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On the principal minors of the powers of a matrix

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Abstract. We show that if A is an $n \times n$-matrix, then the diagonal entries of each power A^m are uniquely determined by the principal minors of A, and can be written as universal (integral) polynomials in the latter. Furthermore, if the latter all equal 1, then so do the former. These results are inspired by Problem B5 on the Putnam contest 2021, and shed a new light on the behavior of minors under matrix multiplication.

Keywords: Principal minors, determinantal identities, determinants, matrices, determinantal ideals, Putnam contest.

MSC: 11C20, 14M12, 15A15.

1. Introduction

Let R be a commutative ring. Let A be an $n \times n$-matrix over R, where n is a nonnegative integer.

A *principal submatrix* of A means a matrix obtained from A by removing some rows and the corresponding columns (i.e., removing the i_1-th, i_2-th, …, i_k-th rows and the i_1-th, i_2-th, …, i_k-th columns for some choice of k integers $i_1, i_2, \ldots, i_k$ satisfying $1 \leq i_1 < i_2 < \cdots < i_k \leq n$). In particular, A itself is a principal submatrix of A (obtained for $k = 0$).

A *principal minor* of A means the determinant of a principal submatrix of A. In particular, each diagonal entry of A is a principal minor of A (being the determinant of a principal submatrix of size 1×1). In total, A has 2^n principal minors, including its own determinant $\det A$ as well as the trivial principal minor 1 (obtained as the determinant of a 0×0 matrix, which is what remains when all rows and columns are removed).

Problem B5 on the Putnam contest 2021 (see [1] or [3]) asked for a proof of the following:

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Theorem 1. *Assume that $R = \mathbb{Z}$. Assume that each principal minor of A is odd. Then, each principal minor of A^m is odd whenever m is a nonnegative integer.*

Without giving the solution away, it shall be noticed that essentially only one proof is known (see [1] or [3] for it), and it is not as algebraic as the statement of Theorem 1 might suggest. In particular, it is unclear if the theorem remains valid if "odd" is replaced by "congruent to 1 modulo 4", or if R is replaced by another ring; the official solution (most of which originates in a result by Dobrinskaya [2, Lemma 3.3]) certainly does not apply to such extensions. An approach that is definitely doomed is to try expressing the principal minors of a power A^m in terms of those of A. The following example shows that the latter do not uniquely determine the former:

Example 2. Set

$$C := \begin{pmatrix} a & b & 1 & 1 \\ c & d & 1 & 1 \\ 1 & 1 & p & q \\ 1 & 1 & r & s \end{pmatrix} \qquad \text{and} \qquad D := \begin{pmatrix} a & b & 1 & 1 \\ c & d & 1 & 1 \\ 1 & 1 & p & r \\ 1 & 1 & q & s \end{pmatrix}$$

for some $a, b, c, d, p, q, r, s \in R$. Then, the matrices C and D have the same principal minors, but their squares C^2 and D^2 differ in their $\{2, 3\}$-principal minor (i.e., their principal minor obtained by removing the 1-st and 4-th rows and columns) unless $(q - r)(b - c) = 0$. Thus, the principal minors of the square of a matrix are not uniquely determined by the principal minors of the matrix itself.

This example is inspired by [4, Example 3], where further related discussion of matrices with equal principal minors can be found.

2. NEVERTHELESS...

However, not all is lost. Among the principal minors of A^m, the simplest ones (besides 1) are those of size 1×1, that is, the diagonal entries of A^m. It turns out that these diagonal entries are indeed uniquely determined by the principal minors of A, and even better, they can be written as universal polynomials[1] in the latter. That is, we have the following:[2]

Theorem 3. *Let n and m be nonnegative integers, and let $i \in \{1, 2, \ldots, n\}$. Then, there exists an integer polynomial $P_{n,i,m}$ in 2^n indeterminates that is independent of R and A, and that has the following property: If A is any $n \times n$-matrix over any commutative ring R, then the i-th diagonal entry of*

[1]A "universal polynomial" means a polynomial with integer coefficients that depends neither on A nor on R (but can depend on m as well as on the location of the diagonal entry).

[2]An *integer polynomial* means a polynomial with integer coefficients.

A^m can be obtained by substituting the principal minors of A into $P_{n,i,m}$. In particular, the principal minors of A uniquely determine this entry.

Let us verify this for $m = 2$: If we denote the (i,j)-th entry of a matrix B by $B_{i,j}$, then each diagonal entry of A^2 has the form

$$\left(A^2\right)_{i,i} = \sum_{j=1}^{n} A_{i,j}A_{j,i} = A_{i,i}^2 + \sum_{j\neq i} \underbrace{A_{i,j}A_{j,i}}_{=A_{i,i}A_{j,j}-\det\begin{pmatrix} A_{i,i} & A_{i,j} \\ A_{j,i} & A_{j,j} \end{pmatrix}}$$
$$= A_{i,i}^2 + \sum_{j\neq i} \left(A_{i,i}A_{j,j} - \det\begin{pmatrix} A_{i,i} & A_{i,j} \\ A_{j,i} & A_{j,j} \end{pmatrix}\right),$$

which is visibly an integer polynomial in the principal minors of A (since all the $A_{i,i}$ and $A_{j,j}$ and $\det\begin{pmatrix} A_{i,i} & A_{i,j} \\ A_{j,i} & A_{j,j} \end{pmatrix}$ are principal minors of A). This verifies Theorem 3 for $m = 2$. Such explicit computations remain technically possible for higher values of m, but become longer and more cumbersome as m increases.

The goal of this note is to prove Theorem 3. We will first show the following theorem, which looks weaker but is essentially equivalent:

Theorem 4. *Let n and m be nonnegative integers. Let R be a commutative ring. Let A be an $n \times n$-matrix over R. Let $\mathcal{P}$ be the subring of R generated by all principal minors of A. Then, all diagonal entries of A^m belong to $\mathcal{P}$.*

Before we prove this, let us explain how Theorem 3 can be easily derived from Theorem 4:

Proof. [Proof of Theorem 3 using Theorem 4.] The notation $B_{i,j}$ shall denote the (i,j)-th entry of any matrix B.

Let $\mathbf{R}$ be the polynomial ring $\mathbb{Z}\left[x_{i,j} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n\right]$ in n^2 independent indeterminates $x_{i,j}$ over $\mathbb{Z}$. (For instance, if $n = 2$, then $\mathbf{R} = \mathbb{Z}\left[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}\right]$.) Let $\mathbf{A}$ be the $n \times n$-matrix over $\mathbf{R}$ whose (i,j)-th entry is $x_{i,j}$ for each $(i,j) \in \{1,2,\ldots,n\}^2$. This matrix $\mathbf{A}$ is known as "the general $n \times n$-matrix", since any matrix A over any commutative ring can be obtained from it by substituting appropriate elements (viz., the entries of A) for the variables $x_{i,j}$. This very property will be crucial to the argument that follows.

Let $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{2^n}$ be the 2^n principal minors of $\mathbf{A}$ (numbered in some order). Let $\mathcal{P}$ denote the subring of $\mathbf{R}$ generated by all these principal minors of $\mathbf{A}$. Theorem 4 (applied to $\mathbf{R}$ and $\mathbf{A}$ instead of R and A) shows that all diagonal entries of $\mathbf{A}^m$ belong to $\mathcal{P}$. In other words, for each $i \in \{1,2,\ldots,n\}$, we have $\left(\mathbf{A}^m\right)_{i,i} \in \mathcal{P}$.

Fix $i \in \{1,2,\ldots,n\}$. As we just showed, we have $\left(\mathbf{A}^m\right)_{i,i} \in \mathcal{P}$. In other words, there exists an integer polynomial $P_{n,i,m}$ in 2^n indeterminates such that $(\mathbf{A}^m)_{i,i} = P_{n,i,m}(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{2^n})$ (since $\mathcal{P}$ is the subring of $\mathbf{R}$ generated by $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{2^n}$). Consider this polynomial $P_{n,i,m}$; note that it is independent of R and A (by its very construction).

Now, consider a commutative ring R and an $n \times n$-matrix A over R. Let $p_1, p_2, \ldots, p_{2^n}$ be the 2^n principal minors of A (numbered in the same order as $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{2^n}$). Let $f : \mathbf{R} \to R$ be the $\mathbb{Z}$-algebra homomorphism that sends each indeterminate $x_{i,j}$ to the (i,j)-th entry $A_{i,j}$ of A. This homomorphism f therefore sends each entry of the matrix $\mathbf{A}$ to the corresponding entry of A, and thus also sends each principal minor of $\mathbf{A}$ to the corresponding principal minor of A (since a principal minor is a certain signed sum of products of entries of the matrix). In other words,

$$f(\mathbf{p}_i) = p_i \qquad \text{for each } i \in \{1, 2, \ldots, 2^n\}. \tag{1}$$

However, $P_{n,i,m}$ is an integer polynomial, and thus "commutes" with any $\mathbb{Z}$-algebra homomorphism – i.e., if $a_1, a_2, \ldots, a_{2^n}$ are any 2^n elements of a commutative ring, and if g is any $\mathbb{Z}$-algebra homomorphism out of that ring, then

$$g(P_{n,i,m}(a_1, a_2, \ldots, a_{2^n})) = P_{n,i,m}(g(a_1),\ g(a_2),\ \ldots,\ g(a_{2^n})).$$

Applying this to $a_i = \mathbf{p}_i$ and $g = f$, we obtain

$$\begin{aligned} f(P_{n,i,m}(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{2^n})) &= P_{n,i,m}(f(\mathbf{p}_1),\ f(\mathbf{p}_2),\ \ldots,\ f(\mathbf{p}_{2^n})) \\ &= P_{n,i,m}(p_1, p_2, \ldots, p_{2^n}) \qquad \text{(by (1))}. \end{aligned}$$

In view of $(\mathbf{A}^m)_{i,i} = P_{n,i,m}(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{2^n})$, we can rewrite this as

$$f\left((\mathbf{A}^m)_{i,i}\right) = P_{n,i,m}(p_1, p_2, \ldots, p_{2^n}). \tag{2}$$

However, the $\mathbb{Z}$-algebra homomorphism f sends each entry of the matrix $\mathbf{A}$ to the corresponding entry of A, and therefore also sends each entry of the matrix $\mathbf{A}^m$ to the corresponding entry of A^m (since the entries of $\mathbf{A}^m$ are certain sums of products of entries of $\mathbf{A}$, whereas the entries of A^m are the same sums of products of entries of A). In other words, $f\left((\mathbf{A}^m)_{u,v}\right) = (A^m)_{u,v}$ for any $u, v \in \{1, 2, \ldots, n\}$. Thus, in particular, $f\left((\mathbf{A}^m)_{i,i}\right) = (A^m)_{i,i}$. Comparing this with (2), we obtain $(A^m)_{i,i} = P_{n,i,m}(p_1, p_2, \ldots, p_{2^n})$. In other words, the i-th diagonal entry of A^m can be obtained by substituting the principal minors of A into $P_{n,i,m}$ (since $p_1, p_2, \ldots, p_{2^n}$ are these principal minors of A). This proves Theorem 3. $\square$

3. Notations

In order to prove Theorem 4, we will need some more notations regarding matrices and their minors:

- If $m \in \mathbb{Z}$, then $[m]$ shall denote the set $\{1, 2, \ldots, m\}$.

- If B is a $u \times v$-matrix and if $i \in [u]$ and $j \in [v]$, then $B_{i,j}$ shall denote the (i,j)-th entry of B.
- If u and v are two nonnegative integers, and if $a_{i,j}$ is an element of a ring for each $i \in [u]$ and $j \in [v]$, then the notation $(a_{i,j})_{1 \le i \le u,\ 1 \le j \le v}$ means the $u \times v$-matrix whose (i,j)-th entry is $a_{i,j}$ for all $i \in [u]$ and $j \in [v]$.
- If B is a $u \times v$-matrix, and if $(i_1, i_2, \ldots, i_p) \in [u]^p$ and $(j_1, j_2, \ldots, j_q) \in [v]^q$ are two sequences of integers, then $\operatorname{sub}_{i_1,i_2,\ldots,i_p}^{j_1,j_2,\ldots,j_q} B$ shall denote the $p \times q$-matrix $\left(B_{i_x,j_y}\right)_{1 \le x \le p,\ 1 \le y \le q}$. If $i_1 < i_2 < \cdots < i_p$ and $j_1 < j_2 < \cdots < j_q$, then this matrix is a submatrix of B.
- If B is a $u \times v$-matrix, and if I is a subset of $[u]$, and if J is a subset of $[v]$, then $\operatorname{sub}_I^J B$ shall denote the submatrix $\operatorname{sub}_{i_1,i_2,\ldots,i_p}^{j_1,j_2,\ldots,j_q} B$ of B, where $i_1, i_2, \ldots, i_p$ are the elements of I in increasing order, and where $j_1, j_2, \ldots, j_q$ are the elements of J in increasing order.

  Thus, in particular, if B is an $n \times n$-matrix, and if I is a subset of $[n]$, then $\operatorname{sub}_I^I B$ is a principal submatrix of B, so that $\det\left(\operatorname{sub}_I^I B\right)$ is a principal minor of B.
- If B is an $n \times n$-matrix, and if $i, j \in [n]$, then $B_{\sim i,\sim j}$ shall denote the submatrix of B obtained by removing the i-th row and the j-th column from B. In other words, $B_{\sim i,\sim j}$ denotes the matrix $\operatorname{sub}_{[n]\setminus\{i\}}^{[n]\setminus\{j\}} B$.
- If B is an $n \times n$-matrix, then $\operatorname{adj} B$ shall mean the *adjugate matrix* of B. This is defined as the $n \times n$-matrix

$$\left((-1)^{i+j} \det\left(B_{\sim j,\sim i}\right)\right)_{1 \le i \le n,\ 1 \le j \le n}.$$

- If m is a nonnegative integer, then I_m denotes the $m \times m$ identity matrix.

We will need the following properties of determinants:

- For any $n \times n$-matrix B, we have

$$B \cdot (\operatorname{adj} B) = (\operatorname{adj} B) \cdot B = (\det B) \cdot I_n. \tag{3}$$

  (This is the main property of adjugates; see, e.g., [5, Theorem 6.100] for a proof.)
- For any commutative ring S, any $m \times m$-matrix B and any element $x \in S$, we have

$$\det\left(B + xI_m\right) = \sum_{P \subseteq [m]} \det\left(\operatorname{sub}_P^P B\right) \cdot x^{m-|P|}. \tag{4}$$

  (This is a folklore result – essentially the explicit formula for the characteristic polynomial of a matrix in terms of its principal minors. The proof is straightforward: Expand the left hand side into a sum of products, and combine products according to "which factors come from B and which factors come from xI_m". See [7, Proposition 6.4.29] or [5, Corollary 6.164] for detailed proofs.)

For any ring S, we consider the univariate polynomial ring $S[t]$ as well as the ring $S[[t]]$ of formal power series. Of course, $S[t]$ is a subring of $S[[t]]$. Note that the ring S needs not be commutative for $S[t]$ and $S[[t]]$ to be defined.

4. Proof of Theorem 4

We now finally step to the proof of Theorem 4.

Proof. [Proof of Theorem 4.] We must prove that all diagonal entries of A^m belong to $\mathcal{P}$. In other words, we must prove that $(A^m)_{i,i} \in \mathcal{P}$ for each $i \in [n]$.

It is well-known that a polynomial over a matrix ring is "essentially the same as" a matrix with polynomial entries. In other words, we can identify the ring $R^{n\times n}[t]$ with the ring $(R[t])^{n\times n}$ using a straightforward ring isomorphism (which sends each $\sum_{i\geq 0} C_i t^i \in R^{n\times n}[t]$ to $\sum_{i\geq 0} C_i t^i \in (R[t])^{n\times n}$). In the same way, we identify the ring $R^{n\times n}[[t]]$ with the ring $(R[[t]])^{n\times n}$.

Let B be the matrix $I_n - tA$ in the power series ring $R^{n\times n}[[t]]$. This matrix $B = I_n - tA$ is invertible, and its inverse is

$$B^{-1} = I_n + tA + t^2A^2 + t^3A^3 + \cdots . \tag{5}$$

(This can be proved by directly verifying that $I_n + tA + t^2A^2 + t^3A^3 + \cdots$ is inverse to $I_n - tA$. Indeed, both products $\left(I_n + tA + t^2A^2 + t^3A^3 + \cdots\right) \cdot (I_n - tA)$ and $(I_n - tA) \cdot \left(I_n + tA + t^2A^2 + t^3A^3 + \cdots\right)$ turn, upon expanding, into sums that telescope to I_n.)

Since the matrix B is invertible, its determinant $\det B$ is invertible as well (since $(\det B) \cdot (\det(B^{-1})) = \det\left(\underbrace{BB^{-1}}_{=I_n}\right) = \det(I_n) = 1$).

Now, recall that we must prove that $(A^m)_{i,i} \in \mathcal{P}$ for each $i \in [n]$. So let us fix $i \in [n]$. Then, (5) yields

$$\begin{aligned}\left(B^{-1}\right)_{i,i} &= \left(I_n + tA + t^2A^2 + t^3A^3 + \cdots\right)_{i,i} \\ &= (I_n)_{i,i} + tA_{i,i} + t^2\left(A^2\right)_{i,i} + t^3\left(A^3\right)_{i,i} + \cdots .\end{aligned}$$

Hence, the t^m-coefficient of the power series $(B^{-1})_{i,i} \in R[[t]]$ is $(A^m)_{i,i}$. Thus, in order to prove that $(A^m)_{i,i} \in \mathcal{P}$ (which is our goal), it suffices to show that all coefficients of the power series $(B^{-1})_{i,i}$ belong to $\mathcal{P}$. In other words, it suffices to show that $(B^{-1})_{i,i} \in \mathcal{P}[[t]]$. This is what we shall now show.

From (3), we obtain $B \cdot (\operatorname{adj} B) = (\operatorname{adj} B) \cdot B = (\det B) \cdot I_n$, so that

$$B^{-1} = \frac{1}{\det B} \cdot \operatorname{adj} B.$$

Hence,

$$\left(B^{-1}\right)_{i,i} = \frac{1}{\det B} \cdot (\operatorname{adj} B)_{i,i}. \qquad (6)$$

Our next goal is to show that both factors $\dfrac{1}{\det B}$ and $(\operatorname{adj} B)_{i,i}$ on the right hand side of this equality belong to $\mathcal{P}[[t]]$. This will then entail that $\left(B^{-1}\right)_{i,i} \in \mathcal{P}[[t]]$ as well, and we will be done.

From $B = I_n - tA = -tA + 1I_n$, we obtain

$$\begin{aligned}
\det B = \det(-tA + 1I_n) &= \sum_{P \subseteq [n]} \det\left(\underbrace{\operatorname{sub}_P^P(-tA)}_{=-t\operatorname{sub}_P^P A}\right) \cdot \underbrace{1^{n-|P|}}_{=1} \\
&\qquad \left(\begin{array}{c}\text{by (4), applied to } n,\ R[[t]],\ -tA \text{ and } 1 \\ \text{instead of } m,\ S,\ B \text{ and } x\end{array}\right) \\
&= \sum_{P \subseteq [n]} \underbrace{\det\left(-t\operatorname{sub}_P^P A\right)}_{=(-t)^{|P|}\det\left(\operatorname{sub}_P^P A\right)} \\
&= \sum_{P \subseteq [n]} (-t)^{|P|} \underbrace{\det\left(\operatorname{sub}_P^P A\right)}_{\substack{\in \mathcal{P} \\ \text{(since } \det\left(\operatorname{sub}_P^P A\right) \text{ is} \\ \text{a principal minor of } A)}} \qquad (7) \\
&\in \mathcal{P}[t] \subseteq \mathcal{P}[[t]].
\end{aligned}$$

Thus, $\det B$ is a formal power series over $\mathcal{P}$. Moreover, (7) shows that this power series has constant term 1 (since the only addend in the sum in (7) that contributes to the constant term is the addend for $P = \varnothing$, but this addend is $\underbrace{(-t)^{|\varnothing|}}_{=1} \underbrace{\det\left(\operatorname{sub}_\varnothing^\varnothing A\right)}_{\substack{=1 \\ \text{(since the } 0\times 0\text{-matrix} \\ \text{has determinant 1)}}} = 1$). Thus, this power series is invertible in $\mathcal{P}[[t]]$. Therefore,

$$\frac{1}{\det B} \in \mathcal{P}[[t]]. \qquad (8)$$

Now, recall the definition of an adjugate matrix. This definition yields

$$
\begin{aligned}
(\operatorname{adj} B)_{i,i} &= \underbrace{(-1)^{i+i}}_{=1} \det\left(B_{\sim i,\sim i}\right) = \det\left(B_{\sim i,\sim i}\right) \\
&= \det\left(\left(-tA + 1I_n\right)_{\sim i,\sim i}\right) \\
&\qquad \left(\text{ since } B = -tA + 1I_n \right) \\
&= \det\left(-tA_{\sim i,\sim i} + I_{n-1}\right) \\
&\qquad \left(\text{ since } \left(-tA + I_n\right)_{\sim i,\sim i} = -tA_{\sim i,\sim i} + I_{n-1} \right) \\
&= \sum_{P\subseteq[n-1]} \det\left(\underbrace{\operatorname{sub}_P^P\left(-tA_{\sim i,\sim i}\right)}_{=-t\operatorname{sub}_P^P\left(A_{\sim i,\sim i}\right)}\right) \cdot \underbrace{1^{n-1-|P|}}_{=1} \\
&\qquad \left(\begin{array}{c}\text{by (4), applied to } n-1,\ R\left[[t]\right],\ -tA_{\sim i,\sim i} \text{ and } 1 \\ \text{instead of } m,\ S,\ B \text{ and } x\end{array}\right) \\
&= \sum_{P\subseteq[n-1]} \underbrace{\det\left(-t\operatorname{sub}_P^P\left(A_{\sim i,\sim i}\right)\right)}_{=(-t)^{|P|}\det\left(\operatorname{sub}_P^P\left(A_{\sim i,\sim i}\right)\right)} \\
&= \sum_{P\subseteq[n-1]} (-t)^{|P|} \det\left(\operatorname{sub}_P^P\left(A_{\sim i,\sim i}\right)\right). \qquad (9)
\end{aligned}
$$

Now, let P be an arbitrary subset of $[n-1]$. Write this subset P in the form $P = \{p_1, p_2, \ldots, p_r\}$, where $p_1 < p_2 < \cdots < p_r$. Furthermore, let $g \in \{0, 1, \ldots, r\}$ be the element that satisfies

$$p_1 < p_2 < \cdots < p_g < i \leq p_{g+1} < p_{g+2} < \cdots < p_r.$$

(Here, g will be 0 if all elements of P are $\geq i$, and g will be r if all elements of P are $< i$.) Then, due to the combinatorial nature of removing rows and columns, we have

$$\operatorname{sub}_P^P\left(A_{\sim i,\sim i}\right) = \operatorname{sub}_{P'}^{P'} A,$$

where P' is the subset $\{p_1, p_2, \ldots, p_g\} \cup \{p_{g+1}+1, p_{g+2}+1, \ldots, p_r+1\}$ of $[n]$. Hence, $\operatorname{sub}_P^P\left(A_{\sim i,\sim i}\right)$ is a principal submatrix of A. Therefore, its determinant $\det\left(\operatorname{sub}_P^P\left(A_{\sim i,\sim i}\right)\right)$ is a principal minor of A, thus belongs to $\mathcal{P}$.

Forget that we fixed P. We thus have shown that $\det\left(\operatorname{sub}_P^P\left(A_{\sim i,\sim i}\right)\right) \in \mathcal{P}$ for each $P \subseteq [n-1]$. Therefore, (9) becomes

$$(\operatorname{adj} B)_{i,i} = \sum_{P\subseteq[n-1]} (-t)^{|P|} \underbrace{\det\left(\operatorname{sub}_P^P\left(A_{\sim i,\sim i}\right)\right)}_{\in\mathcal{P}} \in \mathcal{P}[t] \subseteq \mathcal{P}\left[[t]\right]. \qquad (10)$$

Now, (6) becomes

$$\left(B^{-1}\right)_{i,i} = \underbrace{\frac{1}{\det B}}_{\substack{\in \mathcal{P}[[t]] \\ \text{(by (8))}}} \cdot \underbrace{(\operatorname{adj} B)_{i,i}}_{\substack{\in \mathcal{P}[[t]] \\ \text{(by (10))}}} \in \mathcal{P}\left[[t]\right] \cdot \mathcal{P}\left[[t]\right] \subseteq \mathcal{P}\left[[t]\right].$$

As explained above, this completes our proof of Theorem 4. $\square$

Somewhat regrettably, the above proof is the slickest I am aware of. A more-or-less equivalent proof can be given avoiding the use of power series (using [6, Proposition 3.9 and Lemma 3.11] instead). A more pedestrian (but harder to formalize) proof uses the Cayley–Hamilton theorem and a variant of the inclusion/exclusion principle.

5. Variants

A counterpart of Theorem 4 for the off-diagonal entries of A^m exists as well:

Theorem 5. *Let n, m, R, A and $\mathcal{P}$ be as in Theorem 4.*

*Let i and j be two distinct elements of $[n]$. An (i,j)-*quasiprincipal minor *of A shall mean a determinant of the form $\det\left(\operatorname{sub}_I^J A\right)$, where I and J are two subsets of $[n]$ satisfying*

$$i \in I \text{ and } j \in J \text{ and } |I| = |J| \text{ and } J = (I \setminus \{i\}) \cup \{j\}.$$

(For instance, if $n \geq 7$, then $\det\left(\operatorname{sub}_{\{1,2,7\}}^{\{2,5,7\}} A\right)$ is a $(1,5)$-quasiprincipal minor of A.)

Let $\mathcal{K}_{i,j}$ be the $\mathbb{Z}$-submodule of R spanned by all (i,j)-quasiprincipal minors of A. Then,

$$\left(A^m\right)_{i,j} \in \mathcal{P} \cdot \mathcal{K}_{i,j}.$$

Proof. [Proof outline.] This is similar to our above proof of Theorem 4, but some changes are needed. Most importantly, instead of proving that $(\operatorname{adj} B)_{i,i} \in \mathcal{P}\left[[t]\right]$, we now need to show that $(\operatorname{adj} B)_{i,j} \in \mathcal{K}_{i,j}\left[[t]\right]$ (that is, that all coefficients of the power series $(\operatorname{adj} B)_{j,i}$ belong to $\mathcal{K}_{i,j}$). To do so, we apply the definition of the adjugate matrix to see that

$$(\operatorname{adj} B)_{i,j} = (-1)^{j+i} \det\left(B_{\sim j, \sim i}\right). \tag{11}$$

We can simplify $B_{\sim j,\sim i}$ further to $-tA_{\sim j,\sim i} + (I_n)_{\sim j,\sim i}$ (since $B = I_n - tA = -tA + I_n$), but unfortunately this is not the same as $-tA_{\sim j,\sim i} + 1I_{n-1}$, and thus we can no longer apply (4). Instead, we use a trick:

- We define A' to be the matrix obtained from $-tA$ by replacing the j-th row by $(0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)$, where the only entry equal to 1 is in the i-th position.
- We define I_n' to be the matrix obtained from I_n by replacing the 1 in the j-th row by a 0.

- We define B' to be the matrix obtained from B by replacing the j-th row by $(0,0,\ldots,0,1,0,0,\ldots,0)$, where the only entry equal to 1 is in the i-th position.

Laplace expansion along the j-th row shows that

$$\det(B') = (-1)^{j+i} \det\left((B')_{\sim j,\sim i}\right) = (-1)^{j+i} \det(B_{\sim j,\sim i})$$

(since the matrix B' differs from B only in the j-th row, and thus we have $(B')_{\sim j,\sim i} = B_{\sim j,\sim i}$). Comparing this with (11), we find

$$(\operatorname{adj} B)_{i,j} = \det(B'). \tag{12}$$

Furthermore, recall that $B = I_n - tA = -tA + I_n$. Thus, $B' = A' + I'_n$ (based on how A', I'_n and B' were constructed).

On the other hand, the definition of I'_n shows that I'_n is a diagonal $n\times n$-matrix with diagonal entries $1,1,\ldots,1,0,1,1,\ldots,1$, where the only diagonal entry equal to 0 is in the j-th position. However, another classical fact about determinants ([7, Theorem 6.4.26], [5, Corollary 6.162]) shows that if C is any $n\times n$-matrix, and if D is a diagonal $n\times n$-matrix with diagonal entries $d_1, d_2, \ldots, d_n$, then

$$\det(C+D) = \sum_{P\subseteq[n]} \det\left(\operatorname{sub}_P^P C\right) \cdot \prod_{k\in[n]\setminus P} d_k.$$

We can apply this to $C = A'$, $(d_1, d_2, \ldots, d_n) = \underbrace{(1,1,\ldots,1,0,1,1,\ldots,1)}_{\text{the 0 is in the } j\text{-th position}}$, and $D = I'_n$, and thus obtain

$$\begin{aligned}
\det(A' + I'_n) &= \sum_{P\subseteq[n]} \det\left(\operatorname{sub}_P^P (A')\right) \cdot \prod_{k\in[n]\setminus P} \begin{cases} 1, & \text{if } k \neq j; \\ 0, & \text{if } k = j \end{cases} \\
&\qquad \left(\text{since the } k\text{-th diagonal entry of } I'_n \text{ is } \begin{cases} 1, & \text{if } k \neq j; \\ 0, & \text{if } k = j \end{cases}\right) \\
&= \sum_{P\subseteq[n]} \det\left(\operatorname{sub}_P^P (A')\right) \cdot \begin{cases} 1, & \text{if } j \notin [n]\setminus P; \\ 0, & \text{if } j \in [n]\setminus P \end{cases} \\
&= \sum_{\substack{P\subseteq[n];\\ j\notin[n]\setminus P}} \det\left(\operatorname{sub}_P^P (A')\right) = \sum_{\substack{P\subseteq[n];\\ j\in P}} \det\left(\operatorname{sub}_P^P (A')\right) \\
&= \sum_{\substack{P\subseteq[n];\\ j\in P \text{ and } i\in P}} \det\left(\operatorname{sub}_P^P (A')\right) + \sum_{\substack{P\subseteq[n];\\ j\in P \text{ and } i\notin P}} \underbrace{\det\left(\operatorname{sub}_P^P (A')\right)}_{\substack{=0 \\ \text{(since all entries in the } j\text{-th row of } A' \\ \text{are 0 except for the } i\text{-th entry, and thus} \\ \text{the matrix } \operatorname{sub}_P^P(A') \text{ has a zero row)}}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{P\subseteq[n];\\ j\in P \text{ and } i\in P}} \underbrace{\det\left(\operatorname{sub}_P^P(A')\right)}_{\substack{=\pm\det\left(\operatorname{sub}_{P\setminus\{j\}}^{P\setminus\{i\}}(-tA)\right)\\ \text{(by Laplace expansion along the row}\\ \text{that was the } j\text{-th row of } A')}} \\
&= \sum_{\substack{P\subseteq[n];\\ j\in P \text{ and } i\in P}} \pm \underbrace{\det\left(\operatorname{sub}_{P\setminus\{j\}}^{P\setminus\{i\}}(-tA)\right)}_{=\pm t^{|P|-1}\det\left(\operatorname{sub}_{P\setminus\{j\}}^{P\setminus\{i\}} A\right)} \\
&= \sum_{\substack{P\subseteq[n];\\ j\in P \text{ and } i\in P}} \pm t^{|P|-1} \underbrace{\det\left(\operatorname{sub}_{P\setminus\{j\}}^{P\setminus\{i\}} A\right)}_{\substack{\in\mathcal{K}_{i,j}\\ \text{(since } \det\left(\operatorname{sub}_{P\setminus\{j\}}^{P\setminus\{i\}} A\right) \text{ is}\\ \text{an } (i,j)\text{-quasiprincipal minor of } A)}} \\
&\in \sum_{\substack{P\subseteq[n];\\ j\in P \text{ and } i\in P}} \pm t^{|P|-1}\mathcal{K}_{i,j} \subseteq \mathcal{K}_{i,j}[[t]].
\end{aligned}
$$

In view of $B' = A' + I_n'$, this rewrites as $\det(B') \in \mathcal{K}_{i,j}[[t]]$. Hence, (12) becomes $(\operatorname{adj} B)_{i,j} = \det(B') \in \mathcal{K}_{i,j}[[t]]$. Having showed this, we can finish the proof as we did for Theorem 4. $\square$

Another variant of Theorem 4 is the following:

Theorem 6. *Let n and m be nonnegative integers. Let R be a commutative ring. Let A be an $n\times n$-matrix over R. Assume that all principal minors of A equal 1. Then, all diagonal entries of A^m equal 1.*

Proof. Follow the above proof of Theorem 4. From (7), we obtain

$$\det B = \sum_{P\subseteq[n]} (-t)^{|P|} \underbrace{\det\left(\operatorname{sub}_P^P A\right)}_{\substack{=1\\ \text{(by assumption, since } \operatorname{sub}_P^P A\\ \text{is a principal minor of } A)}} = \sum_{P\subseteq[n]} (-t)^{|P|} = (1-t)^n$$

(since the binomial formula yields $(1-t)^n = \sum_{k=0}^n \binom{n}{k} (-t)^k = \sum_{P\subseteq[n]} (-t)^{|P|}$). Let $i\in\{1,2,\ldots,n\}$. From (9), we obtain

$$(\operatorname{adj} B)_{i,i} = \sum_{P\subseteq[n-1]} (-t)^{|P|} \underbrace{\det\left(\operatorname{sub}_P^P(A_{\sim i,\sim i})\right)}_{\substack{=1\\ \text{(by assumption,}\\ \text{since } \operatorname{sub}_P^P(A_{\sim i,\sim i})\\ \text{is a principal minor of } A)}} = \sum_{P\subseteq[n-1]} (-t)^{|P|} = (1-t)^{n-1}$$

(again by the binomial formula). Now, (6) becomes

$$\left(B^{-1}\right)_{i,i} = \frac{1}{\det B} \cdot (\operatorname{adj} B)_{i,i} = \underbrace{(\operatorname{adj} B)_{i,i}}_{=(1-t)^{n-1}} / \underbrace{(\det B)}_{=(1-t)^{n}} = (1-t)^{n-1} / (1-t)^{n}$$
$$= \frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots .$$

Thus, the t^m-coefficient of the power series $\left(B^{-1}\right)_{i,i} \in R[[t]]$ is 1. However, we have already seen that this coefficient is $(A^m)_{i,i}$. Thus, we conclude that $(A^m)_{i,i} = 1$. This shows that all diagonal entries of A^m equal 1, so that Theorem 6 is proved. $\square$

6. Back to Putnam 2021

As already mentioned, we do not know whether Theorem 1 can be generalized by replacing "odd" by "congruent to 1 modulo 4". More generally, we are tempted to ask the following:

Question 1. *Fix a commutative ring R. Let A be an $n \times n$-matrix over R. Let m be a nonnegative integer. Assume that each principal minor of A is 1. Is it true that each principal minor of A^m is 1 as well?*

For $R = \mathbb{Z}/2$, this would yield Theorem 1; the "congruent to 1 modulo 4" variant would follow for $R = \mathbb{Z}/4$. Theorem 6 corresponds to the case when the principal minor of A^m is a diagonal entry. The argument from [2, Lemma 3.3] shows that Question 1 has a positive answer whenever R is an integral domain; thus, the answer is also positive when R is a product of integral domains. On the other hand, if R can be arbitrary, then the answer to Question 1 is negative, but the only counterexample we know is when R is a certain quotient ring of a polynomial ring (and $n = 4$ and $m = 2$). Here are the details: Let R be the quotient ring

$$\mathbb{Q}[x,y] / \left(x^3 + y^3, xy, x^4, x^3y, x^2y^2, xy^3, y^4\right),$$

and let $A := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & y & x \\ x & 0 & 1 & y \\ y & 0 & x & 1 \end{pmatrix} \in R^{4\times 4}$. Then, all principal minors of A are 1, but the principal minor $\det\left(\operatorname{sub}_{\{2,3\}}^{\{2,3\}}\left(A^2\right)\right) = 1 - x^3 - xy$ is not 1 since $x^3 \neq 0$ in R. Actually, we can replace $\mathbb{Q}$ by any field here (even by $\mathbb{Z}/2$); then, R becomes a finite ring. (But we cannot turn R into $\mathbb{Z}/n$ without changing the construction of A.) The smallest ring R for which the question remains open is $\mathbb{Z}/4$.

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Exotic series with Bernoulli, Harmonic, Catalan, and Stirling numbers

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Abstract. In this paper, we present a formula for generating various "exotic" series in the spirit of Ovidiu Furdui and Alina Sîntămărian [5]. Our new series (evaluated in closed form) involve Bernoulli, harmonic, and Catalan numbers. Also Stirling numbers of the second kind, other special numbers, and exponential polynomials. The results include series identities with Laguerre polynomials and derangement polynomials.

Keywords: Bernoulli numbers, Catalan numbers, harmonic numbers, Stirling numbers, derangement numbers, central binomial coefficients, exponential polynomials, Laguerre polynomials.

MSC: 11B68, 11B73, 11C08, 40D05.

1. Introduction

In a recent paper Ovidiu Furdui and Alina Sîntămărian [5] evaluated several interesting exotic series. For example, they proved that

$$\sum_{n=1}^{\infty}\left(e^x - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}\right) x^n = \frac{e^x - ex}{x-1} + 1 \tag{1}$$

for every $x \neq 1$ with a limit case for $x = 1$

$$\sum_{n=1}^{\infty}\left(e^x - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}\right) = 1.$$

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They also proved the evaluation

$$\sum_{n=k}^{\infty} \binom{n}{k} \left(e - 1 - \frac{1}{1!} - \cdots - \frac{1}{n!}\right) x^n = \\ \frac{x^k e}{(1-x)^{k+1}} \left\{1 - e^{-(1-x)} \sum_{j=0}^{k} \frac{(1-x)^j}{j!}\right\} \tag{2}$$

for $x \neq 1$ with the limit case for $x = 1$

$$\sum_{n=k}^{\infty} \binom{n}{k} \left(e - 1 - \frac{1}{1!} - \cdots - \frac{1}{n!}\right) = \frac{e}{(k+1)!}.$$

The series (2) appeared as Monthly Problem 12012 (Amer. Math. Monthly, vol. 124, December 2017) proposed by the same authors.

Furdui, in another publication [6], evaluated the series

$$\sum_{n=1}^{\infty} n^p \left(e^y - 1 - \frac{y}{1!} - \frac{y^2}{2!} - \cdots - \frac{y^n}{n!}\right) \quad (p \geq 1) \tag{3}$$

which was discussed later by the present author in [2].

In this paper we develop a unified approach for the construction and evaluation of such series. Our general result includes the cases (1), (2), and (3) and produces further interesting exotic series. The main theorem is given in the next section and in Section 3 we present the applications. For illustration, three exotic series proved in Section 3 are

$$\sum_{n=0}^{\infty} B_n \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) (-1)^n = e^y \left(y \ln(1 - e^{-y}) - \mathrm{Li}_2(e^{-y}) + \frac{\pi^2}{6}\right),$$

$$\sum_{n=0}^{\infty} H_n \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) = e^y (y \,\mathrm{Ein}(y) - y + 1) - 1,$$

$$\sum_{n=0}^{\infty} \varphi_n(\lambda) \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) = \frac{e^y}{\lambda}(1 - e^{\lambda(e^{-y}-1)}),$$

where B_n are the Bernoulli numbers, H_n are the harmonic numbers, $\mathrm{Ein}(z)$ is the exponential integral function (see below equation (24)), and $\varphi_n(\lambda)$ are the exponential polynomials. At the end we also prove series identities involving derangement polynomials and Laguerre polynomials (examples 8 and 10).

2. Generating exotic series

Suppose we have a function $F(z)$ analytic in a neighborhood of the origin and written in the form

$$F(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}. \tag{4}$$

We construct a function of two variables using the coefficients of $F(z)$

$$
\begin{aligned}
u(x,y) &= \sum_{n=0}^{\infty} a_n \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) x^n \\
&= a_0(e^y - 1) + a_1 \left(e^y - 1 - \frac{y}{1!}\right) x + \cdots .
\end{aligned}
\tag{5}
$$

To better understand the convergence of the above series we show a rough estimate for its terms. Let $a > 0$ be arbitrary. For $|y| < a$ we have the following estimate from Taylor's formula with reminder in the form of Lagrange

$$
\left| e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!} \right| \le \frac{e^a |y|^{n+1}}{(n+1)!}
$$

(here a does not depend on n). From this

$$
\left| a_n \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) x^n \right| \le \frac{e^a |a_n| |y|^{n+1} |x|^n}{(n+1)!} = \frac{|a_n||x|^n}{n!} \frac{|y|^{n+1} e^a}{n+1}.
$$

We will keep this estimate in mind when we consider the applications.

Theorem 1. *Let $F(z)$ and $u(x,y)$ be as in (4) and (5). Then for all appropriate values of x, y we have the integral representation*

$$
u(x,y) = e^y \int_0^y e^{-t} F(xt)\, \mathrm{d}t. \tag{6}
$$

Proof. We compute the partial derivative of $u(x,y)$ with respect to y

$$
u'_y(x,y) = a_0 e^y + \sum_{n=1}^{\infty} a_n \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^{n-1}}{(n-1)!} \right) x^n.
$$

Then clearly

$$
u_y - u = \sum_{n=0}^{\infty} \frac{a_n}{n!} y^n x^n = F(xy).
$$

For x fixed this is a linear differential equation (with respect to the variable y) with integrating factor e^{-y}. That is,

$$
\frac{\mathrm{d}}{\mathrm{d}y}(u e^{-y}) = e^{-y} F(xy).
$$

From here

$$
u e^{-y} = \int_0^y e^{-t} F(xt)\, \mathrm{d}t + C(x),
$$

where $C(x)$ is the constant of integration with respect to y. With $y = 0$ we find $C(x) = u(x,0) = 0$. This way

$$
u(x,y) = e^y \int_0^y e^{-t} F(xt)\, \mathrm{d}t
$$

and the theorem is proved. □

The theorem can also be proved by using Taylor's formula with integral remainder. That is,

$$\frac{1}{n!}\int_0^y (y-t)^n e^t dt = e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!} \quad (n \geq 0).$$

Example 1. Taking the function

$$F(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad a_n = 1,$$

we find

$$u(x,y) = \sum_{n=0}^{\infty} \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) x^n \ = e^y \int_0^y e^{-t} e^{xt}\, \mathrm{d}t.$$

Then for $x = 1$ we have

$$\sum_{n=0}^{\infty} \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) = ye^y$$

and for $x \neq 1$

$$\sum_{n=0}^{\infty} \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) x^n \ = \frac{e^{xy} - e^y}{x-1} \tag{7}$$

which confirms (1). (Note that in (1) the summation starts from $n = 1$.)

Example 2. Now let $p \geq 0$ be an integer. Taking the exponential generating function for the binomial coefficients

$$F(z) = \frac{z^p e^z}{p!} = \sum_{n=0}^{\infty} \binom{n}{p} \frac{z^n}{n!},$$

we find from (6)

$$\begin{aligned} E_2 := \sum_{n=0}^{\infty} \binom{n}{p} \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) x^n &= e^y \int_0^y e^{-t} \left\{\frac{(xt^p)e^{xt}}{p!}\right\} \mathrm{d}t \\ &= \frac{e^y x^p}{p!} \int_0^y t^p e^{(x-1)t}\, \mathrm{d}t. \end{aligned}$$

For $x = 1$

$$\sum_{n=0}^{\infty} \binom{n}{p} \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) = \frac{e^y y^{p+1}}{(p+1)!}$$

and for $x \neq 1$ integration by parts gives

$$E_2 = \frac{x^p e^y}{(1-x)^{p+1}} \left\{1 - e^{-(1-x)y} \sum_{j=0}^{p} \frac{(1-x)^j y^j}{j!}\right\}$$

which confirms Furdui and Sîntămărian's result (2) by setting $y = 1$.

Example 3. For the series in (3) we consider the function

$$F(z)=\sum_{n=0}^{\infty} n^p \frac{x^n}{n!}=\varphi_p(x)e^x,$$

where $p \geq 0$ is an integer, $\varphi_p(x)$ is the exponential polynomial of order p

$$\varphi_p(x)=\sum_{k=0}^{p} S(p,k)x^k, \tag{8}$$

and $S(p,k)$ are the Stirling numbers of the second kind [1, 3, 7]. Applying the theorem we find the representation (with the notational agreement $0^0=1$)

$$\sum_{n=0}^{\infty} n^p \left(e^y-1-\frac{y}{1!}-\frac{y^2}{2!}-\cdots-\frac{y^n}{n!}\right)x^n = e^y\int_0^y e^{-t}e^{xt}\varphi_p(xt)\,\mathrm{d}t.$$

When $x=1$ this becomes

$$\begin{aligned}\sum_{n=0}^{\infty} n^p \left(e^y-1-\frac{y}{1!}-\frac{y^2}{2!}-\cdots-\frac{y^n}{n!}\right) &= e^y\int_0^y \varphi_p(t)\,\mathrm{d}t\\ &= e^y\sum_{k=0}^{p} S(p,k)\frac{y^{k+1}}{k+1},\end{aligned}$$

which is Furdui's result [5]. For $x \neq 0,1$ we have

$$\begin{aligned}&\sum_{n=0}^{\infty} n^p \left(e^y-1-\frac{y}{1!}-\frac{y^2}{2!}-\cdots-\frac{y^n}{n!}\right)x^n\\ &=\sum_{k=0}^{p} S(p,k)\frac{k!x^k}{(1-x)^{k+1}}\left(e^y-e^{xy}\sum_{j=0}^{k}\frac{y^j(1-x)^j}{j!}\right)\end{aligned}$$

(see also [2]).

3. Further exotic series

In this section we use our theorem to evaluate various exotic series with special numbers.

Example 4. Consider the generating function for the Bernoulli numbers

$$F(z)=\frac{z}{e^z-1}=\sum_{n=0}^{\infty} B_n\frac{z^n}{n!} \quad (|z|<2\pi).$$

The theorem implies the representation

$$\begin{aligned}\sum_{n=0}^{\infty} B_n \left(e^y-1-\frac{y}{1!}-\cdots-\frac{y^n}{n!}\right)x^n &= e^y\int_0^y \frac{e^{-t}xt}{e^{xt}-1}\,\mathrm{d}t\\ &= xe^y\int_0^y \frac{te^{-t}}{e^{xt}-1}\,\mathrm{d}t.\end{aligned} \tag{9}$$

For $x, y > 0$ the integral can be evaluated in terms of series

$$\begin{aligned}\int_0^y \frac{te^{-t}}{e^{xt}-1}\,\mathrm{d}t &= \int_0^y \frac{te^{-t}e^{-xt}}{1-e^{-xt}}\,\mathrm{d}t \\ &= \int_0^y t\left\{\sum_{n=1}^{\infty} e^{-(nx+1)t}\right\}\mathrm{d}t \\ &= \sum_{n=1}^{\infty}\int_0^y te^{-(nx+1)t}\,\mathrm{d}t \\ &= -ye^{-y}\sum_{n=1}^{\infty}\frac{e^{-nxy}}{nx+1} - e^{-y}\sum_{n=1}^{\infty}\frac{e^{-nxy}}{(nx+1)^2} + \sum_{n=1}^{\infty}\frac{1}{(nx+1)^2}.\end{aligned}$$

These series can be expressed through the Lerch transcendent $\Phi(z, s, a)$ (see [4]) and the dilogarithm $\mathrm{Li}_2(z)$

$$\Phi(z, s, a) = \sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}; \quad \mathrm{Li}_2(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^2}.$$

Namely, we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_n\left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right)x^n &= \frac{e^y}{x}\Phi(1, 2, x^{-1}) - y\,\Phi(e^{-xy}, 1, x^{-1}) \\ &\quad - \frac{1}{x}\Phi(e^{-xy}, 2, x^{-1}) + x(y + 1 - e^y).\end{aligned}$$

For $x = 1$ this representation takes the form

$$\begin{aligned}\sum_{n=0}^{\infty} B_n\left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) &= e^y\left(y\ln(1-e^{-y}) - \mathrm{Li}_2(e^{-y}) + \frac{\pi^2}{6} - 1\right) \\ &\quad + 1 + y.\end{aligned}$$

For $x = -1$ the integral in (9) becomes simpler

$$\begin{aligned}-\int_0^y \frac{te^{-t}}{e^{-t}-1}\,\mathrm{d}t = \int_0^y t\,\mathrm{d}(\ln(1-e^{-t})) &= y\ln(1-e^{-y}) - \int_0^y \ln(1-e^{-t})\,\mathrm{d}t \\ &= y\ln(1-e^{-y}) + \frac{\pi^2}{6} - \mathrm{Li}_2(e^{-y})\end{aligned}$$

and we come to the remarkable evaluation

$$\sum_{n=0}^{\infty} B_n\left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right)(-1)^n = e^y\left(y\ln(1-e^{-y}) - Li_2(e^{-y}) + \frac{\pi^2}{6}\right).$$

Example 5. In this example we use the exponential generating function for the Stirling numbers $S(n, k)$ of the second kind [3, 7]

$$F(z) = \frac{1}{k!}(e^z - 1)^k = \sum_{n=0}^{\infty} S(n, k)\frac{z^n}{n!}$$

where $k \geq 0$ is an integer (the summation actually starts from $n = k$, as $S(n,k) = 0$ for $n < k$). From the theorem

$$\sum_{n=0}^{\infty} S(n,k) \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) x^n = \frac{e^y}{k!} \int_0^y e^{-t}(e^{xt} - 1)^k \, \mathrm{d}t.$$

The integral can be evaluated in terms of binomial expressions.

$$\begin{aligned}
\int_0^y e^{-t}(e^{xt} - 1)^k \, \mathrm{d}t &= \int_0^y \left\{ \sum_{j=0}^{k} \binom{k}{j} (-1)^j e^{(jx-1)t} \right\} \mathrm{d}t \\
&= \sum_{j=0}^{k} \binom{k}{j} (-1)^j \int_0^y e^{(jx-1)t} \, \mathrm{d}t \\
&= \sum_{j=0}^{k} \binom{k}{j} (-1)^j \int_0^y e^{(jx-1)t} \, \mathrm{d}t \\
&= \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{e^{(jx-1)y} - 1}{jx - 1}.
\end{aligned}$$

Thus we have the closed form evaluation

$$\sum_{n=0}^{\infty} S(n,k) \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) x^n = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{e^{jxy} - e^y}{jx - 1}.$$

Example 6. Now we construct an exotic series containing the exponential polynomials $\varphi_n(x)$ defined in (8). Their generating function is given by

$$F(z) = e^{x(e^z - 1)} = \sum_{n=0}^{\infty} \varphi_n(x) \frac{z^n}{n!}$$

(see [1]). The theorem implies the representation

$$\begin{aligned}
\sum_{n=0}^{\infty} \varphi_n(\lambda) \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) x^n &= e^y \int_0^y e^{-t} e^{\lambda(e^{xt} - 1)} \, \mathrm{d}t \\
&= e^{y-\lambda} \int_0^y e^{-t} e^{\lambda e^{xt}} \mathrm{d}t.
\end{aligned}$$

We can easily evaluate this integral when $x = -1$

$$\int_0^y e^{-t} e^{\lambda e^{-t}} \, \mathrm{d}t = -\frac{1}{\lambda} \int_0^y e^{\lambda e^{-t}} \mathrm{d}\lambda e^{-t} = -\frac{1}{\lambda}\, e^{\lambda e^{-t}} \Big|_0^y = -\frac{1}{\lambda}(e^{\lambda e^{-y}} - e^{\lambda}).$$

From this

$$e^{y-\lambda} \int_0^y e^{-t} e^{\lambda e^{xt}} \, \mathrm{d}t = -\frac{e^y}{\lambda}(e^{\lambda(e^{-y} - 1)} - 1)$$

and we come to the elegant formula

$$\sum_{n=0}^{\infty} \varphi_n(\lambda) \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) = \frac{e^y}{\lambda}(1 - e^{\lambda(e^{-y} - 1)}).$$

Unexpectedly, we can spot the generating function for the exponential polynomials in the expression on the right hand side. This gives the identity

$$\sum_{n=0}^{\infty} \varphi_n(\lambda) \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!} \right) = \frac{e^y}{\lambda} \left(1 - \sum_{n=0}^{\infty} \varphi_n(\lambda) \frac{(-1)^n y^n}{n!} \right).$$

Example 7. In this application we present an exotic series with harmonic number. The harmonic numbers are defined by

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \quad (n \geq 1);\ H_0 = 0$$

with exponential generating function

$$F(z) = \sum_{n=0}^{\infty} H_n \frac{z^n}{n!} = e^z \operatorname{Ein}(z).$$

Here $\operatorname{Ein}(z)$ is the exponential integral

$$\operatorname{Ein}(z) = \int_0^z \frac{1 - e^{-u}}{u} \, \mathrm{d}u = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n! n}$$

(as the series representation shows, $\operatorname{Ein}(z)$ is an entire function).

From the theorem we obtain

$$\sum_{n=0}^{\infty} H_n \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!} \right) x^n = e^y \int_0^y e^{-t} e^{xt} \operatorname{Ein}(xt) \, \mathrm{d}t.$$

The integral can easily be evaluated in explicit form when $x = 1$

$$\begin{aligned} \int_0^y e^{-t} e^t \operatorname{Ein}(t) \, \mathrm{d}t &= \int_0^y \left\{ \int_0^t \frac{1 - e^{-u}}{u} du \right\} \, \mathrm{d}t \\ &= \int_0^y \left\{ \int_u^y \mathrm{d}t \right\} \frac{1 - e^{-u}}{u} \, \mathrm{d}u \\ &= \int_0^y (y - u) \frac{1 - e^{-u}}{u} \, \mathrm{d}u \\ &= y \operatorname{Ein}(y) - \int_0^y (1 - e^{-u}) \, \mathrm{d}u = y \operatorname{Ein}(y) - y - e^{-y} + 1. \end{aligned}$$

Using the series representation of Ein (t) we find also

$$\int_0^y \operatorname{Ein}(t) \, \mathrm{d}t = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} y^{n+1}}{n! n(n+1)}.$$

Finally, for all y we have the beautiful equation

$$\begin{aligned} \sum_{n=0}^{\infty} H_n \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!} \right) &= e^y (y \operatorname{Ein}(y) - y + 1) - 1 \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} y^{n+1}}{n! n(n+1)}. \end{aligned}$$

Example 8. The representation (6) leads to some interesting identities involving derangement numbers and polynomials. We come to these identities by using the simple exponential generating function for the numbers $n!$

$$F(z)=\sum_{n=0}^{\infty} n!\frac{z^n}{n!}=\sum_{n=0}^{\infty} z^n=\frac{1}{1-z} \quad (|z| < 1).$$

Our theorem gives the representation

$$\sum_{n=0}^{\infty} n!\left(e^y-1-\frac{y}{1!}-\cdots-\frac{y^n}{n!}\right)x^n=e^y\int_0^y \frac{e^{-t}}{1-xt}\,\mathrm{d}t. \tag{10}$$

This result can be related to the derangement numbers

$$\begin{aligned} D_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k! &= (-1)^n\sum_{k=0}^{n}\binom{n}{k}(-1)^k k! \\ &= n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}=n!\left(1-\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{(-1)^n}{n!}\right) \end{aligned}$$

which are popular in combinatorics [3, p. 180], [7, pp. 194–196], [8, 9]. We have from their definition

$$e^{-1}n!\,-D_n=n!\left(e^{-1}-1+\frac{1}{1!}-\frac{1}{2!}+\cdots+\frac{(-1)^n}{n!}\right)$$

and (10) with $y=-1$ gives the representation

$$\sum_{n=0}^{\infty}(e^{-1}n!\,-D_n)x^n=e^{-1}\int_0^{-1}\frac{e^{-t}}{1-xt}\,\mathrm{d}t. \tag{11}$$

Now consider also the derangement polynomials [9]

$$d_n(x)=(-1)^n\sum_{k=0}^{n}\binom{n}{k}(-1)^k k!x^k=n!\sum_{j=0}^{n}\frac{(-1)^j}{j!}x^{n-j}$$

where $d_n(1)=D_n$. The exponential generating function for these polynomials can be computed easily

$$\begin{aligned} \sum_{n=0}^{\infty} d_n(x)\frac{z^n}{n!} &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\left\{(-1)^n\sum_{k=0}^{n}\binom{n}{k}\,k!(-1)^k x^k\right\} \\ &= \sum_{k=0}^{\infty}(-1)^k k!x^k\left\{\sum_{n=k}^{\infty}\binom{n}{k}\frac{(-1)^n z^n}{n!}\right\} \\ &= \sum_{k=0}^{\infty}(-x)^k(-t)^k\left\{\sum_{n=k}^{\infty}\frac{(-z)^{n-k}}{(n-k)!}\right\} \\ &= \sum_{k=0}^{\infty}(xz)^k\left\{\sum_{m=0}^{\infty}\frac{(-z)^m}{m!}\right\}=\frac{e^{-z}}{1-xz}, \end{aligned}$$

that is,

$$\frac{e^{-z}}{1-xz} = \sum_{n=0}^{\infty} d_n(x)\frac{z^n}{n!}. \tag{12}$$

This function appears in (10) and (11). We have by integrating in (12) the representation

$$\int_0^y e^{-t}\frac{1}{1-xt}\,\mathrm{d}t = \sum_{n=0}^{\infty} d_n(x)\frac{y^{n+1}}{(n+1)!}$$

which in view of (10) gives the curious series identity

$$\sum_{n=0}^{\infty} n!\left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right)x^n = e^y\sum_{n=0}^{\infty} d_n(x)\frac{y^{n+1}}{(n+1)!}.$$

For $x=1,\ y=-1$ this identity becomes

$$\begin{aligned}\sum_{n=0}^{\infty} n!\left(e^{-1} - 1 + \frac{1}{1!} - \cdots - \frac{(-1)^n}{n!}\right) &= e^{-1}\sum_{n=0}^{\infty} D_n\frac{(-1)^{n+1}}{(n+1)!}\\ &= \sum_{n=0}^{\infty}(e^{-1}n! - D_n)\\ &= e^{-1}\int_0^{-1}\frac{e^{-t}}{1-t}\,\mathrm{d}t.\end{aligned}$$

Example 9. In this example we construct exotic series with central binomial coefficients $\binom{2n}{n}$ and Catalan numbers $C_n = \binom{2n}{n}\frac{1}{n+1}$. The Catalan numbers especially are very popular in combinatorics ([3, p. 53] and [7, pp. 203 and 358]).

The exponential generating functions for these numbers are

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{z^n}{n!} = e^{2z}\sum_{n=0}^{\infty}\frac{z^{2n}}{(n!)^2} = e^{2z}I_0(2z)$$

$$\sum_{n=0}^{\infty} C_n\frac{z^n}{n!} = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{z^n}{(n+1)!} = e^{2z}\left(I_0(2z) - I_1(2z)\right)$$

where $I_0(x)$ and $I_1(x) = I_0'(x)$ are the modified Bessel functions of the first kind [11, pp. 77-84]

$$I_0(z) = \sum_{n=0}^{\infty}\frac{z^{2n}}{4^n(n!)^2},\quad I_1(z) = \frac{z}{2}\sum_{n=0}^{\infty}\frac{z^{2n}}{4^n(n!)^2(n+1)}.$$

The theorem implies

$$\sum_{n=0}^{\infty}\binom{2n}{n}\left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right)x^n = e^y\int_0^y e^{(2x-1)t}I_0(2xt)\,\mathrm{d}t$$

$$\sum_{n=0}^{\infty} C_n \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) x^n = e^y \int_0^y e^{(2x-1)t}(I_0(2xt) - I_0'(2xt))\,\mathrm{d}t.$$

For $x = \frac{1}{2}$ the integrals become simpler and can be evaluated explicitly. Thus

$$\int_0^y I_0(t)\,\mathrm{d}t = yI_0(y) + \frac{\pi y}{2}[I_0(y)L_1(y) - I_1(y)L_0(y)]$$

where $L_0(y)$, $L_1(y)$ are the modified Struve functions [10, entry 1.11.1(4)]. We come to the identities

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^n} \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) = e^y \sum_{n=0}^{\infty} \frac{y^{2n+1}}{4^n (n!)^2 (2n+1)}$$
$$= e^y \left(yI_0(y) + \frac{\pi y}{2}[I_0(y)L_1(y) - I_1(y)L_0(y)]\right)$$

$$\sum_{n=0}^{\infty} \frac{C_n}{2^n} \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) = e^y \left(1 - I_0(y) + \sum_{n=0}^{\infty} \frac{y^{2n+1}}{4^n (n!)^2 (2n+1)}\right)$$
$$= e^y \left(1 + (y-1)I_0(y) + \frac{\pi y}{2}[I_0(y)L_1(y) - I_1(y)L_0(y)]\right).$$

Example 10. Equation (7) can be viewed as the ordinary generating function for the functions

$$e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}$$

We will show now that the exponential generating function for these expressions is very close to the exponential generating function for the Laguerre polynomials $L_n(x)$. Namely, the following series identity holds

$$\sum_{n=0}^{\infty} \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) \frac{x^n}{n!} = e^y \sum_{n=0}^{\infty} (-1)^n L_n(x) \frac{y^{n+1}}{(n+1)!}. \tag{13}$$

Here is the proof of this identity. In order to compute the left hand side in (13) we consider the exponential generating function for the numbers $\frac{1}{n!}$

$$F(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{(n!)} = I_0(2\sqrt{z})$$

where $I_0(x)$ is the modified Bessel function of zero order (see (33)). According to (6)

$$\sum_{n=0}^{\infty} \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!}\right) \frac{x^n}{n!} = e^y \int_0^y e^{-t} I_0(2\sqrt{xt})\,\mathrm{d}t. \tag{14}$$

At the same time, by using the Cauchy rule for multiplication of power series we compute

$$
\begin{aligned}
e^{-t} I_0(2\sqrt{xt}) = e^{-t} \sum_{n=0}^{\infty} \left\{ \frac{x^n}{n!} \right\} \frac{t^n}{n!} &= \left(\sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} \right) \sum_{n=0}^{\infty} \left\{ \frac{x^n}{n!} \right\} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k} x^k}{k!} \right\} \frac{t^n}{n!}.
\end{aligned} \tag{15}
$$

The Laguerre polynomials $L_n(x)$ have the binomial representation

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k x^k}{k!}$$

and from (14) and (15) we obtain the identity

$$\sum_{n=0}^{\infty} \left(e^y - 1 - \frac{y}{1!} - \cdots - \frac{y^n}{n!} \right) \frac{x^n}{n!} = e^y \int_0^y \left\{ \sum_{n=0}^{\infty} L_n(x) \frac{(-1)^n t^n}{n!} \right\} \mathrm{d}t.$$

Now term by term integration leads to (13).

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15th South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2021

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Abstract. The 15th South Eastern European Mathematical Olympiad for University Students took place on July 19–24, 2021, in Agros, Cyprus. We present the problems from the contest and their solutions as given by the corresponding authors, together with alternative solutions provided by the members of problem solving committee (PSC).

Keywords: Similar matrices, Hermitian matrices, minimal polynomial, diagonalizable matrix, normal matrix, eigenvalues, eigenvectors, Riemann integral, continuous functions, sequences of real numbers, series of real numbers.

MSC: 15A03, 15A15, 15A21, 26A15, 26D15.

The COVID-19 pandemic affected SEEMOUS 2021 twice: the competition had to be postponed until July 2021 and reduced drastically the number of participants to 41, representing 10 universities: 6 from Romania, 3 from Greece, and one from Cyprus. The jury awarded 5 gold medals, 10 silver medals and 14 bronze medals. The student Sergiu-Ionuţ Novac from University of Bucharest was the winner of the competition with 37 points out of 40.

We present the problems from the contest and their solutions as given by the corresponding authors, together with alternative solutions provided by the members of problem solving committee (PSC).

Problem 1. Let $f : [0,1] \longrightarrow \mathbb{R}$ be a continuous increasing function such that

$$\lim_{x\to 0^+} \frac{f(x)}{x} = 1.$$

(a) Prove that the sequence $(x_n)_{n\geq 1}$ defined by

$$x_n = f\left(\frac{1}{1}\right) + f\left(\frac{1}{2}\right) + \cdots + f\left(\frac{1}{n}\right) - \int_1^n f\left(\frac{1}{x}\right)\,\mathrm{d}x$$

is convergent.

(b) Find the limit of the sequence $(y_n)_{n\geq 1}$ defined by

$$y_n = f\left(\frac{1}{n+1}\right) + f\left(\frac{1}{n+2}\right) + \cdots + f\left(\frac{1}{2021n}\right).$$

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Author's solution. (a) We write

$$x_n = \sum_{k=1}^{n-1} \left(f\left(\frac{1}{k}\right) - \int_k^{k+1} f\left(\frac{1}{x}\right)\,\mathrm{d}x \right) + f\left(\frac{1}{n}\right).$$

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Because f is increasing, for all $k \geqslant 1$ and $x \in [k, k+1]$ we have

$$f\left(\frac{1}{k+1}\right) \leqslant f\left(\frac{1}{x}\right) \leqslant f\left(\frac{1}{k}\right)$$

and therefore

$$f\left(\frac{1}{k+1}\right) \leqslant \int_k^{k+1} f\left(\frac{1}{x}\right) \mathrm{d}x \leqslant f\left(\frac{1}{k}\right). \tag{1}$$

Summing up for $k = 1$ up to $n-1$ we obtain

$$f\left(\frac{1}{n}\right) \leqslant x_n \leqslant f(1).$$

Since f is increasing, (x_n) is bounded below by $f(0)$.

It is easy to see that (x_n) is decreasing since using (1) we have

$$x_{n+1} - x_n = f\left(\frac{1}{n+1}\right) - \int_n^{n+1} f\left(\frac{1}{x}\right) \mathrm{d}x \leqslant 0\,.$$

We conclude that (x_n) is convergent.

(b) Since

$$y_n = x_{2021n} - x_n + \int_n^{2021n} f\left(\frac{1}{x}\right) \mathrm{d}x\,,$$

from part (a), it is enough to find

$$\lim_{n\to\infty} \int_n^{2021n} f\left(\frac{1}{x}\right) \mathrm{d}x.$$

With the change of variable $x = \dfrac{1}{t}$ we obtain

$$\int_n^{2021n} f\left(\frac{1}{x}\right) \mathrm{d}x = \int_{\frac{1}{2021n}}^{\frac{1}{n}} \frac{f(t)}{t^2} \mathrm{d}t.$$

Since $\lim_{x\to 0^+} \frac{f(x)}{x} = 1$, given $\varepsilon > 0$, there is $\delta > 0$ such that $1 - \varepsilon < \frac{f(x)}{x} < 1 + \varepsilon$ for every $0 < x < \delta$. In particular, for every $n > \frac{1}{\delta}$, we have $0 < \frac{1}{2021n} < \frac{1}{n} < \delta$ and therefore

$$(1-\varepsilon) \int_{\frac{1}{2021n}}^{\frac{1}{n}} \frac{1}{t} \mathrm{d}t \leq \int_{\frac{1}{2021n}}^{\frac{1}{n}} \frac{f(t)}{t^2} \mathrm{d}t \leq (1+\varepsilon) \int_{\frac{1}{2021n}}^{\frac{1}{n}} \frac{1}{t} \mathrm{d}t.$$

Since ε is arbitrary and since

$$\int_{\frac{1}{2021n}}^{\frac{1}{n}} \frac{1}{t} \mathrm{d}t = \ln 2021,$$

we conclude that

$$\lim_{n\to\infty} y_n = \ln 2021\,.$$

Alternative solution by PSC. (b) Since $\lim_{x\to 0^+} \frac{f(x)}{x} = 1$, given $\varepsilon > 0$, there is $\delta > 0$ such that $1 - \varepsilon < \frac{f(x)}{x} < 1 + \varepsilon$ for every $0 < x < \delta$. In particular, for every $n > \frac{1}{\delta}$ and every $k \geq 1$, we have $0 < \frac{1}{n+k} < \frac{1}{n} < \delta$ and therefore

$$(1-\varepsilon)\frac{1}{n+k} < f\left(\frac{1}{n+k}\right) < (1+\varepsilon)\frac{1}{n+k}.$$

Summing up the above inequalities from $k = 1$ to $2020n$ we get

$$(1-\varepsilon)\left(\frac{1}{n+1}+\cdots+\frac{1}{2021n}\right) < y_n < (1+\varepsilon)\left(\frac{1}{n+1}+\cdots+\frac{1}{2021n}\right).$$

It is well-known that

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\cdots+\frac{1}{2021n}\right) = \ln 2021\,,$$

and since ε is arbitrary we get $\lim\limits_{n\to\infty} y_n = \ln 2021$.

Problem 2. Let $n \geq 2$ be a positive integer and let $A \in \mathcal{M}_n(\mathbb{R})$ be a matrix such that $A^2 = -I_n$. If $B \in \mathcal{M}_n(\mathbb{R})$ and $AB = BA$, prove that $\det B \geqslant 0$.

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Authors' solution. Since $A^2 = -I_n$, the only possible eigenvalues of A are $\pm i$. From $A \in \mathcal{M}_n(\mathbb{R})$ it follows that $n = 2k$ and A has k eigenvalues equal to i and k eigenvalues equal to $-i$. Its minimal polynomial is x^2+1, which has distinct roots, therefore A is diagonalizable and is therefore similar to

$$X = \begin{bmatrix} iI_k & 0_k \\ 0_k & -iI_k \end{bmatrix}.$$

Similarly, if $P = \begin{bmatrix} 0_k & I_k \\ -I_k & 0_k \end{bmatrix}$, then P is also a real matrix with $P^2 = -I_n$ and so P is also similar to X. Therefore A and P are similar and so there is an invertible matrix $U \in \mathcal{M}_n(\mathbb{R})$ such that $P = U^{-1}AU$. For $C = U^{-1}BU \in \mathcal{M}_n(\mathbb{R})$ we get

$$CP = U^{-1}BAU \quad \text{and} \quad PC = U^{-1}ABU\,. \tag{2}$$

Since $AB = BA$, by (2) it follows that $CP = PC$.

Writing C into block form $C = \begin{bmatrix} X & Y \\ Z & T \end{bmatrix}$, where $X, Y, Z, T \in \mathcal{M}_k(\mathbb{R})$, and using $CP = PC$, it follows that $X = T$ and $Z = -Y$. Hence $C = \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$.

We now see that

$$\begin{aligned} \begin{vmatrix} X & Y \\ -Y & X \end{vmatrix} = \begin{vmatrix} X+iY & Y-iX \\ -Y & X \end{vmatrix} &= \begin{vmatrix} X+iY & (Y-iX)+i(X+iY) \\ -Y & X-iY \end{vmatrix} \\ &= \begin{vmatrix} X+iY & 0 \\ -Y & X-iY \end{vmatrix}. \end{aligned}$$

Therefore

$$\det B = \det C = \begin{vmatrix} X & Y \\ -Y & X \end{vmatrix} = \det(X+iY)\det(X-iY) = |\det(X+iY)|^2 \geqslant 0.$$

Alternative solution by PSC. Let λ be a real eigenvalue of B and let G_λ be its generalized eigenspace considered as a real vector space, i.e.,

$$G_\lambda = \{\mathbf{v} \in \mathbb{R}^n : (B - \lambda I_n)^n \mathbf{v} = \mathbf{0}\}.$$

We have $AB^2 = (AB)B = (BA)B = B(AB) = B(BA) = B^2 A$. Inductively we get $AB^k = B^k A$ for every natural number k and from this we deduce that $Ap(B) = p(B)A$ for every polynomial $p(x)$. In particular, $A(B - \lambda I_n)^n = (B - \lambda I_n)^n A$.

Now if $\mathbf{v} \in G_\lambda$, then $(B - \lambda I_n)^n (A\mathbf{v}) = A(B - \lambda I_n)^n \mathbf{v} = \mathbf{0}$, so $A\mathbf{v} \in G_\lambda$. Therefore we can define a linear map $\alpha : G_\lambda \to G_\lambda$ by $\alpha(\mathbf{v}) = A\mathbf{v}$.

Pick a basis of G_λ and let A' be the matrix of α with respect to this basis. Then $A' \in \mathcal{M}_n(\mathbb{R})$ and $(A')^2 = -I_{n'}$, where $n' = \dim(G_\lambda)$. As in the previous solution, we get that n' is even.

Since $\dim(G_\lambda)$ is even for every real eigenvalue of B and since its complex eigenvalues come in conjugate pairs, it results then $\det(B) \geqslant 0$.

Remark. The contestant Sergiu-Ionuţ Novac has found a similar solution using generalized eigenspaces.

Problem 3. Let $A \in \mathcal{M}_n(\mathbb{C})$ be a matrix such that $(AA^*)^2 = A^*A$, where $A^* = \overline{A}^t$ denotes the Hermitian transpose (i.e., the conjugate transpose) of A.

(a) Prove that $AA^* = A^*A$.
(b) Show that the non-zero eigenvalues of A have modulus one.

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Authors' solution. (a) The matrix AA^* is Hermitian and all its eigenvalues are non-negative real numbers.

If $\lambda \in \sigma(AA^*)$, then $\lambda^2 \in \sigma\left((AA^*)^2\right) = \sigma\left(A^*A\right) = \sigma(AA^*)$, hence $\lambda^2 \in \sigma(AA^*)$. It follows by induction that $\lambda^{2^k} \in \sigma(AA^*)$, for all $k \in \mathbb{N}$. Since $\lambda \geqslant 0$, the last relation assures us that $\lambda \in \{0, 1\}$, so AA^* will have eigenvalues 0 or 1. On the other hand, since AA^* is Hermitian, it is also diagonalizable, thus

$$AA^* = U^{-1} \begin{bmatrix} I_k & O_{k,n-k} \\ O_{n-k,k} & O_{n-k} \end{bmatrix} U.$$

Using the above statement, we conclude that

$$A^*A = (AA^*)^2 = AA^*.$$

(b) Using (a), the equality of our hypothesis can be transformed into $A^*A \cdot (AA^* - I_n) = O_n$. Letting $B = A \cdot (AA^* - I_n)$ we obtain

$$B^*B = (AA^* - I_n)A^*A(AA^* - I_n) = O_n,$$

which gives $B = O_n$. Thus

$$A^2A^* = A\,. \tag{3}$$

Since $A^*A = AA^*$, it follows that the matrix A is normal, hence it is a unitary diagonalizable matrix. It follows that there is a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that $A = U^*DU$, where $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Then $A^2A^* = U^*D^2UU^*\overline{D}U = U^*D^2\overline{D}U$ and using (3) we get

$$\begin{aligned} A^2A^* = A \iff D^2\overline{D} = D &\iff \lambda_i^2 \cdot \overline{\lambda_i} = \lambda_i \text{ for all } i \in \{1, 2, \dots, n\} \\ &\iff \lambda_i(|\lambda_i|^2 - 1) = 0 \text{ for all } i \in \{1, 2, \dots, n\}\,. \end{aligned}$$

Hence the conclusion.

Alternative solution by PSC. (a) Let $X = AA^*$ and $Y = A^*A$. Since X is Hermitian, it is diagonalizable, so $P^{-1}XP = D$ for some matrices P, D with D diagonal. Let $Z = P^{-1}YP$. The initial condition gives $Z = D^2$. Since X and Y have the same characteristic polynomial, so do $Z = D^2$ and D. As in the original proof we deduce that every entry of D must be 0 or 1. Then $Z = D$ and so $X = Y$ as required.

(b) Writing $A = U^*DU$ as in the original proof and using $(AA^*)^2 = A^*A$ (rather than $A^2A^* = A$) we get $(D\overline{D})^2 = \overline{D}D$. From this we get that $|\lambda|^4 = |\lambda|^2$ for each eigenvalue λ of A and the conclusion follows.

Problem 4. Let $p \in \mathbb{R}$ and let $(a_n)_{n\geq 1}$ be the sequence defined by

$$a_n = \frac{1}{n^p} \int_0^n \Big|\sin(\pi x)\Big|^x \,\mathrm{d}x\,.$$

Determine all possible values of p for which the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Tiberiu Trif, Babeş-Bolyai University, Cluj-Napoca, Romania

Author's solution. For every positive integer n, let

$$I_n = \int_0^n \Big|\sin(\pi x)\Big|^x \,\mathrm{d}x = \sum_{k=0}^{n-1} \int_k^{k+1} \Big|\sin(\pi x)\Big|^x \,\mathrm{d}x.$$

Then we have

$$\sum_{k=0}^{n-1} \int_k^{k+1} \Big|\sin(\pi x)\Big|^{k+1} \,\mathrm{d}x < I_n < \sum_{k=0}^{n-1} \int_k^{k+1} \Big|\sin(\pi x)\Big|^{k} \,\mathrm{d}x\,.$$

Substituting $t = \pi x - k\pi$, we deduce that

$$\int_k^{k+1} \Big|\sin(\pi x)\Big|^m \,\mathrm{d}x = \frac{1}{\pi} \int_0^{\pi} \sin^m t \,\mathrm{d}t$$

for every nonnegative integer m. Therefore

$$\frac{1}{\pi}\sum_{k=1}^{n} J_k < I_n < \frac{1}{\pi}\sum_{k=0}^{n-1} J_k\,, \tag{4}$$

where $J_k = \int_0^\pi \sin^k t\,\mathrm{d}t$. For $k \geqslant 2$, integration by parts yields

$$\begin{aligned}
J_k &= \int_0^\pi (-\cos t)' \sin^{k-1} t\,\mathrm{d}t \\
&= \Big[-\cos t \sin^{k-1} t\Big]_0^\pi + (k-1)\int_0^\pi \sin^{k-2} t \cos^2 t\,\mathrm{d}t \\
&= 0 + (k-1)\int_0^\pi \sin^{k-2} t(1-\sin^2 t)\,\mathrm{d}t \\
&= (k-1)J_{k-2} - (k-1)J_k\,,
\end{aligned}$$

whence

$$J_k = \frac{k-1}{k}\,J_{k-2}\,.$$

Since $J_0 = \pi$ and $J_1 = 2$, we obtain

$$J_{2k} = \pi\,\frac{(2k-1)!!}{(2k)!!} \quad \text{and} \quad J_{2k+1} = 2\,\frac{(2k)!!}{(2k+1)!!}\,.$$

We observe that

$$J_{2k-1}J_{2k} = \frac{2\pi}{2k} \quad \text{and} \quad J_{2k}J_{2k+1} = \frac{2\pi}{2k+1}\,.$$

Since (J_n) is a decreasing sequence, we deduce that

$$\frac{2\pi}{2k+1} = J_{2k}J_{2k+1} \le J_{2k}^2 \le J_{2k-1}J_{2k} = \frac{2\pi}{2k}\,.$$

It follows that

$$\sqrt{2\pi}\sqrt{\frac{2k}{2k+1}} \le \sqrt{2k}J_{2k} \le \sqrt{2\pi}$$

and therefore

$$\lim_{k\to\infty} \sqrt{2k}\,J_{2k} = \sqrt{2\pi}\,. \tag{5}$$

Similarly

$$\sqrt{2\pi}\sqrt{\frac{2k+1}{2k+2}} \le \sqrt{2k+1}J_{2k+1} \le \sqrt{2\pi}$$

and therefore

$$\lim_{k\to\infty} \sqrt{2k+1}\,J_{2k+1} = \sqrt{2\pi}\,. \tag{6}$$

By (5) and (6) it follows that

$$\lim_{n\to\infty} \sqrt{n}\,J_n = \sqrt{2\pi}\,. \tag{7}$$

By virtue of (7) and the Cesàro-Stolz theorem we have

$$\begin{aligned}\lim_{n\to\infty}\frac{J_1+\cdots+J_n}{\sqrt{n}} &= \lim_{n\to\infty}\frac{J_{n+1}}{\sqrt{n+1}-\sqrt{n}}\\ &= \lim_{n\to\infty}\left(\sqrt{n+1}+\sqrt{n}\right)J_{n+1}\\ &= 2\sqrt{2\pi}\,.\end{aligned}\tag{8}$$

Now relations (4) and (8) ensure that

$$\lim_{n\to\infty}\frac{I_n}{\sqrt{n}}=\frac{1}{\pi}\cdot 2\sqrt{2\pi}=2\sqrt{\frac{2}{\pi}}\,.$$

Taking into consideration that

$$a_n=\frac{I_n}{n^p}=\frac{I_n}{\sqrt{n}}\cdot\frac{1}{n^{p-\frac{1}{2}}}\,,$$

we deduce that the series $\sum\limits_{n=1}^{\infty}a_n$ has the same nature as $\sum\limits_{n=1}^{\infty}\frac{1}{n^{p-\frac{1}{2}}}$. In conclusion, the series $\sum\limits_{n=1}^{\infty}a_n$ converges if and only if $p>\frac{3}{2}$.

Comments by PSC. (i) One could use Wallis' formula or Stirling's approximation in order to deduce (7).
(ii) One could avoid the use of Cesàro-Stolz as follows: By (7) we have $J_n=\Theta(\frac{1}{\sqrt{n}})$. Since also (e.g., by considering Riemann sums) $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}=\Theta(\sqrt{n})$, one has $a_n=\Theta\left(\frac{1}{n^{p-1/2}}\right)$ and the conclusion follows as before.

MATHEMATICAL NOTES

When must a locally integrable function be integrable?

GEORGE STOICA[1)]

Abstract. We present two conditions on the decay at infinity of locally integrable functions, and discuss their optimality, assuring that the latter functions are, in fact, integrable.

Keywords: Locally integrable functions, decay conditions.

MSC: 26A12, 26A42.

A function is called locally integrable on $\mathbb{R}$ (cf. [1]) if it is Lebesgue integrable (so its integral is finite) on every compact subset of $\mathbb{R}$. Unlike integrable functions, locally integrable functions can grow arbitrarily fast at infinity, but are still manageable in a way similar to ordinary integrable functions. They play a prominent role in distribution theory (cf. [2], [3]) and occur in the definition of various classes of functions and function spaces, like functions of bounded variation; moreover, every locally integrable function defines an absolutely continuous measure and, conversely, every absolutely continuous measure defines a locally integrable function (cf. the celebrated Radon-Nikodym theorem).

Our purpose is to find out conditions on how fast must locally integrable functions decay at infinity, to become, in fact, integrable. This type of conditions is relevant in functional analysis, as a locally integrable function with at most polynomial growth at infinity has a Fourier transform — as a distribution, in general (cf. [2], [3]). The main difficulty in finding such conditions comes from the fact that not all integrable functions on $\mathbb{R}$ decay to 0 at infinity. Nevertheless, by correlating the rate of decay with the size of the integration domain therein, we were able to find such conditions (Theorem 1) *and* to prove that they are optimal for integrability (Theorem 2), as will be described below.

Our main result follows. For simplicity, we denote below by $L^1_{\mathrm{loc}}(\mathbb{R})$ and $L^1(\mathbb{R})$ the spaces of locally integrable and of integrable functions on $\mathbb{R}$, respectively.

Theorem 1. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. If, for some $\alpha > \beta \geq 0$, there exists $c = c(\alpha, \beta) > 0$ such that*

$$\int_{\{|x|<a\}} |x^\alpha f(x)| \, \mathrm{d}x \leq c \cdot a^\beta \text{ for all } a > 0 \tag{1}$$

[1)] 5 Deveber Terrace, Saint John NB, E2K 2B5 Canada, gstoica2015@gmail.com *then* $f \in L^1(\mathbb{R})$*. The same conclusion holds if, for some* $\alpha > \beta \geq 0$*, we have*

$$\lim_{a\to\infty} \frac{1}{a^\beta} \int_{\{|x|<a\}} |x^\alpha f(x)|\,\mathrm{d}x = 0. \tag{2}$$

Before the proof, we shall present some illustrative examples.

Example 1. The function $f(x) = x^{-\gamma}$ for $|x| > 1$ and 0 if $|x| \leq 1$ satisfies condition (1) in Theorem 1 for any $\alpha \geq \gamma - 1$ and $\beta = \alpha - \gamma + 1$; similarly, it satisfies condition (2) above for any $\alpha \geq \gamma - 1$ and $\beta > \alpha - \gamma + 1$. In both cases, the requirements $\alpha > \beta \geq 0$ in conditions (1) and (2) are satisfied precisely when $\gamma > 1$ (which is, by no means, accidental). Indeed, one can easily check that the above function f is in $L^1_{\mathrm{loc}}(\mathbb{R})$ for all $\gamma > 0$, but is in $L^1(\mathbb{R})$ if and only if $\gamma > 1$.

Example 2. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $g \in L^1_{\mathrm{loc}}(\mathbb{R})$, where $g(x) := x^\gamma f(x)$ for some $\gamma > 0$. Theorem 1 says that $g \in L^1(\mathbb{R})$ provided that, for some $\alpha > \beta \geq 0$, there is $c = c(\alpha, \beta) > 0$ such that

$$\int_{\{|x|<a\}} |x^{\alpha+\gamma} f(x)|\,\mathrm{d}x \leq c \cdot a^\beta \text{ for all } a > 0,$$

or, for some $\alpha > \beta \geq 0$,

$$\lim_{a\to\infty} \frac{1}{a^\beta} \int_{\{|x|<a\}} |x^{\alpha+\gamma} f(x)|\,\mathrm{d}x = 0.$$

This example is a two-way improvement (weaker hypotheses and stronger conclusion) of the following easy exercise: let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ be such that $g \in L^1(\mathbb{R})$, where $g(x) := x^\gamma f(x)$ for some $\gamma > 0$; then $f \in L^1(\mathbb{R})$.

Example 3. The inequality $\alpha > \beta \geq 0$ in conditions (1) and (2) is necessary. Indeed, the function $f(x) = x^{-1}$ for $|x| > 1$ and 0 if $|x| \leq 1$ satisfies condition (1) but not condition (2) with $\alpha = \beta = 2$, is in $L^1_{\mathrm{loc}}(\mathbb{R})$, but not in $L^1(\mathbb{R})$. Also note that the function $f(x) = x^{-3}$ for $|x| > 1$ and 0 if $|x| \leq 1$ satisfies condition (2) but not condition (1) with $\alpha = 2$ and $\beta = 0$, is in $L^1_{\mathrm{loc}}(\mathbb{R})$, but not in $L^1(\mathbb{R})$

Proof of Theorem 1. We need to show that $\int_{|x|>a\}} |f(x)|\,\mathrm{d}x < \infty$ for all $a > 0$. Actually, we shall prove more, namely that condition (1) implies the existence of another constant $c_1 = c_1(\alpha, \beta) > 0$ such that

$$\int_{\{|x|>a\}} |f(x)|\,\mathrm{d}x \leq c_1 \cdot a^{\beta-\alpha} \text{ for all } a > 0. \tag{3}$$

Indeed, let us fix $n = 0, 1, 2, \ldots$ and $a > 0$. We have

$$(2^n a)^\alpha \int_{\{2^n a<|x|<2^{n+1}a\}} |f(x)|\,\mathrm{d}x \leq \int_{\{2^n a<|x|<2^{n+1}a\}} |x^\alpha f(x)|\,\mathrm{d}x \leq c\cdot(2^{n+1}a)^\beta,$$

so

$$\int_{\{2^n a<|x|<2^{n+1}a\}} |f(x)|\,\mathrm{d}x \leq c \cdot 2^\beta \cdot a^{\beta-\alpha} \cdot 2^{(\beta-\alpha)n}.$$

From the identity

$$\{|x| > a\} = \bigcup_{n=0}^{\infty} \{2^n a < |x| < 2^{n+1} a\} \cup \bigcup_{n=0}^{\infty} \{|x| = 2^{n+1} a\}$$

and using that the last union of sets is countable, hence Lebesgue negligible, it follows that

$$\int_{\{|x|>a\}} |f(x)| \,\mathrm{d}x = \sum_{n=0}^{\infty} \int_{\{2^n a<|x|<2^{n+1}a\}} |f(x)| \,\mathrm{d}x.$$

Using the above estimates, we obtain

$$\int_{\{|x|>a\}} |f(x)| \,\mathrm{d}x \le c \cdot 2^{\beta} \cdot a^{\beta-\alpha} \sum_{n=0}^{\infty} 2^{(\beta-\alpha)n} = \frac{c \cdot 2^{\alpha}}{2^{\alpha-\beta} - 1} a^{\beta-\alpha}$$

(we used that $\alpha > \beta$ for the convergence of the latter series), so we can take $c_1 := c \cdot 2^{\alpha}/(2^{\alpha-\beta} - 1)$ in formula (3).

Similarly, we can prove that condition (2) implies

$$\lim_{a\to\infty} a^{\alpha-\beta} \int_{\{|x|>a\}} |f(x)| \,\mathrm{d}x = 0, \tag{4}$$

and the latter clearly implies that $f \in L^1(\mathbb{R})$. Indeed, by (2), for any $\varepsilon > 0$ there exists $a_0 := a_0(\varepsilon)$ such that

$$\int_{\{|x|<a\}} |x^{\alpha} f(x)| \,\mathrm{d}x < \varepsilon \cdot a^{\beta} \text{ for } a > a_0.$$

The arguments and computations from the first part of the proof show that

$$\int_{\{|x|>a\}} |f(x)| \,\mathrm{d}x < c_1 \cdot \varepsilon \cdot a^{\beta-\alpha} \text{ for } a > a_0,$$

with the same constant c_1 as above, and condition (4) now follows. □

Remark. As the proof of Theorem 1 shows, in conditions (1) and (2) we may replace x^{α} by any other function $h : \mathbb{R} \to \mathbb{R}$ satisfying $h \cdot f \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $|h(x)| \ge \mathrm{const} \cdot |x|^{\alpha}$.

In the proof of Theorem 1 we showed that (1) $\Rightarrow$ (3), (2) $\Rightarrow$ (4), and used that conditions (3) and (4) obviously imply the required integrability. As we shall see below, the interesting part with conditions (1) and (2) is that they are *optimal* for integrability, in the sense that (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4), provided that $\alpha > \beta > 0$. Specifically, we shall prove the following

Theorem 2. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. With the same notations as above, and if $\alpha \ge \beta > 0$, we have that* (3) $\Rightarrow$ (1) *and* (4) $\Rightarrow$ (2).

Proof. The trick is to perform majorizations involving *negative* integers, instead of positive ones. Namely, assume (3) and fix $a > 0, n = 0, -1, -2, \ldots$ We have

$$\begin{aligned}\frac{1}{(2^n a)^\alpha} \int_{\{2^{n-1}a<|x|<2^n a\}} |x^\alpha f(x)| \,\mathrm{d}x &\leq \int_{\{2^{n-1}a<|x|<2^n a\}} |f(x)| \,\mathrm{d}x \\ &\leq c_1 \cdot (2^{n-1}a)^{\beta-\alpha},\end{aligned}$$

so

$$\int_{\{2^{n-1}a<|x|<2^n a\}} |x^\alpha f(x)| \,\mathrm{d}x \leq c_1 \cdot 2^{\alpha-\beta} \cdot a^\beta \cdot 2^{\beta n}.$$

From the identity

$$\{|x| < a\} = \bigcup_{n=-\infty}^{0} \{2^{n-1}a < |x| < 2^n a\} \cup \bigcup_{n=-\infty}^{0} \{|x| = 2^{n+1}a\}$$

and using that the last union of sets is countable, hence Lebesgue negligible, it follows that

$$\int_{\{|x|<a\}} |x^\alpha f(x)| \,\mathrm{d}x = \sum_{n=-\infty}^{0} \int_{\{2^{n-1}a<|x|<2^n a\}} |x^\alpha f(x)| \,\mathrm{d}x.$$

Using the above estimates, we obtain:

$$\int_{\{|x|<a\}} |x^\alpha f(x)| \,\mathrm{d}x \leq c_1 \cdot 2^{\alpha-\beta} \cdot a^\beta \sum_{n=-\infty}^{0} 2^{\beta n} = \frac{c_1 \cdot 2^\alpha}{2^\beta - 1} a^\beta$$

(we used that $\beta > 0$ for the convergence of the latter series), so we can take $c := c_1 \cdot 2^\alpha/(2^\beta - 1)$ in formula (1). A similar argument shows that (4) implies (2). □

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[3] L. Schwartz, *Théorie des Distributions*, Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX–X, Hermann Editeurs, Paris, 1998.

PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of November 2022**.

PROPOSED PROBLEMS

521. Prove that

$$\lim_{p\to\infty}\sum_{i,j=0}^{p}\frac{(-4a^2)^{i+j}}{2i+2j+1}\binom{p+i}{p-i}\binom{p+j}{p-j}=\frac{1}{4a}\ln\left(\frac{1+a}{1-a}\right)\qquad\forall a\in(0,1).$$

Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.

522. Evaluate the integral

$$\int_0^\infty \frac{\log\left(1+x^{10}\right)}{1+x^2}\,\mathrm{d}x.$$

Proposed by Seán M. Stewart, King Abdullah University of Science and Technology (KAUST), Saudi Arabia.

523. Prove that

$$\operatorname{rank}(A-ABA)-\operatorname{rank}(B-BAB)=\operatorname{rank}(A)-\operatorname{rank}(B)$$

for all $A,B\in\mathcal{M}_n(\mathbb{C})$.

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania and Mihai Opincariu, Avram Iancu National College, Brad, Romania.

524. Determine all bijective and differentiable functions $f:\mathbb{R}\to\mathbb{R}$, with $f'(x)\neq 0\ \forall x\in\mathbb{R}$, satisfying

$$f(x)+f\left(\frac{1}{f'(x)}-x\right)=1,\quad\text{for every } x\in\mathbb{R}.$$

Proposed by Mircea Rus, Technical University of Cluj-Napoca, Romania.

525. Let $n\geq 4$ and let $a_1,\ldots,a_n>0$.

(i) Prove that if $\dfrac{1}{a_1} + \cdots + \dfrac{1}{a_n} \le 1$, then

$$2 \sum_{1\le i<j\le n} a_i a_j - n^3(n-1) \ge 2(n-1)^2(a_1 + \cdots + a_n - n^2).$$

(ii) Prove that if $\dfrac{1}{a_1} + \cdots + \dfrac{1}{a_n} \ge 1$ and $k = \dfrac{n^2-n+1}{(n-1)^3}$, then

$$a_1^2 + \cdots + a_n^2 - n^3 \ge k\left(2 \sum_{1\le i<j\le n} a_i a_j - n^3(n-1)\right).$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploieşti, Romania and Leonard Giugiuc, Traian National College, Drobeta-Turnu Severin, Romania.

526. If $a, b \in \mathbb{C}$ and $k \ge 2$, then we define

$$\mathcal{F}_{a,b}^k = \{f : \mathbb{C} \to \mathbb{C} \mid f^{(k)}(z) = az + b\},$$

where $f^{(k)} = f \circ \cdots \circ f$, with k copies of f.

For given $a, b \in \mathbb{C}$ and $k \ge 2$ find necessary and sufficient condition such that a set $A \subseteq \mathbb{C}$ can be written as $A = \mathrm{Fix}(f)$ for some $f \in \mathcal{F}_{a,b}^k$.

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania and Constantin-Nicolae Beli, IMAR, Bucharest, Romania.

527. For every non-negative integer n, evaluate the integral

$$I_n = \int_0^\infty \frac{\log^n(x)\sin(x)}{x}\,dx.$$

The answer may be given either in a closed form or recursively. Also it may include values of the zeta function at integers ≥ 2.

Proposed by Seán M. Stewart, King Abdullah University of Science and Technology (KAUST), Saudi Arabia.

528. If $a_1, a_2, a_3, a_4, a_5, a_6$ are non-negative real numbers such that

$$a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_6 + a_6a_1 = 6,$$
$$a_1 \ge a_2 \ge a_3 \ge a_4 \ge a_5 \ge a_6,$$

then

$$\frac{1}{a_1+3} + \frac{1}{a_2+3} + \frac{1}{a_3+3} + \frac{1}{a_4+3} + \frac{1}{a_5+3} + \frac{1}{a_6+3} \ge \frac{3}{2}.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploieşti, Romania.

SOLUTIONS

504. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with f' continuous on $[0,1]$, such that $|f'(x)| \le 1$ $\forall x \in [0,1]$. Prove that if $2\left|\int_0^1 f(x)\,\mathrm{d}x\right| \le 1$ then $(n+2)\left|\int_0^1 x^n f(x)\,\mathrm{d}x\right| \le 1$ $\forall n \ge 1$.

Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.

Solution by the author. For $n \ge 0$ we denote $I_n = \int_0^1 x^n f(x)\,\mathrm{d}x$. We obviously have $|I_0| \le \frac{1}{2}$ and we prove by induction on n that $|I_n| \le \frac{1}{n+2}$.

For $n = 1$ we use integration by parts and we get

$$\begin{aligned} I_1 &= \frac{1}{2}\left(I_0 + \int_0^1 (2x-1)f(x)\,\mathrm{d}x\right) \\ &= \frac{1}{2}\left(I_0 + (x^2-x)f(x)\Big|_0^1 - \int_0^1 (x^2-x)f'(x)\,\mathrm{d}x\right) \\ &= \frac{1}{2}\left(I_0 + \int_0^1 (x-x^2)f'(x)\,\mathrm{d}x\right) \end{aligned}$$

Since $|I_0| \le \frac{1}{2}$ and $x - x^2 \ge 0$ and $|f'(x)| \le 1$ $\forall x \in [0,1]$, we get

$$|I_1| \le \frac{1}{2}\left(|I_0| + \int_0^1 (x-x^2)|f'(x)|\,\mathrm{d}x\right) \le \frac{1}{2}\left(\frac{1}{2} + \int_0^1 (x-x^2)\,\mathrm{d}x\right) = \frac{1}{3}.$$

Suppose now that $n \ge 2$ and $|I_k| \le \frac{1}{k+2}$ for $k = 0, 1, \ldots, n-1$. We prove that $|I_n| \le \frac{1}{n+2}$. For every $k \ge 0$ we have

$$I_k = \frac{x^{k+1}}{k+1}f(x)\Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1}f'(x)\ \mathrm{d}x = \frac{f(1)}{k+1} - \frac{1}{k+1}\int_0^1 x^{k+1}f'(x)\,\mathrm{d}x.$$

It follows that $f(1) = (k+1)I_k + \int_0^1 x^{k+1}f'(x)\,\mathrm{d}x$. We add these relations for $k = 0, \ldots, n-1$ and we divide by $n(n+1)$. We get

$$nf(1) = \sum_{k=0}^{n-1}\left((k+1)I_k + \int_0^1 x^{k+1}f'(x)\,\mathrm{d}x\right).$$

We multiply the formula for I_n by $n(n+1)$ and we get

$$\begin{aligned} n(n+1)I_n &= nf(1) - n\int_0^1 x^{n+1}f'(x)\,\mathrm{d}x \\ &= \sum_{k=0}^{n-1}\left((k+1)I_k + \int_0^1 x^{k+1}f'(x)\,\mathrm{d}x\right) - n\int_0^1 x^{n+1}f'(x)\,\mathrm{d}x \\ &= \sum_{k=0}^{n-1}(k+1)I_k + \int_0^1\left(\sum_{k=0}^{n-1} x^{k+1} - nx^{n+1}\right)f'(x)\,\mathrm{d}x. \end{aligned}$$

If $x \in [0,1]$ then for $k = 0, \dots, n-1$ we have $x^{k+1} \geq x^{n+1}$ so $\sum_{k=0}^{n-1} x^{k+1} - nx^{n+1} \geq 0$. We also have $I_k \leq \frac{1}{k+2}$ for $k = 0, \dots, n-1$ and $|f'(x)| \leq 1$ for $x \in [0,1]$. It follows that

$$\begin{aligned}
n(n+1)I_n &\leq \sum_{k=0}^{n-1}(k+1)|I_k| + \int_0^1 \left(\sum_{k=0}^{n-1} x^{k+1} - nx^{n+1}\right)|f'(x)|\,\mathrm{d}x \\
&\leq \sum_{k=0}^{n-1}\frac{k+1}{k+2} + \int_0^1 \left(\sum_{k=0}^{n-1} x^{k+1} - nx^{n+1}\right)\mathrm{d}x \\
&= \sum_{k=0}^{n-1}\frac{k+1}{k+2} + \sum_{k=0}^{n-1}\frac{1}{k+2} - \frac{n}{n+2} = \sum_{k=0}^{n-1} 1 - \frac{n}{n+2} = n - \frac{n}{n+2} \\
&= \frac{n(n+1)}{n+2}.
\end{aligned}$$

From this we conclude that $|I_n| \leq \frac{1}{n+2}$. $\square$

We have received the same proof from Daniel Văcaru.

Editor's note. The problem can be solved without resorting to mathematical induction, by adapting the same idea from the $n = 1$ step. Namely, we have

$$\begin{aligned}
I_n &= \frac{1}{n+1}\left(I_0 + \int_0^1 ((n+1)x^n - 1)f(x)\,\mathrm{d}x\right) \\
&= \frac{1}{n+1}\left(I_0 + (x^{n+1} - x)f(x)\Big|_0^1 - \int_0^1 (x^{n+1} - x)f'(x)\,\mathrm{d}x\right) \\
&= \frac{1}{n+1}\left(I_0 + \int_0^1 (x - x^{n+1})f'(x)\,\mathrm{d}x\right).
\end{aligned}$$

But $|I_0| \leq \frac{1}{2}$ and on $[0,1]$ we have $x - x^{n+1} \geq 0$ and $|f'(x)| \leq 1$. Hence,

$$\begin{aligned}
|I_n| &\leq \frac{1}{n+1}\left(|I_1| + \int_0^1 (x - x^{n+1})|f'(x)|\,\mathrm{d}x\right) \\
&= \frac{1}{n+1}\left(\frac{1}{2} + \int_0^1 (x - x^{n+1})\,\mathrm{d}x\right) = \frac{1}{n+2}.
\end{aligned}$$

This is, essentially, the solution we received from Moubinool Omarjee.

Solution by Moubinool Omarjee, Paris, France. We have

$$\begin{aligned}
\left|(n+1)\int_0^1 x^n f(x)\,\mathrm{d}x\right| &\leq \left|(n+1)\int_0^1 x^n f(x)\,\mathrm{d}x - \int_0^1 f(x)\,\mathrm{d}x\right| \\
&\qquad + \left|\int_0^1 f(x)(x)\right|\mathrm{d}x - \int_0^1 f(x)\,\mathrm{d}x \\
&= \int_0^1 m'(x)f(x)\,\mathrm{d}x = -\int_0^1 m(x)f'(x)\,\mathrm{d}x.
\end{aligned}$$

Since $|f'(x)| \leq 1$ and $-m(x) = x - x^{n+1} \geq 0$ $\forall x \in [0,1]$, we get

$$\left|(n+1)\int_0^1 x^n f(x)\,\mathrm{d}x - \int_0^1 f(x)\,\mathrm{d}x\right| \leq \int_0^1 |m(x)f'(x)|\,\mathrm{d}x \leq \int_0^1 (x - x^{n+1})\,\mathrm{d}x$$
$$= \frac{1}{2} - \frac{1}{n+2}.$$

Together with $\left|\int_0^1 f(x)\,\mathrm{d}x\right| \leq \frac{1}{2}$, this implies

$$\left|(n+1)\int_0^1 x^n f(x)\,\mathrm{d}x\right| \leq \frac{1}{2} - \frac{1}{n+2} + \frac{1}{2} = \frac{n+1}{n+2} < 1.$$

505. Let $n, p, q \in \mathbb{N}$ such that $1 \leq q < p < n$. If there exists $A \in \mathcal{M}_n(\{0,1\})$ such that $A \cdot A^t$ has all the elements on the diagonal equal to p and all the other elements equal to q, prove that:

a) $p(p-1) = q(n-1)$;
b) $A \cdot A^t = A^t \cdot A$;
c) if n is even, then $p - q$ is a perfect square.

`Proposed by Vasile Pop and Mircea Rus, Technical University of Cluj-Napoca, Romania.`

Solution by the authors. Let $A = (a_{ij})$ and $B = A \cdot A^t = (b_{ij})$. We have

$$p = b_{ii} = \sum_{j=1}^n a_{ij}^2 = \sum_{j=1}^n a_{ij} \quad \text{for all } i \in \{1, 2, \ldots, n\},$$

hence $AU = pU$, where U is the column vector of order n with all elements equal to 1. Denoting by J_n the $n \times n$ matrix with all elements equal to 1, it follows that

$$A \cdot J_n = pJ_n \tag{1}$$

and, by hypothesis,

$$A \cdot A^t = (p-q)I_n + qJ_n, \tag{2}$$

where I_n denotes the $n \times n$ identity matrix.

From (1) and (2) it follows

$$A \cdot A^t = (p-q)I_n + \frac{q}{p}A \cdot J_n,$$

that is

$$\frac{1}{p-q}A \cdot \left(A^t - \frac{q}{p}J_n\right) = I_n,$$

which proves that $A^t - \frac{q}{p}J_n$ is the inverse of $\frac{1}{p-q}A$, hence they commute:

$$\left(A^t - \frac{q}{p}J_n\right) \cdot \frac{1}{p-q}A = I_n. \tag{3}$$

By multiplying (3) to the right with $(p-q)J_n$, it follows by (1) that

$$\left(A^t - \frac{q}{p}J_n\right) pJ_n = (p-q)J_n$$

and, by taking into account that $J_n^2 = nJ_n$, we obtain that

$$A^t \cdot J_n = \frac{p-q+qn}{p} J_n. \tag{4}$$

By (4), the sum of elements in any line of A^t is $\dfrac{p-q+qn}{p}$. Hence, the sum $\sum A^t$ of all elements of A^t is $\dfrac{p-q+qn}{p} \cdot n$. Similarly, from (1) we get that $\sum A = pn$. But $\sum A = \sum A^t$ and so $\dfrac{p-q+qn}{p} \cdot n = pn$, which finally leads to

$$p(p-1) = q(n-1), \tag{5}$$

which concludes the proof of part a).

Now, by (5), we can rewrite (4) as

$$A^t \cdot J_n = pJ_n,$$

hence

$$J_n \cdot A = pJ_n. \tag{6}$$

Then, by (3),

$$A^t \cdot A - \frac{q}{p}J_n \cdot A = (p-q)I_n,$$

hence, by (6) and (2),

$$A^t \cdot A = \frac{q}{p}J_n \cdot A + (p-q)I_n = qJ_n + (p-q)I_n = A \cdot A^t,$$

which concludes the proof of b).

Finally, computing the circulant determinant $\det B$ (e.g., subtract the first column from each of the other columns, then add to the first row each of the other rows), it follows that

$$(\det A)^2 = \det\left(A \cdot A^t\right) = \det B = (p-q)^{n-1}\left(p + (n-1)q\right) = (p-q)^{n-1} p^2,$$

hence $p - q$ is a perfect square (since n is even), which concludes the proof of c).

Remarks. (i) For every $n \geq 3$, there exist p, q and A which satisfy the required conditions. For example, let A be the circulant matrix $C(\underbrace{1, 1, \ldots, 1}_{n-1 \text{ values of } 1}, 0)$; then $p = n-1$ and $q = n-2$.

In general, given n, p, q, it is not always possible to find such a matrix A, even when n, p, q satisfy properties a) and c). For example, it is possible to show that for $q = 1$, $p = 7$ and $n = 43$ (which satisfy the required properties), there is no such matrix A (this example is related to the non-existence of finite projective planes of order 6).

(ii) It is possible to interpret A as the adjacency matrix of a directed graph (or, equivalently, of a binary relation). By letting $X = \{1, 2, \ldots, n\}$, define a binary relation R on X as follows:

$$(i, j) \in R \Longleftrightarrow a_{ij} = 1 \quad (i, j \in X).$$

By hypothesis, for all $i \in X$ one has

$$p = b_{ii} = \sum_{j=1}^{n} a_{ij}^2 = \sum_{j=1}^{n} a_{ij} = |\{j \in X : a_{ij} = 1\}| = |R(i)|,$$

while for all $i, j \in X$, $i \neq j$

$$q = b_{ij} = \sum_{k=1}^{n} a_{ik}a_{jk} = |\{k \in X : a_{ik} = a_{jk} = 1\}| = |R(i) \cap R(j)|,$$

where, as usual, $R(x) := \{y \in X : (x, y) \in R\}$ $(x \in X)$.

These ideas are also strongly connected to the theory of symmetric balanced incomplete block design, where A is the incidence matrix.

506. Let $N > 1$ be a squarefree integer. For every integer k we denote $q_N(k) = \gcd(N, k)$. Prove that there is a finite subset S of the unit circle such that for every polynomial $f = \sum_{k=0}^{n} a_k X^k \in \mathbb{C}[X]$ we have

$$\mu(N) \sum_{k=0}^{n} \mu(q_N(k))\phi(q_N(k))a_k = \sum_{\zeta \in S} f(\zeta).$$

(Here μ and ϕ denote the Möbius function and Euler's totient function, respectively.)

`Proposed by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bârlad, Romania.`

Solution by the author. Let U_t, and P_t be the set of all roots of unity of order t and of the primitive roots of unity of order t, respectively. We actually prove that our result holds with $S = P_N$.

By linearity, it is enough to prove the identity when f belongs to the basis $\{X^m \mid m \geq 0\}$ of $\mathbb{C}[X]$. That is, we must show that

$$\sum_{\zeta \in P_N} \zeta^m = \mu(N)\mu(q_N(m))\varphi(q_N(m)) \qquad \forall m \geq 0.$$

Note first, that, if $t \geq 1$ and m are integers, then we have the well-known relation

$$\sum_{\zeta \in U_t} \zeta^m = \begin{cases} 0, & \text{if } t \nmid m, \\ t, & \text{if } t \mid m. \end{cases}$$

In particular, if $t = p$ is a prime, then $U_p = P_p \cup \{1\}$, so the above equality becomes

$$1 + \sum_{\zeta \in P_p} \zeta^m = \begin{cases} 0, & \text{if } p \nmid m, \\ p, & \text{if } p \mid m. \end{cases}$$

But if $p \nmid m$ then $\mu(q_p(m))\varphi(q_p(m)) = \mu(1)\varphi(1) = 1$ and if $p \mid m$ then $\mu(q_p(m))\varphi(q_p(m)) = \mu(p)\varphi(p) = -(p-1)$. Hence,

$$\sum_{\zeta \in P_p} \zeta^m = -\mu(q_p(m))\varphi(q_p(m)).$$

This proves the desired identity in this (simplest) case.

Let now $N = p_1 \cdots p_s$, where $p_1, \ldots, p_s$ are distinct primes, so that $\mu(N) = (-1)^s$. Now it is well known that any $\zeta \in U_N$ writes uniquely as $\zeta = \zeta_1 \cdots \zeta_s$, where $\zeta_l \in U_{p_l}$ for each $1 \le l \le s$. Moreover, ζ is a primitive in U_N iff each ζ_l is primitive in U_{p_l}. Therefore

$$\begin{aligned} \sum_{\zeta \in P_N} \zeta^m &= \sum_{\zeta_1 \in P_{p_1}, \ldots \zeta_s \in P_{p_s}} (\zeta_1 \cdots \zeta_s)^m = \Big(\sum_{\zeta_1 \in P_{p_1}} \zeta_1^m \Big) \cdots \Big(\sum_{\zeta_s \in P_{p_s}} \zeta_s^m \Big) \\ &= \big(-\mu(q_{p_1}(m))\varphi(q_{p_1}(m))\big) \cdots \big(-\mu(q_{p_s}(m))\varphi(q_{p_s}(m))\big) \\ &= (-1)^s \mu(q_N(m))\varphi(q_N(m)), \end{aligned}$$

as desired. The last equality follows from the multiplicativity of the arithmetic functions μ and φ, the fact that $q_{p_1}, \ldots, q_{p_s}$ are mutually prime and the obvious equality

$$q_{p_1}(m) \cdots q_{p_s}(m) = q_N(m).$$

Remark. The Ramanujan sum $c_q(m)$ is the sum of the m-powers of the qth primitive roots of unity. The formula

$$c_q(m) = \frac{\varphi(q)}{\varphi\left(\dfrac{q}{(m,q)}\right)} \mu\left(\frac{q}{(m,q)}\right)$$

is well-known and if N is squarefree it immediately implies the identity

$$c_N(m) = \sum_{\zeta \in P_N} \zeta^m = \mu(N)\mu(q_N(m))\varphi(q_N(m)).$$

So, basically, we proved a particular case of the formula for the Ramanujan sum. (Nevertheless, the formula is *not* the starting point for the problem!) The solver who is familiar with this formula will probably find the problem to be rather easy.

507. Calculate the integral

$$\int_1^\infty \frac{\ln x}{x^3 + x\sqrt{x} + 1} \mathrm{d}x.$$

Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.

Solution by the author. Let us denote $I = \int_1^{\infty} \frac{\ln x}{x^3 + x\sqrt{x} + 1}\, \mathrm{d}x$.

In this integral we substitute $x = \frac{1}{y}$, which gives

$$I = \int_1^{\infty} \frac{\ln x}{x^3 + x\sqrt{x} + 1}\, \mathrm{d}x = -\int_0^1 \frac{y \ln y}{y^3 + y\sqrt{y} + 1}\, \mathrm{d}y.$$

Let us also denote $J = \int_0^1 \frac{y \ln y}{y^3 + y\sqrt{y} + 1}\, \mathrm{d}y$.

In this integral we substitute $y = z^{\frac{2}{3}}$, which gives

$$J = \frac{4}{9} \int_0^1 \frac{z^{\frac{1}{3}} \ln z}{z^2 + z + 1}\, \mathrm{d}z.$$

We have successively:

$$\begin{aligned}
\frac{9}{4} J &= \int_0^1 \frac{(1-z) z^{\frac{1}{3}} \ln z}{1 - z^3}\, \mathrm{d}z = \int_0^1 \frac{z^{\frac{1}{3}} \ln z}{1 - z^3}\, \mathrm{d}z - \int_0^1 \frac{z^{\frac{4}{3}} \ln z}{1 - z^3}\, \mathrm{d}z \\
&= \int_0^1 \sum_{n=0}^{\infty} z^{3n + \frac{1}{3}} \ln z\, \mathrm{d}z - \int_0^1 \sum_{n=0}^{\infty} z^{3n + \frac{4}{3}} \ln z\, \mathrm{d}z \\
&= \sum_{n=0}^{\infty} \int_0^1 z^{3n + \frac{1}{3}} \ln z\, \mathrm{d}z - \sum_{n=0}^{\infty} \int_0^1 z^{3n + \frac{4}{3}} \ln z\, \mathrm{d}z.
\end{aligned}$$

Now we will use the following relationship

$$\int_0^1 x^a \ln x\, \mathrm{d}x = -\frac{1}{(a+1)^2} \quad \forall a \in \mathbb{R},\ a \geq -1.$$

We obtain

$$J = \frac{4}{9} \sum_{n=0}^{\infty} \left(\frac{1}{\left(3n + \frac{7}{3}\right)^2} - \frac{1}{\left(3n + \frac{4}{3}\right)^2} \right) = \frac{4}{9} \sum_{n=0}^{\infty} \left(\frac{\frac{1}{9}}{\left(n + \frac{7}{9}\right)^2} - \frac{\frac{1}{9}}{\left(n + \frac{4}{9}\right)^2} \right).$$

We will now use the following relationship

$$\psi_1(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2},$$

where $\psi_1(x)$ is the trigamma function. This gives

$$J = \frac{4}{81} \left(\psi_1\left(\frac{7}{9}\right) - \psi_1\left(\frac{4}{9}\right) \right).$$

We obtained the value of the integral required in the problem statement

$$I = -J = \frac{4}{81} \left(\psi_1\left(\frac{4}{9}\right) - \psi_1\left(\frac{7}{9}\right) \right).$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA. We use the substitution $u = 1/x$ and we get

$$\int_1^\infty \frac{\ln x}{x^3 + x^{3/2} + 1}\,\mathrm{d}x = -\int_1^0 \frac{\ln(1/u)}{u^{-3} + u^{-3/2} + 1} \cdot \frac{\mathrm{d}u}{u^2} = -\int_0^1 \frac{u \ln u}{1 + u^{3/2} + u^3}\,\mathrm{d}u.$$

Now, multiply numerator and denominator of the integrand in the last integral by $1 - u^{3/2}$ to obtain

$$\int_1^\infty \frac{\ln x}{x^3 + x^{3/2} + 1}\,\mathrm{d}x = -\int_0^1 \frac{(u - u^{5/2}) \ln u}{1 - u^{9/2}}\,\mathrm{d}u.$$

With the series representation

$$\frac{1}{1 - u^{9/2}} = \sum_{k=0}^\infty u^{9k/2},$$

this becomes

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x^3 + x^{3/2} + 1}\,\mathrm{d}x &= -\int_0^1 \sum_{k=0}^\infty (u^{1+9k/2} - u^{5/2+9k/2}) \ln u\,\mathrm{d}u \\
&= -\sum_{k=0}^\infty \int_0^1 (u^{1+9k/2} - u^{5/2+9k/2}) \ln u\,\mathrm{d}u \\
&= \sum_{k=0}^\infty \left(\frac{1}{(2 + 9k/2)^2} - \frac{1}{(7/2 + 9k/2)^2} \right) \\
&= \frac{4}{81} \sum_{k=0}^\infty \left(\frac{1}{(k + 4/9)^2} - \frac{1}{(k + 7/9)^2} \right) \\
&= \frac{4}{81} \left(\psi_1\left(\frac{4}{9}\right) - \psi_1\left(\frac{7}{9}\right) \right),
\end{aligned}$$

where $\psi_1(x)$ is the trigamma function.

(Here we used the formula $\int_0^\infty u^a \ln u\,\mathrm{d}u = -\frac{1}{(a+1)^2}$, which holds for $a > -1$.)

508. Let K be a field and let $A, B \in M_n(K)$, with $n \geq 1$, such that $AB - BA = c(A - B)$ for some $c \in K \setminus \{0\}$.

(i) Prove that if $\operatorname{char} K = 0$ or $\operatorname{char} K > n$, then A and B have the same eigenvalues.

(ii) Prove that if $0 < \operatorname{char} K \leq n$, then (i) is no longer true.

Remark. The statement (i) is an extension of the statement b) of problem 495, whose solution was published in GMA **38 (117)** 3–4/2020, 61–64.

Proposed by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.

Solution by the author. (i) We denote by f_X the characteristic polynomial of a square matrix X. We prove that $f_A = f_B$, which implies that A and B have the same eigenvalues.

By the same reasoning as in the solution of problem 495 b), we get $f_A(x)f_B(x+c) = f_B(x)f_A(x+c)$, so it is enough to prove that if $P, Q \in K[X]$ are monic of degree n with $P(x)Q(x+c) = Q(x)P(x+c)$ and $\operatorname{char} K = 0$ or $\operatorname{char} K > n$, then $P = Q$. Let $R = \gcd(P, Q)$, with R monic, and let $P = P'R$, $Q = Q'R$. Then $\gcd(P', Q') = 1$ and P', Q' are monic with $\deg P' = \deg Q' =: n' \leq n$. After dividing the relation $P(x)Q(x+c) = Q(x)P(x+c)$ by $R(x)R(x+c)$, we get $P'(x)Q'(x+c) = Q'(x)P'(x+c)$. If $\deg R = n$, then $P = Q = R$, so we are done. So, we may assume that $\deg R < n$ and $n' > 0$. Hence, P' has a root x_0. Since $\gcd(P', Q') = 1$, we have $Q'(x_0) \neq 0$. Hence, $0 = P'(x_0)Q'(x_0+c) = Q'(x_0)P'(x_0+c)$ implies that $P'(x_0+c) = 0$, so x_0+c is another root of P. Inductively, $x_0, x_0+c, x_0+2c, \ldots$ are all roots of P. If $\operatorname{char} K = 0$, then $x_0, x_0+c, x_0+2c, \ldots$ are mutually distinct, so P has an infinity of roots. Contradiction. If $\operatorname{char} K = p > n$, then $x_0, x_0+c, \ldots, x_0+(p-1)c$ are p distinct roots of P', with $\deg P' = n' \leq n < p$. Again, contradiction.

(ii) We consider first the case when $n = p$. Before finding a counterexample, we give some hints about how this counterexample should look like.

Same as for (i), we put $P = f_B$, $Q = f_A$. Then P, Q are monic of degree $n = p$ and $P(x)Q(x+1) = Q(x)P(x+1)$. We search for some P, Q of this kind such that $P \neq Q$. Again, we write $P = P'R$ and $Q = Q'R$, where $R = \gcd(P, Q)$ is monic. If $\deg R = p$, then $P = Q = R$, so we may assume that $\deg R < p$. Then, as seen from the proof of (i), we have $\deg P' = \deg Q' =: n' \leq p$ and P' has p distinct roots of the form $x_0, x_0+c, \ldots, x_0+(p-1)c$. It follows that $n' = p$, i.e., $\gcd(P, Q) = R = 1$ and $P = P'$, $Q = Q'$, and $x_0, x_0+c, \ldots, x_0+(p-1)c$ are all roots of P. Thus, $P(x) = (x-x_0)(x-x_0-c)\cdots(x-x_0-(p-1)c) = c^pT(\frac{x-x_0}{c})$, where $T(y) = y(y-1)\cdots(y-(p-1)) = y^p - y$. (By Fermat's little theorem, in characteristic p we have that $0, 1\ldots, p-1$ are roots of $y^p - y$.) Thus $P(x) = (x-x_0)^p - c^{p-1}(x-x_0) = x^p - c^{p-1}x - (x_0^p - c^{p-1}x_0)$.

First we consider the case when $c = 1$. In order that $P \neq Q$, i.e., $f_A \neq f_B$, the roots of $P = f_B$ must be of the form $x_0, x_0+1, \ldots, x_0+p-1$. We choose $x_0 = 0$, so we take $B \in M_p(K)$ whose eigenvalues are $0, 1, \ldots, p-1$ and we have $P(x) = f_B(x) = x(x-1)\cdots(x-(p-1)) = x^p - x$. Since B has p mutually distinct eigenvalues, it is diagonalizable. Therefore, we may assume that $B = \operatorname{diag}(0, 1, \ldots, p-1) \in M_p(\mathbb{F}_p) \subseteq M_p(K)$.

Before searching for a suitable A, note that if we denote $C = A - B$, that is, $A = B + C$, then the condition $AB - BA = A - B$ writes as $(B + C)B - B(B+C) = C$, i.e., as $CB - BC = C$.

For convenience, we index the rows and columns of matrices from $M_p(K)$ not by $1, 2 \ldots, p$, but by $\mathbb{F}_p = \{0, \ldots, p-1\}$. We denote by $\{e_{i,j} \mid i, j \in \mathbb{F}_p\}$ the canonical basis of $M_p(K)$, where $e_{i,j}$ has 1 on the position (i, j) and 0 everywhere else. We have $e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l}$.

Then $B = \operatorname{diag}(0, 1, \ldots, p-1) \in M_p(\mathbb{F}_p)$ writes as $\sum_{i\in\mathbb{F}_p} ie_{i,i}$. We write $C = \sum_{j,k\in\mathbb{F}_p} a_{j,k}e_{j,k}$. Then

$$\begin{aligned} CB - BC &= \sum_{j,k\in\mathbb{F}_p} a_{j,k}e_{j,k} \sum_{i\in\mathbb{F}_p} ie_{i,i} - \sum_{i\in\mathbb{F}_p} ie_{i,i} \sum_{j,k\in\mathbb{F}_p} a_{j,k}e_{j,k} \\ &= \sum_{i,j,k\in\mathbb{F}_p} ia_{j,k}\delta_{k,i}e_{j,i} - \sum_{i,j,k\in\mathbb{F}_p} ia_{j,k}\delta_{i,j}e_{i,k} \\ &= \sum_{j,k\in\mathbb{F}_p} ka_{j,k}e_{j,k} - \sum_{j,k\in\mathbb{F}_p} ja_{j,k}e_{j,k} = \sum_{j,k\in\mathbb{F}_p} (k-j)a_{j,k}e_{j,k}. \end{aligned}$$

Then the relation $CB-BC = C$ writes as $(k-j)a_{j,k} = a_{j,k}\ \forall j, k \in \mathbb{F}_p$, which is equivalent to $a_{j,k} = 0$ when $k - j \neq 1$. Hence, we have $CB - BC = C$ iff C has the form $C = \sum_{j\in\mathbb{F}_p} a_{j,j+1}e_{j,j+1} = \sum_{i\in\mathbb{F}_p} a_ie_{i,i+1}$. (Here $a_i := a_{i,i+1}$.) Thus the only nonzero entries of C are above the main diagonal and in the left bottom corner, in the position $(p-1, 0)$.

If $C \in M_p(K)$ is of this type, then $CB-BC = C$, so $A = B+C$ satisfies $AB - BA = A - B$. As seen above, $f_B = x(x-1)\cdots(x-(p-1)) = x^p - x$. On the other hand, the matrix A has $0, 1, \ldots, p-1$ on the main diagonal, $a_0, \ldots, a_{p-2}$ above the main diagonal, and a_{p-1} in the bottom left corner. Thus, $xI-A$ has $x, x-1, \ldots, x-(p-1)$ on the main diagonal, $-a_0, \ldots, -a_{p-2}$ above the main diagonal, and $-a_{p-1}$ in the bottom left corner. When we develop $\det(xI_p - A)$ along the first column, we get $f_A(x) = \det(xI_p - A) = x(x-1)\cdots(x-(p-1)) + (-1)^{p+1}(-a_{p-1})(-a_0)\cdots(-a_{p-2}) = x^p - x - a$, where $a = a_0 \cdots a_{p-1}$. If we take $a_0, \ldots, a_{p-1} \neq 0$, then $a \neq 0$, so f_A and f_B are mutually prime, so that A and B have no common eigenvalues.

If $c \in K \setminus \{0\}$ is arbitrary, then we define $A' = cA$ and $B' = cB$, so the relation $AB - BA = A - B$ implies $A'B' - B'A' = c(A' - B')$. On the other hand, $P_{B'}(x) = c^pP_B(x/c) = x^p - c^{p-1}x$ and $P_{A'}(x) = c^pP_A(x/c) = x^p - c^{p-1}x - c^pa$, so again $\gcd(P_{A'}, P_{B'}) = 1$ and thus A' and B' have different eigenvalues. Thus we have counterexamples for arbitrary c.

If $n > p$, then, by the case $n = p$, there are $A', B' \in M_p(K)$ with $A'B' - B'A' = c(A' - B')$, $f_{B'}(x) = x^p - c^{p-1}x$ and $f_{A'}(x) = x^p - c^{p-1}x - c^pa$, with $a \neq 0$. Then we define $A, B \in M_n(K)$ by $A = A' \oplus 0_{n-p}$, $B = B' \oplus 0_{n-p}$, where 0_{n-p} is the zero element of $M_{n-p}(K)$. As a consequence of $A'B' - B'A' = c(A' - B')$, we have $AB - BA = c(A - B)$. We also have $f_A(x) = f_{A'}(x)f_{0_{n-p}} = (x^p - c^{p-1}x)x^{n-p}$ and $f_B(x) = f_{B'}(x)f_{0_{n-p}} = (x^p - c^{p-1}x - c^pa)x^{n-p}$. Hence, the only common eigenvalue of A and B is 0. Thus, they don't have the same eigenvalues.

509. Let m and n be positive integers and let $A_1, A_2, \ldots, A_m \in \mathcal{M}_n(\mathbb{R})$. For every $i \in \{1, 2, \ldots, m\}$ denote by $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in} \in \mathbb{C}$ the eigenvalues of A_i.

Prove that there exist $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \in \{-1, 1\}$ such that the eigenvalues $\mu_1, \mu_2, \ldots, \mu_n \in \mathbb{C}$ of the matrix $\varepsilon_1 A_1 + \varepsilon_2 A_2 + \cdots + \varepsilon_m A_m \in \mathcal{M}_n(\mathbb{R})$ satisfy the inequality

$$\sum_{j=1}^{n} \mu_j^2 \geq \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}^2. \tag{5}$$

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania.

Solution by the author. Note that if $A \in \mathcal{M}_n(\mathbb{R})$ has the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$, then $\sum_{i=1}^{n} \lambda_i^2 = \operatorname{Tr}(A^2)$, so every sum in (5) is a real number, hence the inequality makes sense. Also, the problem can be restated as follows: for any matrices $A_1, A_2, \ldots, A_m \in \mathcal{M}_n(\mathbb{R})$ there exist $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \in \{-1, 1\}$ such that

$$\operatorname{Tr}\left((\varepsilon_1 A_1 + \varepsilon_2 A_2 + \cdots + \varepsilon_m A_m)^2\right) \geq \operatorname{Tr}\left(A_1^2\right) + \operatorname{Tr}\left(A_2^2\right) + \cdots + \operatorname{Tr}\left(A_m^2\right). \tag{1$'$}$$

First, we claim that the function

$$f : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}, \quad f(A) = \operatorname{Tr}\left(A^2\right),\ A \in \mathcal{M}_n(\mathbb{R}),$$

verifies

$$f(A+B) + f(A-B) = 2(f(A) + f(B)), \quad \text{for all } A, B \in \mathcal{M}_n(\mathbb{R}). \tag{2}$$

Indeed,

$$\begin{aligned} f(A+B) + f(A-B) &= \operatorname{Tr}\left((A+B)^2 + (A-B)^2\right) = \operatorname{Tr}\left(2A^2 + 2B^2\right) \\ &= 2\left(\operatorname{Tr}(A^2) + \operatorname{Tr}(B^2)\right) = 2(f(A) + f(B)). \end{aligned}$$

Now, we prove (1$'$) by induction on $m \geq 1$. The statement is obviously true for $m = 1$ by letting $\varepsilon_1 = 1$. For the inductive step $m \to m+1$ let $A_1, A_2, \ldots, A_m, A_{m+1} \in \mathcal{M}_n(\mathbb{R})$. According to the hypothesis of the induction, there exist $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \in \{-1, 1\}$ such that

$$\operatorname{Tr}\left((\varepsilon_1 A_1 + \varepsilon_2 A_2 + \cdots + \varepsilon_m A_m)^2\right) \geq \operatorname{Tr}\left(A_1^2\right) + \operatorname{Tr}\left(A_2^2\right) + \cdots + \operatorname{Tr}\left(A_m^2\right) \tag{3}$$

and denote $A := \varepsilon_1 A_1 + \varepsilon_2 A_2 + \cdots + \varepsilon_m A_m$. Based on (2), it follows that

$$f(A + A_{m+1}) + f(A - A_{m+1}) = 2(f(A) + f(A_{m+1})),$$

which means that at least one of the inequalities

$$\begin{aligned} f(A + A_{m+1}) &\geq f(A) + f(A_{m+1}), \\ f(A - A_{m+1}) &\geq f(A) + f(A_{m+1}), \end{aligned}$$

is true. Concluding, there exists $\varepsilon_{m+1} \in \{-1, 1\}$ such that

$$f(A + \varepsilon_{m+1} A_{m+1}) \geq f(A) + f(A_{m+1}),$$

which translates to

$$\operatorname{Tr}\left(\left(\sum_{k=1}^{m+1}\varepsilon_k A_k\right)^2\right) \geq \operatorname{Tr}\left(\left(\sum_{k=1}^{m}\varepsilon_k A_k\right)^2\right) + \operatorname{Tr}\left(A_{m+1}^2\right). \tag{4}$$

Finally, by combining (3) and (4), we obtain

$$\operatorname{Tr}\left(\left(\sum_{k=1}^{m+1}\varepsilon_k A_k\right)^2\right) \geq \operatorname{Tr}\left(A_1^2\right) + \operatorname{Tr}\left(A_2^2\right) + \cdots + \operatorname{Tr}\left(A_m^2\right) + \operatorname{Tr}\left(A_{m+1}^2\right),$$

which concludes the argument.

Editor's note. This problem is a restatement of Problem 2 from SEEMOUS 2019; see GMA **37 (116)** 1–2/2019, pages 20–22.

510. Prove that

$$\sum_{n=1}^{\infty}\binom{4n}{2n}\frac{1}{16^n n^2(2n+1)} = 4\operatorname{Li}_2\left(\frac{1-\sqrt{2}}{2}\right) + \frac{\pi^2}{3} + 4\log\left(\frac{1+\sqrt{2}}{4}\right) - 2\log^2\left(\frac{1+\sqrt{2}}{2}\right) - \log^2(4) + 4\left(\sqrt{2}-1\right).$$

Here $\operatorname{Li}_2(x)$ is the dilogarithm with integral representation given by

$$\operatorname{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\,\mathrm{d}t.$$

Proposed by Seán M. Stewart, Bomaderry, NSW, Australia.

Solution by the author. We begin with a classical result for absolutely convergent series, namely

$$\sum_{n=1}^{\infty} a_{2n} = \frac{1}{2}\sum_{n=1}^{\infty} a_n + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n a_n.$$

Applying this result to the given series we see that

$$\begin{aligned}\sum_{n=1}^{\infty}\binom{4n}{2n}\frac{1}{16^n n^2(2n+1)} &= 4\sum_{n=1}^{\infty}\binom{4n}{2n}\frac{1}{4^{2n}(2n)^2(2n+1)} \\ &= 2\sum_{n=1}^{\infty}\binom{2n}{n}\frac{1}{4^n n^2(n+1)} + 2\sum_{n=1}^{\infty}\binom{2n}{n}\frac{(-1)^n}{4^n n^2(n+1)} \\ &= 2S_+ + 2S_-. \end{aligned} \tag{1}$$

To find S_+ and S_- we will first find a generating function for

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^n}{n^2(n+1)}$$

starting with the well-known generating function for the central binomial coefficients, namely

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}, \quad |x| < \frac{1}{4}.$$

After summing out the first term in the series and dividing by x, we have

$$\sum_{n=1}^{\infty} \binom{2n}{n} x^{n-1} = \frac{1}{x} \cdot \left(\frac{1}{\sqrt{1-4x}} - 1 \right) = \frac{1-\sqrt{1-4x}}{x\sqrt{1-4x}}.$$

Replacing x with t before integrating with respect to t from 0 to x yields

$$\begin{aligned} \sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^n}{n} &= \int_0^x \frac{1-\sqrt{1-4t}}{t\sqrt{1-4t}} \,\mathrm{d}t \\ &= -2\log\left(\sqrt{1-4t}+1\right)\Big|_0^x \\ &= 2\log(2) - 2\log\left(1+\sqrt{1-4x}\,\right). \end{aligned}$$

Replacing x with t before integrating with respect to t from 0 to x gives

$$\begin{aligned} \sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^{n+1}}{n(n+1)} &= 2\log(2)\int_0^x \mathrm{d}x - 2\int_0^x \log\left(1+\sqrt{1-4t}\right) \mathrm{d}t \\ &= 2x\log(2) - 2t\log\left(1+\sqrt{1-4t}\right)\Big|_0^x \\ &\quad - 4\int_0^x \frac{t}{\sqrt{1-4t}(1+\sqrt{1-4t})} \,\mathrm{d}t \quad \text{(by parts)} \end{aligned}$$

But

$$\begin{aligned} \int_0^x \frac{-4t}{\sqrt{1-4t}(\sqrt{1-4t}+1)} \,\mathrm{d}t &= \int_0^x \frac{\sqrt{1-4t}-1}{\sqrt{1-4t}} \,\mathrm{d}t = \int_0^x \left(1 - \frac{1}{\sqrt{1-4t}}\right) \mathrm{d}t \\ &= \left[t + \frac{1}{2}\sqrt{1-4t}\right]_0^x = x + \frac{1}{2}\sqrt{1-4x} - \frac{1}{2} \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^{n+1}}{n(n+1)} = 2x\log(2) - 2x\log\left(1+\sqrt{1-4x}\right) + x + \frac{1}{2}\sqrt{1-4x} - \frac{1}{2},$$

so

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^{n-1}}{n(n+1)} = \frac{2\log(2)+1}{x} - \frac{1}{2x^2} + \frac{\sqrt{1-4x}}{2x^2} - \frac{2\log\left(1+\sqrt{1-4x}\right)}{x},$$

after having divided throughout by x^2.

A final integration is needed in order to reach our desired generating function. This is best achieved using indefinite integrals before finding the constant of integration afterwards by letting $x \to 0$. We have

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^n}{n^2(n+1)} = (2\log(2)+1)\log(x) + \frac{1}{2x} + \frac{1}{2}\int \frac{\sqrt{1-4x}}{x^2}\,\mathrm{d}x - 2\int \frac{\log\left(1+\sqrt{1-4x}\right)}{x}\,\mathrm{d}x. \tag{2}$$

We denote by I_1 and I_2 the two integrals appearing to the right of the equality in (2). In both integrals we substitute $u = \sqrt{1-4x}$, so $x = \frac{1-u^2}{4}$. We get

$$I_1 = -8\int \frac{u^2}{(1-u^2)^2}\,\mathrm{d}u \quad \text{and} \quad I_2 = -2\int \frac{u}{1-u^2}\log(1+u)\,\mathrm{d}u.$$

Making use of the partial fraction decomposition of

$$\frac{u^2}{(1-u^2)^2} = -\frac{1}{4(u+1)} + \frac{1}{4(u+1)^2} + \frac{1}{4(u-1)} + \frac{1}{4(u-1)^2},$$

we have

$$\begin{aligned} I_1 &= 2\int \frac{\mathrm{d}u}{u+1} - 2\int \frac{\mathrm{d}u}{(u+1)^2} - 2\int \frac{\mathrm{d}u}{u-1} - 2\int \frac{\mathrm{d}u}{(u-1)^2} \\ &= 2\log(1+u) - 2\log|u-1| + \frac{2}{u+1} + \frac{2}{u-1} + C_1 \\ &= 2\log\left(1+\sqrt{1-4x}\right) - 2\log\left(1-\sqrt{1-4x}\right) - \frac{\sqrt{1-4x}}{x} + C_1. \end{aligned}$$

Here C_1 is a constant of integration.

For the second integral we use the partial fraction decomposition

$$\frac{u}{1-u^2} = -\frac{1}{2(u+1)} - \frac{1}{2(u-1)},$$

and we get

$$\begin{aligned} I_2 &= \int \frac{\log(1+u)}{1+u}\,\mathrm{d}u + \int \frac{\log(1+u)}{u-1}\,\mathrm{d}u \\ &= \frac{1}{2}\log^2(1+u) + \int \frac{\log(u+1)}{u-1}\,\mathrm{d}u. \end{aligned}$$

Making a substitution of $u = 2t - 1$ in the remaining integral, we get

$$\begin{aligned}
I_2 &= \frac{1}{2}\log^2\left(1+\sqrt{1-4x}\right) + \int \frac{\log(2t)}{t-1}\,\mathrm{d}t \\
&= \frac{1}{2}\log^2\left(1+\sqrt{1-4x}\right) + \log(2)\int \frac{\mathrm{d}t}{t-1} + \int \frac{\log(t)}{t-1}\,\mathrm{d}t \\
&= \frac{1}{2}\log^2\left(1+\sqrt{1-4x}\right) + \log(2)\log|t-1| - \mathrm{Li}_2(1-t) + C_2 \\
&= \frac{1}{2}\log^2\left(1+\sqrt{1-4x}\right) + \log(2)\log\left(1-\sqrt{1-4x}\right) - \log^2(2) \\
&\qquad - \mathrm{Li}_2\left(\frac{1-\sqrt{1-4x}}{2}\right) + C_2.
\end{aligned}$$

Here C_2 is a constant of integration. We also note the following integral representation for the dilogarithm has been used:

$$-\int_1^t \frac{\log(u)}{1-u}\,\mathrm{d}u = \mathrm{Li}_2(1-t).$$

On collecting all the pieces together for the two integrals, the sum in (2) becomes

$$\begin{aligned}
\sum_{n=1}^\infty \binom{2n}{n}\frac{x^n}{n^2(n+1)} &= -\log^2\left(1+\sqrt{1-4x}\right) + \frac{1-\sqrt{1-4x}}{2x} \\
&\quad + \log\left(1+\sqrt{1-4x}\right) + (2\log(2)+1)\log\left(\frac{x}{1-\sqrt{1-4x}}\right) \\
&\quad + 2\,\mathrm{Li}_2\left(\frac{1-\sqrt{1-4x}}{2}\right) + C,
\end{aligned}$$

where $C = \frac{1}{2}C_1 - 2(-2\log^2(2) + C_2)$. In order to find the constant C we let $x \to 0$. Noting that

$$\lim_{x\to 0}\frac{1-\sqrt{1-4x}}{2x} = 1 \quad \text{and} \quad \lim_{x\to 0}\log\left(\frac{x}{1-\sqrt{1-4x}}\right) = -\log(2),$$

doing so yields $C = 3\log^2(2) - 1$. Our sought after generating function is thus

$$\begin{aligned}
\sum_{n=1}^\infty \binom{2n}{n}\frac{x^n}{n^2(n+1)} &= 3\log^2(2) - 1 - \log^2\left(1+\sqrt{1-4x}\right) + \log\left(1+\sqrt{1-4x}\right) \\
&\quad + \frac{1-\sqrt{1-4x}}{2x} + (2\log(2)+1)\log\left(\frac{x}{1-\sqrt{1-4x}}\right) \\
&\quad + 2\,\mathrm{Li}_2\left(\frac{1-\sqrt{1-4x}}{2}\right), \quad |x| \leqslant \frac{1}{4}.
\end{aligned} \tag{3}$$

Setting $x = \frac{1}{4}$ in (3) gives S_+. Here

$$S_+ = 1 + 3\log^2(2) - 2\log(2)(2\log(2)+1) + 2\operatorname{Li}_2\left(\frac{1}{2}\right).$$

Noting that (see, for example, Eq. (1.16) on page 6 of *Polylogarithms and Associated Functions* by L. Lewin (North Holland, New York, 1981))

$$\operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2(2),$$

the sum for S_+ reduces to

$$S_+ = \frac{\pi^2}{6} + 1 - 2\log(2) - 2\log^2(2).$$

Setting $x = -\frac{1}{4}$ in (3) gives S_-. Here

$$S_- = 2\operatorname{Li}_2\left(\frac{1-\sqrt{2}}{2}\right) + 2\log\left(\frac{1+\sqrt{2}}{2}\right) - \log^2\left(\frac{1+\sqrt{2}}{2}\right) + 2\sqrt{2} - 3.$$

On combining the two results found for the sums S_+ and S_- into (1), the claimed result then follows.

Editor's note. At the end of the proof he author skipped some steps in the calculations, so the reader may have some difficulties in understanding where the term $-\log^2\left(\frac{1+\sqrt{2}}{2}\right)$ from the formula for S_- comes from. It is obtained after setting $x = -\frac{1}{4}$ in the terms

$$3\log^2(2) - \log^2\left(1+\sqrt{1-4x}\right) + 2\log(2)\log\left(\frac{x}{1-\sqrt{1-4x}}\right)$$

from (3).

What we get is

$$\begin{aligned}
&3\log^2(2) - \log^2(1+\sqrt{2}) + 2\log(2)\log\left(\frac{1}{4(\sqrt{2}-1)}\right)\\
&\quad= 3\log^2(2) - \log^2(1+\sqrt{2}) + 2\log(2)\log\left(\frac{1+\sqrt{2}}{4}\right)\\
&\quad= 3\log^2(2) - \log^2(1+\sqrt{2}) + 2\log(2)(\log(1+\sqrt{2}) - 2\log(2))\\
&\quad= -\log^2(2) - \log^2(1+\sqrt{2}) + 2\log(2)\log(1+\sqrt{2})\\
&\quad= -\left(\log(1+\sqrt{2}) - \log(2)\right)^2 = -\log^2\left(\frac{1+\sqrt{2}}{2}\right).
\end{aligned}$$

Solution by Narendra Bhandari, Bajura, Nepal, and Daniel Văcaru, Pitești, Romania. We note that

$$\frac{1}{n^2(2n+1)} = -\frac{2}{n} + \frac{1}{n^2} + \frac{4}{2n+1}$$

and, moreover, the generating function for central binomial coefficients is given by

$$f\left(x\right)=\sum_{n\geq 0}\frac{x^{n}}{4^{n}}\binom{2n}{n}=\frac{1}{\sqrt{1-x}},\quad |x|<1,$$

and we note that

$$\sum_{n=0}^{\infty}\frac{x^{2n}}{16^{n}}\binom{4n}{2n}=\frac{1}{2}\sum_{n=1}^{\infty}x^{n}\binom{2n}{n}\frac{1+(-1)^{n}}{4^{n}}=\frac{f\left(x\right)+f\left(-x\right)}{2}. \quad (1)$$

We subtract 1 (the first term) from (1), multiply by 4 and we integrate from 0 to 1 to get

$$\sum_{n=1}^{\infty}\frac{4}{2n+1}\binom{4n}{2n}=2\int_{0}^{1}\left(\frac{1}{\sqrt{1-x}}+\frac{1}{\sqrt{1+x}}-1\right)\mathrm{d}x=4\left(\sqrt{2}-1\right). \quad (2)$$

Next we evaluate the generating functions

$$g(y)=\sum_{1}^{\infty}\frac{y^{n}}{4^{n}}\binom{2n}{n}\frac{1}{n}\quad\text{and}\quad h(z)=\sum_{1}^{\infty}\frac{z^{n}}{4^{n}}\binom{2n}{n}\frac{1}{n^{2}}.$$

To evaluate $g\left(y\right)$, we subtract the first term, 1, from $f\left(x\right)$, we divide by x and we integrate from 0 to y. We get

$$g\left(y\right)=\int_{0}^{y}\left(\frac{1}{\sqrt{1-x}}-1\right)\frac{\mathrm{d}x}{x}=2\log\left(\frac{2}{1+\sqrt{1-y}}\right),$$

Since $g\left(1\right)=2\log 2$ and $g\left(-1\right)=2\log\left(\frac{2}{1+\sqrt{2}}\right)$, we get

$$\sum_{n=1}^{\infty}\frac{-2}{n}\binom{4n}{2n}\frac{1}{16^{n}}=-4\left(\frac{g\left(1\right)+g\left(-1\right)}{2}\right)=4\log\left(\frac{1+\sqrt{2}}{4}\right). \quad (3)$$

Next, we divide $g\left(y\right)$ by y and we integrate from 0 to z and we get

$$h(z)=2\int_{0}^{z}\log\left(\frac{2}{1+\sqrt{1-y}}\right)\frac{\mathrm{d}y}{y}.$$

To calculate the primitive of the last integrand we make the substitution $u^{2}=1-y$, $u\geq 0$, and we get

$$2\log 2\log y-2\int\frac{\log\left(1+\sqrt{1-y}\right)}{y}\,\mathrm{d}y=2\log 2\log y+4\int\frac{u\log(1+u)}{1-u^{2}}\,\mathrm{d}u.$$

By partial fraction decomposition, the last integral writes as

$$-2\int\left(\frac{\log\left(1+u\right)}{1+u}+\frac{\log\left(1+u\right)}{u-1}\right)\mathrm{d}u=-\log^{2}(1+u)-2\int\frac{\log\left(1+u\right)}{u-1}\,\mathrm{d}u.$$

After making the substitution $u=1-2v$, we get

$$-2\int\frac{\log\left(1+u\right)}{u-1}\,\mathrm{d}u=-2\int\frac{\log\left(2-2v\right)}{v}\,\mathrm{d}v=-2\log 2\log v+2\mathrm{Li}_{2}\left(v\right)+\mathcal{C}.$$

By reversing all substitutions and putting together all parts, we get

$$\begin{aligned}
2\int \log\left(\frac{2}{1+\sqrt{1-y}}\right)\frac{\mathrm{d}y}{y} \\
&= 2\log 2\log y - \log^2\left(1+\sqrt{1-y}\right) - 2\log 2\log\left(\frac{1-\sqrt{1-y}}{2}\right) \\
&\qquad + 2\mathrm{Li}_2\left(\frac{1-\sqrt{1-y}}{2}\right) + \mathcal{C} \\
&= 2\mathrm{Li}_2\left(\frac{1-\sqrt{1-y}}{2}\right) - \log^2\left(1+\sqrt{1-y}\right) + 2\log 2\log\left(1+\sqrt{1-y}\right) + \mathcal{C}.
\end{aligned}$$

Here we used the fact that $\frac{y}{1-\sqrt{1-y}} = 1+\sqrt{1-y}$ and the extra constant term $2\log^2 2$ was ignored, as it can be absorbed into the constant of integration $\mathcal{C}$.

The limit as $y \searrow 0$ of the above antiderivative is $2\mathrm{Li}_2(0) + \log^2 2 = \log^2 2$. We get

$$\begin{aligned}
h(z) = 2\mathrm{Li}_2\left(\frac{1-\sqrt{1-z}}{2}\right) - \log^2\left(1+\sqrt{1-z}\right) + 2\log 2\log\left(1+\sqrt{1-z}\right) \\
- \log^2 2.
\end{aligned}$$

Since $\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2 2$, we have $h(1) = \frac{\pi^2}{6} - 2\log^2 2$. We also have

$$\begin{aligned}
h(-1) &= 2\mathrm{Li}_2\left(\frac{1-\sqrt{2}}{2}\right) - \log^2\left(1+\sqrt{2}\right) + 2\log 2\log\left(1+\sqrt{2}\right) - \log^2 2 \\
&= 2\mathrm{Li}_2\left(\frac{1-\sqrt{2}}{2}\right) - \log^2\left(\frac{1+\sqrt{2}}{2}\right)^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^2}\binom{4n}{2n}\frac{1}{16^n} &= 4\left(\frac{h(1)+h(-1)}{2}\right) \\
&= 4\mathrm{Li}_2\left(\frac{1-\sqrt{2}}{2}\right) - 2\log^2\left(\frac{1+\sqrt{2}}{2}\right)^2 + \frac{\pi^2}{3} - 4\log^2 2. \quad (4)
\end{aligned}$$

By adding the formulas (2), (3) and (4), we get the desired result. (Note that $4\log^2 2 = (2\log 2)^2 = \log^2 4$.) □

511. Find the best lower and upper bounds for $\sum_{i=1}^{n}\cos(\angle A_i)$ over all convex n-gons $A_1A_2\ldots A_n$.

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Solution by C.N. Beli. The necessary and sufficient conditions such that $x_i = \angle A_i$ with $1 \le i \le n$ are the angles of a convex n-gon are $0 < x_i < \pi$ and $x_1 + \cdots + x_n = (n-2)\pi$. So we must determine the lower and upper bounds of $\cos x_1 + \cdots + \cos x_n$ where $(x_1, \ldots, x_n) \in S := \{(x_1, \ldots, x_n) \in (0,\pi)^n \mid x_1 + \cdots + x_n = (n-2)\pi\}$. These upper and lower bounds coincide with the maximum and minimum of $\cos x_1 + \cdots + \cos x_n$ where $(x_1, \ldots, x_n)$ belongs to the compact set $\bar{S} := \{(x_1, \ldots, x_n) \in [0,\pi]^n \mid x_1 + \cdots + x_n = (n-2)\pi\}$.

We use the property that the cosine function is strictly concave on $[0, \pi/2]$ and strictly convex on $[\pi/2, \pi]$. We need some preliminary results.

Lemma 1. Let $f : [a,b] \to \mathbb{R}$ be a function, let $n \ge 1$ be an integer and let $\kappa \in \mathbb{R}$, $na \le \kappa \le nb$. Put $\bar{S} = \{(x_1, \ldots, x_n) \in [a,b]^n \mid x_1 + \cdots + x_n = \kappa\}$.

(i) If f is strictly convex, then the minimum of f on $\bar{S}$ is reached only at $(\kappa/n, \ldots, \kappa/n)$ and the maximum is reached at the only element $(x_1, \ldots, x_n) \in \bar{S}$ (up to a permutation) where all but one of $x_1, \ldots, x_n$ are a or b.

(ii) If f is strictly concave, then the maximum of f on $\bar{S}$ is reached only at $(\kappa/n, \ldots, \kappa/n)$ and the minimum is reached at the only element $(x_1, \ldots, x_n) \in \bar{S}$ (up to a permutation) where all but one of $x_1, \ldots, x_n$ are a or b.

Proof. (i) The first statement follows from Jensen's theorem.

For the second statement assume that $(x_1, \ldots, x_n) \in \bar{S}$ and there are the indices $j \ne l$ such that $a < x_j \le x_l < b$. By Karamata's theorem, if $x'_j, x'_l \in [a,b]$ with $x'_j < x_j \le x_l < x'_l$ and $x'_j + x'_l = x_j + x_l$, then $f(x'_j) + f(x'_l) > f(x_j) + f(x_l)$. Hence, if in $(x_1, \ldots, x_n)$ we replace x_j, x_l by x'_j, x'_l, then we get a new element $(x_1, \ldots, x_n) \in \bar{S}$ with a larger sum $f(x_1) + \cdots + f(x_n)$. Note that we can take such a pair x'_j, x'_l where $x'_j = a$ or $x'_l = b$. Namely, if $x_j + x_l \le a + b$, then we take $(x'_j, x'_l) = (a, x_j + x_l - a)$, and if $x_j + x_l > a + b$, then we take $(x'_j, x'_l) = (x_j + x_l - b, b)$. Then in the new n-tuple $(x_1, \ldots, x_n)$ the number of indices i with $x_i \ne a, b$ is smaller. We repeat the procedure until we get an n-tuple $(y_1, \ldots, y_n) \in \bar{S}$ where all but one of the y_i's are a or b and $f(x_1) + \cdots + f(x_n) < f(y_1) + \cdots + f(y_n)$.

In conclusion, the maximum of $f(x_1) + \cdots + f(x_n)$ is reached for an n-tuple $(x_1, \ldots, x_n) \in \bar{S}$ such that all but one of the x_i's are a or b. For the unicity up to permutations of such $(x_1, \ldots, x_n)$ we denote by s the number of indices i such that $x_i = b$. We claim that $s = \left[\frac{\kappa - na}{b-a}\right]$. Indeed, if $\kappa = nb$, then $\bar{S} = \{(b, \ldots, b)\}$, so $s = n$ follows trivially. If $\kappa < nb$, then we have $s < n$, so there are $n - s - 1$ indices i with $x_i = a$ and an index m such that $a \le x_m < b$. It follows that $\kappa = (n-s-1)a + x_m + sb$. But $a \le x_m < b$, so $(n-s)a + sb \le \kappa < (n-s-1)a + (s+1)b$. It follows that $s(b-a) \le \kappa - na < (s+1)(b-1)$, so $s \le \frac{\kappa - na}{b-a} < s + 1$, which implies $s = \left[\frac{\kappa - na}{b-a}\right]$. Thus the number s of indices i with $x_i = b$ is well defined. The remaining $n - s$ entries of $(x_1, \ldots, x_n)$ are also well defined. Namely, a appears $n - s - 1$ times and we have an extra index m with $x_m = \kappa - (n-s-1)a - sb$. This concludes the proof.

The proof of (ii) is similar. (We may also apply (i) to the function $-f$.) $\square$

Lemma 2. Let $f : [a,b] \to \mathbb{R}$, κ and $\bar{S}$ be the same as in Lemma 1. Suppose that there exists $c \in (a,b)$ such that f is strictly concave on (a,c) and strictly convex on (c,b).

(i) If $(x_1,\dots,x_n) \in \bar{S}$ such that $f(x_1)+\cdots+f(x_n)$ is maximum, then there is $0 \le s \le n$ such that, up to a permutation of $x_1,\dots,x_n$, we have

$$a \le x_1 = \cdots = x_s \le c < x_{s+1} \le x_{s+2} = \cdots = x_n = b.$$

(ii) If $(x_1,\dots,x_n) \in \bar{S}$ such that $f(x_1)+\cdots+f(x_n)$ is minimum, then there is $0 \le s \le n$ such that, up to a permutation of $x_1,\dots,x_n$, we have

$$a = x_1 = \cdots = x_{s-1} \le x_s < c \le x_{s+1} = \cdots = x_n \le b.$$

Proof. We may assume that $x_1 \le \cdots \le x_n$.

(i) Let $(x_1,\dots,x_n) \in \bar{S}$ be such that $f(x_1)+\cdots+f(x_n)$ is maximum. Let $0 \le s \le n$ be such that $x_1,\dots,x_s \in [a,c]$ and $x_{s+1},\dots,x_n \in (c,b]$. We have $\kappa = \kappa' + \kappa''$, where $\kappa' = x_1+\cdots+x_s$, $\kappa'' = x_{s+1}+\cdots+x_n$. Then $(x_1,\dots,x_s) \in \bar{S}' := \{(y_1,\dots,y_s) \in [a,c] \mid y_1+\cdots+y_s = \kappa'\}$ and $(x_{s+1},\dots,x_n) \in \bar{S}'' := \{(y_{s+1},\dots,y_n) \in [a,c] \mid y_{s+1}+\cdots+y_n = \kappa''\}$.

Since $\kappa = \kappa'+\kappa''$, if $(y_1,\dots,y_s) \in \bar{S}'$ and $(y_{s+1},\dots,y_n) \in \bar{S}''$, then $(y_1,\dots,y_n) \in \bar{S}$. From the fact that

$$f(x_1)+\cdots+f(x_n) = \max\{f(y_1)+\cdots+f(y_n) \mid (y_1,\dots,y_n) \in \bar{S}\}$$

we deduce that

$$f(x_1)+\cdots+f(x_s) = \max\{f(y_1)+\cdots+f(y_s) \mid (y_1,\dots,y_s) \in \bar{S}\}$$

and

$$f(x_{s+1})+\cdots+f(x_n) = \max\{f(y_{s+1})+\cdots+f(y_n) \mid (y_{s+1},\dots,y_n) \in \bar{S}\}.$$

But f is concave on $[a,c]$ so, by Lemma 1(ii), we have $x_1 = \cdots = x_s$. And f is convex on $[c,b]$ so, by Lemma 1(i), we have that all but one of $x_{s+1},\dots,x_n$ are c or b. Since $c < x_{s+1} \le \cdots \le x_n \le b$, this is equivalent to $x_{s+2} = \cdots = x_n = b$. This concludes the proof of (i).

Similarly, for (ii) we assume $(x_1,\dots,x_n) \in \bar{S}$ is such that $f(x_1)+\cdots+f(x_n)$ is minimum. This time we take s such that $x_1,\dots,x_s \in [a,c)$ and $x_{s+1},\dots,x_n \in [c,b]$. Then we define κ', κ'', $\bar{S}'$ and $\bar{S}''$ as in the proof of (i). Since $f(x_1)+\cdots+f(x_n) = \min\{f(y_1)+\cdots+f(y_n) \mid (y_1,\dots,y_n) \in \bar{S}\}$, we deduce that $f(x_1)+\cdots+f(x_s) = \min\{f(y_1)+\cdots+f(y_s) \mid (y_1,\dots,y_s) \in \bar{S}\}$ and $f(x_{s+1})+\cdots+f(x_n) = \min\{f(y_{s+1})+\cdots+f(y_n) \mid (y_{s+1},\dots,y_n) \in \bar{S}\}$. As f is concave on $[a,c]$, by Lemma 1(ii), we have that all but one of $x_1,\dots,x_s$ are a or c. Since $a \le x_1 \le \cdots \le x_s < c$, this implies that $a = x_1 = \cdots = x_{s-1}$.

As f is convex on $[c, b]$, by Lemma 1(i), we get $x_{s+1} = \cdots = x_n$ and we are done. $\square$

The function $f : [0, \pi] \to \mathbb{R}$, $f(x) = \cos x$, is strictly concave on $[0, \pi/2]$ and strictly convex on $[\pi/2, \pi]$, so in Lemma 2 we may take $a = 0$, $b = \pi$ and $c = \pi/2$. We also take $\kappa = (n-2)\pi$.

Let $\Sigma : \mathbb{R}^n \to \mathbb{R}$, $\Sigma(x_1, \ldots, x_n) = \cos x_1 + \cdots + \cos x_n$. We want to determine the maximum and minimum of $\{\Sigma(x) \mid x \in \bar{S}\}$. They exist because Σ is continuous and $\bar{S}$ is compact.

We may restrict ourselves to the case when $x_1 \leq \cdots \leq x_n$.

First we assume that $x = (x_1, \ldots, x_n) \in \bar{S}$ is such that

$$\Sigma(x) = \max\{\Sigma(y) \mid y \in \bar{S}\}.$$

We have $x_1 + \cdots + x_n = (n-2)\pi$ and, by Lemma 2(i), there is $0 \leq s \leq n$ such that

$$0 \leq x_1 = \cdots = x_s \leq \pi/2 < x_{s+1} \leq x_{s+2} = \cdots = x_n = \pi.$$

Then $(n-2)\pi = x_1 + \cdots + x_n \leq s\frac{\pi}{2} + (n-s)\pi = (n - \frac{s}{2})\pi$, which implies that $s \leq 4$. In particular, if $s = 4$, then all inequalities must become equalities, i.e., $x_1 = x_2 = x_3 = x_4 = \pi/2$ and $x_5 = \cdots = x_n = \pi$.

We also note that $(n-2)\pi = x_1 + \cdots + x_n > s \cdot 0 + \frac{\pi}{2} + (n-s-1)\pi$, which implies that $s > 3/2$, i.e., $s \geq 2$.

We consider the three cases, $s = 2, 3, 4$ separately.

$\mathbf{s = 4}$. Note that this case occurs only for $n \geq 4$. Then we must have $x_1 = x_2 = x_3 = x_4 = \pi/2$ and $x_5 = \cdots = x_n = \pi$, so that we obtain $\Sigma = 4\cos\pi/2 + (n-4)\cos\pi = 4 - n$.

$\mathbf{s = 3}$. If $n = 3$, then $x_1 = x_2 = x_3 = 2\pi/3$ and $\Sigma = 3\cos 2\pi/3 = -3/2$.

If $n \geq 4$, then $x_1 = x_2 = x_3 =: t$ and $x_5 = \cdots = x_n = \pi$, so that $x_1 + \cdots + x_n = (n-2)\pi$ implies that $x_4 = (n-2)\pi - 3x - (n-4)\pi = 2\pi - 3t$. Hence, the n-tuples y of this type are parametrized by a parameter u, as $y = h(u) := (u, u, u, 2\pi - u, \pi, \ldots, \pi)$. In order that $y = h(u)$ be of this type, we must have $0 \leq u \leq \pi/2$ and $\pi/2 < 2\pi - 3u \leq \pi$. Hence, the domain of h is the interval $[\pi/3, \pi/2)$. Let $g : [\pi/3, \pi/2) \to \mathbb{R}$, $g(s) = \Sigma(h(s))$. We have $g(u) = 3\cos u + \cos(2\pi - 3u) + (n-4)\cos\pi = 3\cos u + \cos 3u + 4 - n$.

We have $x = h(t)$, so, by the maximality of $\Sigma(x) = \Sigma(h(t)) = g(t)$, we have $g(t) = \max\{g(u) \mid u \in [\pi/3, \pi/2)\}$. We have $g'(u) = -3\sin u - 3\sin 3u = -3\sin u + 3\sin(3u - \pi)$. But, for every $u \in [\pi/3, \pi/2)$, we have $0 \leq 3u - \pi < u < \pi/2$, which implies that $\sin(3u - \pi) < \sin u$, so $g'(u) < 0$, i.e., g is decreasing. Thus, the maximality of $g(t)$ implies that $t = \pi/3$. We have $2\pi - 3u = \pi$, that is $x = h(\pi/3) = (\pi/3, \pi/3, \pi/3, \pi, \ldots, \pi)$, and therefore $\Sigma(x) = 3\cos\pi/3 + (n-3)\cos\pi = 3 \cdot 1/2 + (n-3)(-1) = 9/2 - n$.

$\mathbf{s = 2}$. We proceed similarly as in the case $s = 3$. We have $x_1 = x_2 =: t$ and $x_4 = \cdots = x_n = \pi$. Thus, $x_3 = (n-2)\pi - 2t - (n-3)\pi = \pi - 2t$. So, the n-tuples of this type are parameterized by $h(u) = (u, u, \pi - 2u, \pi \cdots, \pi)$ and

$g(u) := \Sigma(h(u)) = 2\cos u + \cos(\pi - 2u) + (n-3)\cos\pi = 2\cos u - \cos 2u + 3 - n$. In order that $y = h(u)$ be of the required type, we must have $0 \le u \le \pi$ and $\pi/2 < \pi - 2u \le \pi$, so that the domain of h and g is $[0, \pi/4)$.

Again, from the maximality of $\Sigma(x) = g(t)$, we get that

$$g(t) = \max\{g(u) \mid u \in [0, \pi/4)\}.$$

We have $g'(u) = -2\sin u + 2\sin 2u$. But for $u \in (0, \pi/4)$ we clearly have $0 < u < 2u < \pi/2$, so $\sin u < \sin 2u$ and hence $g'(u) > 0$. Thus, g is strictly increasing. Since its interval of definition, $[0, \pi/4)$, is open to the right, g has no maximum. Hence, we cannot have a maximum of this type.

In conclusion, if $n = 3$, then the maximum is $3/2$ and is reached at $(\pi/3, \pi/3, \pi/3)$. If $n \ge 4$, then we have two possible maxima, one in the case $s = 4$, which is equal to $4 - n$, and one in the case $s = 3$, which is equal to $9/2 - n$. Of the two, the larger is $9/2 - n$. It is reached for $(x_1, \dots, x_n) = (\pi/3, \pi/3, \pi/3, \pi, \dots, \pi)$ and all of its permutations.

We now assume that $x = (x_1, \dots, x_n) \in \bar{S}$ is such that $\Sigma(x) = \cos x_1 + \dots + \cos x_n$ is minimum. By Lemma 2(ii), there is some $0 \le s \le n$ such that

$$0 = x_1 = \dots = x_{s-1} \le x_s < \pi/2 \le x_{s+1} = \dots = x_n \le \pi.$$

We first consider the case $n = 3$. If $0 < x_1$, then we must have $s \le 1$. If $s = 1$, then $\pi = x_1 + x_2 + x_3 \ge x_1 + \pi/2 + \pi/2 > \pi$, contradiction. If $s = 0$, then $\pi = x_1 + x_2 + x_3 \ge \pi/2 + \pi/2 + \pi/2 > \pi$, again contradiction. So, we must have $x_1 = 0$ and $x_2 + x_3 = \pi$. Then $\Sigma = \cos 0 + \cos x_2 + \cos(\pi - x_2) = 1$, regardless of the value of x_2. Thus, in the case $n = 3$ the minimum is 1 and it is reached when one of x_1, x_2, x_3 is 0.

Suppose now that $n \ge 4$. We have $(n-2)\pi = x_1 + \dots + x_n < (s-1)0 + \pi/2 + (n-s)\pi$, so $s < 5/2$, that is, $s \le 2$. We consider the three possible cases separately.

s = 0. We have $x_1 = \dots = x_n = \frac{(n-2)\pi}{n} = \pi - 2\pi/n$. Since $n \ge 4$, we have $\pi/2 \le \pi - 2\pi/n \le \pi$, as required. Then $\Sigma(x) = n\cos(\pi - 2\pi/n) = -n\cos 2\pi/n$.

For the cases $s = 1, 2$ we denote $\pi - t = x_{s+1} = \dots = x_n$. We have $(n-2)\pi = x_1 + \dots + x_n = (s-1)0 + x_s + (n-s)(\pi - t)$, so $x_s = (s-2)\pi + (n-s)t$.

s = 1. We have $x_1 = (n-1)t - \pi$, so the n-tuples of this type are parametrized by $h(u) = ((n-1)u - \pi, \pi - u, \dots, \pi - u)$. Hence, $g(u) = \Sigma(h(u)) = \cos((n-1)u - \pi) + (n-1)\cos(\pi - u) = \cos((n-1)u - \pi) - (n-1)\cos u$. In order that $h(u)$ be of the required type, we need that $0 \le (n-1)u - \pi < \pi/2$ and $\pi/2 \le \pi - u \le \pi$. Thus, the domain of h and of g is $\left[\frac{\pi}{n-1}, \frac{3\pi}{2(n-1)}\right)$. By the minimality of $\Sigma(x) = \Sigma(h(t)) = g(t)$, we have

$$g(t) = \min\left\{g(u) \mid u \in \left[\frac{\pi}{n-1}, \frac{3\pi}{2(n-1)}\right)\right\}.$$

We have $g'(u) = -(n-1)\sin((n-1)u-\pi) + (n-1)\sin u$. But, when $u \in \left[\frac{\pi}{n-1}, \frac{3\pi}{2(n-1)}\right)$, both u and $(n-1)u-\pi$ belong to the interval $[0,\pi/2]$, where the sine function is increasing. Hence, if $u < \frac{\pi}{n-2}$, then $u > (n-1)u-\pi$, so $\sin u > \sin((n-1)u-\pi)$ and therefore $g'(u) > 0$. When $u > \frac{\pi}{n-2}$ we have the reverse inequalities, so that $g'(u) < 0$. Thus, g is increasing on $\left[\frac{\pi}{n-1}, \frac{\pi}{n-2}\right]$ and decreasing on $\left[\frac{\pi}{n-2}, \frac{3\pi}{2(n-1)}\right)$. Since t is a point of minimum for g, which is defined on $\left[\frac{\pi}{n-1}, \frac{3\pi}{2(n-1)}\right)$, we can only have $u = \frac{\pi}{n-1}$. (The interval is open to de right.) Then $(n-1)u-\pi = 0$ and $\pi - u = \frac{(n-2)\pi}{n-1}$, so that

$$x = h(u) = \left(0, \frac{(n-2)\pi}{n-1}, \dots, \frac{(n-2)\pi}{n-1}\right)$$

and

$$\Sigma(x) = g(u) = \cos 0 + (n-1)\cos\frac{(n-2)\pi}{n-1} = 1-(n-1)\cos\frac{\pi}{n-1}.$$

s = 2. We have $x_2 = (n-2)t$, so the n-tuples of this type are parameterized by $h(u) = (0,(n-2)u, \pi-u, \dots, \pi-u)$ and we have $g(u) = \cos 0 + \cos(n-2)u + (n-2)\cos(\pi-u) = 1+\cos(n-2)u-(n-2)\cos u$. In order that $h(u)$ be of the required type, we need that $0 \le (n-2)u < \pi/2$ and $\pi/2 \le \pi-u \le \pi$, so the domain of h and of g is $\left[0, \frac{\pi}{2(n-2)}\right)$. As in the previous case, we have

$$g(t) = \min\left\{g(u) \mid u \in \left[0, \frac{\pi}{2(n-2)}\right)\right\}.$$

Note that $g'(u) = -(n-2)\sin(n-2)u + (n-2)\sin u$ and for any $u \in \left(0, \frac{\pi}{2(n-2)}\right)$ we have $0 < u < (n-2)u < \pi/2$, so that $\sin u < \sin(n-2)u$. Hence, $g'(u) < 0$ and therefore g is strictly decreasing on the whole domain of definition. Since g is decreasing and the interval $\left[0, \frac{\pi}{2(n-2)}\right)$, where it is defined, is open to the right, it has no minimum. Hence Σ cannot have a minimum of this type.

So, for $n \ge 4$ the only possible minima for $\Sigma(x)$ with $x \in \bar{S}$ are $-n\cos\frac{2\pi}{n}$, from the case $s=0$, and $1-(n-1)\cos\frac{\pi}{n-1}$, from the case $s=1$. We have to determine which of the two is smaller.

If $0 < z < \pi/2$, by the reminder formula for Taylor series, there is some $\xi \in [0,z] \subseteq (0,\pi/2)$ such that $\cos z = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4\cos\xi > 1-\frac{1}{2}z^2$. In particular, it holds $\cos\frac{2\pi}{n} \ge 1-\frac{1}{2}\left(\frac{2\pi}{n}\right)^2 = 1-\frac{2\pi^2}{n^2}$, that is, $-n\cos\frac{2\pi}{n} \le -n+\frac{2\pi^2}{n}$. If $n \ge 10 > \pi^2$, then $-n\cos\frac{2\pi}{n} < -n+\frac{2\pi^2}{\pi^2} = 2-n < 1-(n-1)\cos\frac{\pi}{n-1}$. For $4 \le n \le 9$ we check case by case and we get that one has

$1-(n-1)\cos\frac{\pi}{n-1} < -n\cos\frac{2\pi}{n}$ precisely when $4 \le n \le 6$. In conclusion,

$$\min_{x\in\bar{S}} \Sigma(x) = \begin{cases} 1-(n-1)\cos\frac{\pi}{n-1} & \text{for } n \le 6. \\ -n\cos\frac{2\pi}{n} & \text{for } n \ge 7. \end{cases}$$

(If $n=3$ then $1-(n-1)\cos\frac{\pi}{n-1} = 1$, so the top formula also applies to the case $n=3$.)

When $n=3$, the minimum is reached when one of x_1, x_2, x_3 is 0. For $4 \le n \le 6$, it is reached for $x = \left(1, \frac{(n-2)\pi}{n-1}, \dots, \frac{(n-2)\pi}{n-1}\right)$ and all permutations. And for $n \ge 7$ the minimum is reached when $x = \left(\frac{(n-2)\pi}{n}, \dots, \frac{(n-2)\pi}{n}\right)$.

Editor's note. The authors' approach for finding the lower bound is essentially the same. They prove the analogous of Lemma 2(ii), but only for the given function $f : [0,\pi] \to \mathbb{R}$, $f(x) = \cos x$, where a, b and c are 0, π and $\pi/2$, respectively. From here they get the same two possible minima, $1-(n-1)\cos\frac{\pi}{n-1}$ and $-n\frac{2\pi}{n}$. However, for deciding which of the two is the smaller, they use a different approach, which is somewhat lengthier and more complicated.

For determining the upper bound the authors use an inductive reasoning that goes along the following lines.

When $n=3$ we have a geometrical approach. In a triangle ABC we have $\cos A+\cos B+\cos C = 1+\frac{r}{R}$. By Euler's inequality, $\frac{r}{R} \le \frac{1}{2}$, with equality iff ABC is equilateral. Thus, $\cos A+\cos B+\cos C \le \frac{3}{2}$, with equality when and only when $A=B=C=\pi/3$.

(Incidentally, this also gives the lower bound. We have $\cos A+\cos B+\cos C \ge 1$, with equality iff $r=0$, which happens in the degenerate case when $(A,B,C) \in \bar{S}\setminus S$, i.e., when A, B or C is 0.)

Next the induction is used to prove that if $n \ge 3$, then the maximum of $\cos A_1+\dots+\cos A_n$, where $0 \le A_1 \le \dots \le A_n \le \pi$ and $A_1+\dots+A_n = (n-2)\pi$, is $\frac{9}{2}-n$ and it is reached only at $(\pi/3, \pi/3, \pi/3, \pi, \dots, \pi)$. The cases $n=3$ were treated above, so we assume that $n \ge 4$. We have

$$\sum_{k=1}^{n} \cos A_k = \sum_{j=1}^{n-2} \cos A_j + \cos X + \left(\cos Y + \cos A_{n-1} + \cos A_n\right),$$

where $X = A_{n-1}+A_n-\pi$ and $Y = 2\pi - A_{n-1} - A_n$. (We have $X+Y=\pi$, so $\cos X + \cos Y = 0$.)

Since A_{n-1}, A_n are the two largest of the elements in the sequence $A_1, \dots, A_n$, their arithmetic mean is larger than the mean of the whole sequence, i.e., $\frac{1}{2}(A_{n-1}+A_n) \ge \frac{1}{n}(A_1+\dots+A_n) = \frac{(n-2)\pi}{n}$. Since $n \ge 4$, we get $A_{n-1}+A_n \ge \frac{2n-4}{n}\pi \ge \pi$. On the other hand, $A_{n-1}+A_n \le \pi+\pi = 2\pi$. From these inequalities we conclude that $0 \le X, Y \le \pi$.

Since $0 \le A_1, \dots, A_{n-2}, X \le \pi$ and $A_1+\dots+A_{n-2}+X = A_1+\dots+A_n-\pi = (n-3)\pi$, by the induction step, we have that $\cos A_1+\dots+\cos A_{n-2}+$

$\cos X \leq 9/2 - (n-1)$, with equality iff $A_1, \ldots, A_{n-2}, X$ are, in some order, $\pi/3, \pi/3, \pi/3, \pi, \ldots, \pi$. On the other hand, $Y + A_{n-1} + A_n = 2\pi$, so that

$$1 + \cos Y + \cos A_{n-1} + \cos A_n = -4\cos\frac{Y}{2}\cos\frac{A_{n-1}}{2}\cos\frac{A_n}{2}.$$

From $0 \leq Y/2 \leq \pi/2$ and $0 \leq A_{n-1}/2 \leq A_n/2 \leq \pi/2$ it follows that $\cos(Y/2)\cos(A_{n-1}/2)\cos(A_n/2) \geq 0$, with equality iff $Y/2$ or $A_n/2 = \pi/2$, i.e., iff Y or $A_n = \pi$. Hence, $\cos Y + \cos A_{n-1} + \cos A_n \leq -1$, with equality iff Y or $A_n = \pi$. In conclusion, $\cos A_1 + \cdots + \cos A_n \leq (9/2 - (n-1)) + (-1) = 9/2 - n$. For $(A_1, \ldots, A_n) = (\pi/3, \pi/3, \pi/3, \pi, \ldots, \pi)$ we have equality because $3\cos\pi/3 + (n-3)\cos\pi = 3/2 - (n-3) = 9/2 - n$. Conversely, assume that $\cos A_1 + \cdots + \cos A_n = 9/2 - n$. Then $A_1, \ldots, A_{n-2}, X$ are, in some order, $\pi/3, \pi/3, \pi/3, \pi, \ldots, \pi$ and Y or $A_n = \pi$. As $X \in \{\pi/3, \pi\}$, we have $Y = \pi - X \in \{0, 2\pi/3\}$ and so $Y \neq \pi$. Therefore, $A_n = \pi$. It follows that $A_1 + \cdots + A_{n-1} = (n-2)\pi - A_n = (n-3)\pi$ and $\cos A_1 + \cdots + \cos A_{n-1} = 9/2 - n - \cos A_n = 9/2 - (n-1)$. By the induction hypothesis, this implies that $(A_1, \ldots, A_{n-1}) = (\pi/3, \pi/3, \pi/3, \pi, \ldots, \pi)$. Thus $(A_1, \ldots, A_n) = (\pi/3, \pi/3, \pi/3, \pi, \ldots, \pi)$.

512. Evaluate the series

$$\sum_{n=1}^{\infty}\left(n\left(n\left(n\sum_{k=n}^{\infty}\frac{1}{k^2} - 1\right) - \frac{1}{2}\right) - \frac{1}{6}\right).$$

`Proposed by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bârlad, Romania.`

Solution by the author. Our sum writes as

$$S = \sum_{n=1}^{\infty} n^3\left(\sum_{k=n}^{\infty}\frac{1}{k^2} - \frac{1}{n} - \frac{1}{2n^2} - \frac{1}{6n^3}\right).$$

We prove that $S = -\frac{1}{24}$. To do that we use Abel's summation formula

$$\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N-1}(a_1 + \cdots + a_n)(b_n - b_{n+1}) + (a_1 + \cdots + a_N)b_N$$

for

$$a_n = n^3 \qquad \text{and} \qquad b_n = \sum_{k=n}^{\infty}\frac{1}{k^2} - \frac{1}{n} - \frac{1}{2n^2} - \frac{1}{6n^3}.$$

We have $a_1 + \cdots + a_n = \frac{n^2(n+1)^2}{4}$ and

$$b_n - b_{n+1} = \frac{1}{n^2} - \frac{1}{n(n+1)} - \frac{2n+1}{2n^2(n+1)^2} - \frac{3n^2+3n+1}{6n^3(n+1)^3} = -\frac{1}{6n^3(n+1)^3}.$$

It follows that the Nth partial sum of the given series is

$$S_N = \sum_{n=1}^{N} a_n b_n = -\frac{1}{24}\sum_{n=1}^{N} \frac{1}{n(n+1)} + \frac{N^2(N+1)^2}{4} b_N$$
$$= -\frac{1}{24}\left(1 - \frac{1}{N}\right) + \frac{1}{4}\left(1 + \frac{1}{N}\right)^2 N^4 b_N.$$

Now we have (by using the Stolz-Cesàro theorem)

$$\lim_{N\to\infty} N^4 b_N = \lim_{N\to\infty} \frac{b_N}{\frac{1}{N^4}} = \lim_{N\to\infty} \frac{b_N - b_{N+1}}{\frac{1}{N^4} - \frac{1}{(N+1)^4}}$$
$$= \lim_{N\to\infty} \frac{\frac{-1}{6N^3(N+1)^3}}{\frac{4N^3+6N^2+4N+1}{N^4(N+1)^4}} = \lim_{N\to\infty} \frac{-N(N+1)}{6(4N^3+6N^2+4N+1)}$$
$$= 0,$$

whence we conclude

$$\sum_{n=1}^{\infty}\left(n\left(n\left(n\sum_{k=n}^{\infty}\frac{1}{k^2} - 1\right) - \frac{1}{2}\right) - \frac{1}{6}\right) = \lim_{N\to\infty}\sum_{n=1}^{N} a_n b_n$$

$$= \lim_{N\to\infty}\left(-\frac{1}{24}\left(1 - \frac{1}{N}\right) + \frac{1}{4}\left(1 + \frac{1}{N}\right)^2 N^4 b_N\right) = -\frac{1}{24},$$

as claimed at the beginning of the solution.

Remark. The problem is in the same vein as problem 12134 from *The American Mathematical Monthly*. One can also prove (similarly) that

$$\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{1}{k^2} - \frac{1}{n}\right) = 1,$$

but this one is, probably, widely known.

Solution by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA. As in the author's solution, one shows, by Stolz-Cesàro theorem, that

$$n^4\left(\sum_{k=n}^{\infty}\frac{1}{k^2} - \frac{1}{n} - \frac{1}{2n^2} - \frac{1}{6n^3}\right) \to 0$$

as $n \to \infty$, i.e.,

$$\sum_{k=n}^{\infty}\frac{1}{k^2} = \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{6n^3} + o\left(\frac{1}{n^4}\right).$$

From here one can proceed as follows. For $N \gg 0$ we have

$$\begin{aligned}
\sum_{n=1}^{N}\left(n\left(n\left(n\sum_{k=n}^{\infty}\frac{1}{k^2}-1\right)-\frac{1}{2}\right)-\frac{1}{6}\right) &= \sum_{n=1}^{N}\left(n^3\sum_{k=n}^{\infty}\frac{1}{k^2}-n^2-\frac{n}{2}-\frac{1}{6}\right) \\
&= \sum_{n=1}^{N}n^3\sum_{k=n}^{N}\frac{1}{k}^2+\sum_{n=1}^{N}n^3\sum_{k=N+1}^{\infty}\frac{1}{k^2}-\frac{N(N+1)(2N+1)}{6} \\
&= \qquad -\frac{N(N+1)}{4}-\frac{N}{6} \\
&= \sum_{k=1}^{N}\frac{1}{k^2}\sum_{n=1}^{k}n^3+\frac{N^2(N+1)^2}{4}\left(\frac{1}{N}+\frac{1}{2N^2}+\frac{1}{6N^3}+o\left(\frac{1}{N^4}\right)-\frac{1}{N^2}\right) \\
&\qquad -\frac{N(N+1)(2N+1)}{6}-\frac{N(N+1)}{4}-\frac{N}{6} \\
&= \sum_{k=1}^{N}\frac{(k+1)^2}{4}+\frac{N^2(N+1)^2}{4}\left(\frac{1}{N}-\frac{1}{2N^2}+\frac{1}{6N^3}+o\left(\frac{1}{N^4}\right)\right) \\
&\qquad -\frac{N(N+1)(2N+1)}{6}-\frac{N(N+1)}{4}-\frac{N}{6} \\
&= \frac{N(N+1)(2N+1)}{24}+\frac{(N+1)^2}{4}-\frac{1}{4}+\frac{N(N+1)^2}{4}-\frac{(N+1)^2}{8} \\
&\qquad +\frac{(N+1)^2}{24N}+o(1)-\frac{N(N+1)(2N+1)}{6}-\frac{N(N+1)}{4}-\frac{N}{6} \\
&= -\frac{1}{24}+\frac{1}{24N}+o(1)=-\frac{1}{24}+o(1).
\end{aligned}$$

It follows that

$$\sum_{n=1}^{\infty}\left(n\left(n\left(n\sum_{k=n}^{\infty}\frac{1}{k^2}-1\right)-\frac{1}{2}\right)-\frac{1}{6}\right)=-\frac{1}{24}.$$

We also received a solution from Nicuşor Minculete, from Braşov, and Daniel Văcaru, from Piteşti, Romania, which uses the Abel summation, too.