\frac{\sin(s)}{s}\,\mathrm{d}s.$$

The integral appearing on the right is just the Dirichlet integral. Its value is $\frac{\pi}{2}$. Substituting for this value, after replacing the dummy variable s with x and rearranging, we find

$$\int_0^\infty \frac{\log(x)\sin x}{x}\,\mathrm{d}x = -\frac{\gamma\pi}{2},$$

as announced. □

References

[1] F. Oberhettinger and L. Badii, *Tables of Laplace transforms*, Springer-Verlag, New York–Heidelberg, 1973.

Editor's note. We don't need two initial terms of the sequence $(I_n)_{n\geq 0}$ in order to use the recurrence relation. It suffices to determine I_0. The recurrence relation at $n = 0$ gives $I_1 = -\gamma I_0 = -\frac{\gamma\pi}{2}$.

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA. Note,

$$\begin{aligned} I_n &= \int_0^\infty \frac{\log^n x \sin x}{x}\,\mathrm{d}x = \int_0^\infty \left(\int_0^\infty f_n(s)e^{-sx}\mathrm{d}s\right)\sin x\,\mathrm{d}x \\ &= \int_0^\infty \left(\int_0^\infty e^{-sx}\sin x\,\mathrm{d}x\right) f_n(s)\mathrm{d}s = \int_0^\infty \frac{f_n(s)}{1+s^2}\mathrm{d}s, \end{aligned}$$

where

$$\mathcal{L}\left\{f_n(s)\right\} = \frac{\log^n x}{x}.$$

Here $\mathcal{L}\{f\}$ denotes the Laplace transform of f. For $n = 0$ we obviously have $\mathcal{L}\{f_0(s)\} = \dfrac{1}{x}$, so $f_0(s) = 1$. Hence

$$I_0 = \int_0^\infty \frac{1}{1+s^2}\mathrm{d}s = \frac{\pi}{2}.$$

For $n > 0$ we start with $\mathcal{L}\{s^a\} = \dfrac{\Gamma(a+1)}{x^{a+1}}$. By differentiating with respect to a, we get

$$\begin{aligned} \mathcal{L}\{s^a \log^n s\} &= \frac{\mathrm{d}^n}{\mathrm{d}a^n}\left(\frac{\Gamma(a+1)}{x^{a+1}}\right) \\ &= \frac{1}{x^{a+1}}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\log^j x\frac{\mathrm{d}^{n-j}}{\mathrm{d}a^{n-j}}\Gamma(a+1), \end{aligned}$$

so

$$\begin{aligned}\mathcal{L}\{\log^n s\} &= \lim_{a\to 0}\mathcal{L}\{s^a\log^n x\} = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{\log^j x}{x}\Gamma^{(n-j)}(1)\\ &= \sum_{j=0}^{n}(-1)^j\binom{n}{j}\mathcal{L}\{f_j(s)\}\Gamma^{(n-j)}(1)\\ &= \sum_{j=0}^{n-1}(-1)^j\binom{n}{j}\mathcal{L}\{f_j(s)\}\Gamma^{(n-j)}(1)+(-1)^n\mathcal{L}\{f_n(s)\}.\end{aligned}$$

It then follows that

$$\begin{aligned}f_n(s) &= (-1)^n\log^n s-\sum_{j=0}^{n-1}(-1)^{j-n}\binom{n}{j}\Gamma^{(n-j)}(1)f_j(s)\\ &= (-1)^n\log^n s+\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}\Gamma^{(j)}(1)f_{n-j}(s),\end{aligned}$$

and

$$I_n = (-1)^n\int_0^\infty\frac{\log^n s}{1+s^2}\mathrm{d}s+\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}\Gamma^{(j)}(1)I_{n-j}.$$

Now, for n odd,

$$\begin{aligned}\int_0^\infty\frac{\log^n s}{1+s^2}\mathrm{d}s &= \int_0^1\frac{\log^n s}{1+s^2}\mathrm{d}s+\int_1^\infty\frac{\log^n s}{1+s^2}\mathrm{d}s\\ &= \int_0^1\frac{\log^n s}{1+s^2}\mathrm{d}s-\int_0^1\frac{\log^n s}{1+s^2}\mathrm{d}s=0,\end{aligned}$$

while for n even,

$$\begin{aligned}\int_0^\infty\frac{\log^n s}{1+s^2}\mathrm{d}s &= 2\int_0^1\frac{\log^n s}{1+s^2}\mathrm{d}s = 2\sum_{k=0}^{\infty}\int_0^1(-1)^k s^{2k}\log^n s\mathrm{d}s\\ &= 2n!\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)^{n+1}} = 2n!\beta(n+1)\\ &= \frac{(-1)^{n/2}E_n\pi^{n+1}}{2^{n+1}},\end{aligned}$$

where $\beta(n)$ is the Dirichlet beta function and E_n is the nth Euler number. Finally, for the values of the derivatives of the Gamma function evaluated at 1, there is the recursion

$$\Gamma^{(n+1)}(1) = -\gamma\Gamma^{(n)}(1)+n!\sum_{k=1}^{n}\frac{(-1)^{k+1}}{(n-k)!}\zeta(k+1)\Gamma^{(n-k)}(1),$$

see equation (2.2) in J. Choi and H.M. Srivastava, Evaluation of higher-order derivatives of the Gamma function, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* **11** (2000), 9–18. Thus,

$$I_0 = \frac{\pi}{2}$$

and for $n \geq 1$,

$$I_{2n-1} = \sum_{j=1}^{2n-1} (-1)^{j-1} \binom{2n-1}{j} \Gamma^{(j)}(1) I_{2n-1-j}$$

and

$$I_{2n} = \frac{(-1)^n E_{2n} \pi^{2n+1}}{2^{2n+1}} + \sum_{j=1}^{2n} (-1)^{j-1} \binom{2n}{j} \Gamma^{(j)}(1) I_{2n-j}.$$

We have two recurrences, one for the derivatives of Γ at 1 and one for the values of I_n. We start with $\Gamma'(1) = -\gamma\Gamma(1) = -\gamma$. Next we get

$$\begin{aligned}
\Gamma'(1) &= -\gamma\Gamma(1) = -\gamma, \\
\Gamma''(1) &= -\gamma\Gamma'(1) + \zeta(2)\Gamma(1) = \gamma^2 + \zeta(2), \\
\Gamma'''(1) &= -\gamma\Gamma''(1) + 2\left(\zeta(2)\Gamma'(1) - \zeta(3)\Gamma(1)\right) = -\gamma^3 - 3\gamma\zeta(2) - 2\zeta(3), \\
\Gamma^{(4)}(1) &= -\gamma\Gamma'''(1) + 6\left(\frac{1}{2}\zeta(2)\Gamma''(1) - \zeta(3)\Gamma'(1) + \zeta(4)\Gamma(1)\right) \\
&= \gamma^4 + 6\gamma^2\zeta(2) + 3\zeta^2(2) + 8\gamma\zeta(3) + 6\zeta(4).
\end{aligned}$$

For the first values of I_n we start with $I_0 = \frac{\pi}{2}$ and we get

$$\begin{aligned}
I_1 &= \Gamma'(1)I_0 = -\frac{\gamma\pi}{2}, \\
I_2 &= -\frac{E_2\pi^3}{2^3} + 2\Gamma'(1)I_1 - \Gamma''(1)I_0 = \frac{\pi^3}{8} + \gamma^2\pi - (\gamma^2 + \zeta(2))\frac{\pi}{2} = \frac{\pi^3}{24} + \frac{\gamma^2\pi}{2}, \\
I_3 &= 3\Gamma'(1)I_2 - 3\Gamma''(1)I_1 + \Gamma'''(1)I_0 \\
&= -3\gamma\left(\frac{\pi^3}{24} + \frac{\gamma^2\pi}{2}\right) + \frac{3\gamma\pi}{2}(\gamma^2 + \zeta(2)) - (\gamma^3 + 3\gamma\zeta(2) - 2\zeta(3))\frac{\pi}{2} \\
&= -\frac{\gamma\pi^3}{8} - \frac{\gamma^3\pi}{2} - \pi\zeta(3), \\
I_4 &= \frac{E_4\pi^5}{2^5} + 4\Gamma'(1)I_3 - 6\Gamma''(1)I_2 + 4\Gamma'''(1)I_1 - \Gamma^{(4)}(1)I_0 \\
&= \frac{19\pi^5}{480} + \frac{\gamma^2\pi^3}{4} + \frac{\gamma^4\pi}{2} + 4\gamma\pi\zeta(3).
\end{aligned}$$

528. If $a_1, a_2, a_3, a_4, a_5, a_6$ are non-negative real numbers such that

$$a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_6 + a_6a_1 = 6,$$

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6,$$

then

$$\frac{1}{a_1+3}+\frac{1}{a_2+3}+\frac{1}{a_3+3}+\frac{1}{a_4+3}+\frac{1}{a_5+3}+\frac{1}{a_6+3} \geq \frac{3}{2}.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploieşti, Romania.

Solution by the author. We apply Jensen's inequality to the function $f(x) = \dfrac{1}{x+3}$, which is strictly convex on $(-3, \infty)$. We have

$$\frac{1}{a_2+3}+\frac{1}{a_3+3}+\frac{1}{a_4+3}+\frac{1}{a_5+3} \geq \frac{4}{b+3},$$

where

$$b = \frac{a_2+a_3+a_4+a_5}{4}, \qquad a_2 \geq b \geq a_5.$$

Thus, it suffices to show that

$$\frac{1}{a_1+3}+\frac{1}{a_6+3}+\frac{4}{b+3} \geq \frac{3}{2}.$$

First we will show that

$$3b^2 + b(a_1+a_6) + a_1a_6 \leq 6.$$

Using the constraint from the hypothesis, the inequality writes as

$$3b^2 + b(a_1+a_6) + a_1a_6 \leq a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_6 + a_6a_1$$

$$\iff 3b^2 + a_1(b-a_2) + a_6(b-a_5) \leq a_2a_3 + a_3a_4 + a_4a_5.$$

Since $b - a_2 \leq 0$, $b - a_5 \geq 0$, $a_1 \geq a_2$, and $a_5 \geq a_6$, it suffices to show that

$$3b^2 + a_2(b-a_2) + a_5(b-a_5) \leq a_2a_3 + a_3a_4 + a_4a_5,$$

which can be rewritten as

$$a_2a_3 + a_3a_4 + a_4a_5 \geq 3b^2 + b(a_2+a_5) - a_2^2 - a_5^2.$$

Since the sequences a_2, a_3, a_4 and a_3, a_4, a_5 are decreasing, as a consequence of the rearrangement inequality we have

$$3(a_2a_3 + a_3a_4 + a_4a_5) \geq (a_2+a_3+a_4)(a_3+a_4+a_5) = (4b-a_5)(4b-a_2).$$

Thus, it suffices to show that

$$(4b-a_5)(4b-a_2) \geq 9b^2 + 3b(a_2+a_5) - 3(a_2^2+a_5^2),$$

which is equivalent to $7b^2 - 7b(a_2+a_5) + 3(a_2^2+a_5^2) + a_2a_5 \geq 0$ and

$$7(2b-a_2-a_5)^2 + 5(a_2-a_5)^2 \geq 0.$$

Denote now a_1 and a_6 by a and c, respectively. So, we need to show that

$$\frac{1}{a+3}+\frac{1}{c+3}+\frac{4}{b+3} \geq \frac{3}{2}$$

for $a \ge b \ge c \ge 0$ such that $3b^2 + b(a+c) + ac \le 6$. From $6 \ge 3b^2 + b(a+c) + ac > 3b^2$, it follows that $b < \sqrt{2}$. Denoting $S = \dfrac{a+c}{2}$, we need to prove that

$$\frac{2S+6}{ac+6S+9} + \frac{4}{b+3} \ge \frac{3}{2}.$$

Since $ac \le 6 - 2bS - 3b^2$, it suffices to show that

$$\frac{2S+6}{15-3b^2+2(3-b)S} - \frac{1+3b}{2(b+3)} \ge 0.$$

After multiplying by $3-b>0$, we get the equivalent inequalities

$$\frac{18-6b+2(3-b)S}{15-3b^2+2(3-b)S} + \frac{3b^2-8b-3}{2(b+3)} \ge 0,$$

$$1 + \frac{3(b-1)^2}{15-3b^2+2(3-b)S} + \frac{3b^2-8b-3}{2(b+3)} \ge 0,$$

$$\frac{3(b-1)^2}{15-3b^2+2(3-b)S} + \frac{3(b-1)^2}{2(b+3)} \ge 0.$$

The proof is completed. The equality occurs for $a_2 = a_3 = a_4 = a_5 = 1$ and $a_1 + a_6 + a_1 a_6 = 3$ $(a_1 \ge 1 \ge a_6)$. □