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Pearls of quadratic series

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Abstract. In this paper we calculate three remarkable quadratic series, as well as several linear series involving the tail of $\zeta(2)$.

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The paper is dedicated to the 125th appearance of Gazeta Matematică.
La mulţi ani, Gazeta Matematică!!!

1. Introduction and the main results

In this paper we prove the following three remarkable quadratic series formulae

$$\sum_{n=1}^{\infty}(-1)^n\left[n^2\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-1\right]=\frac{1}{2}+\frac{\zeta(2)}{2}+\frac{5}{8}\zeta(3)-\frac{\pi^2\ln 2}{4},$$

$$\sum_{n=1}^{\infty}\left[n\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-\frac{1}{n}\right]=\frac{3}{2}-\frac{\zeta(2)}{2}-\frac{3}{2}\zeta(3),$$

and

$$\sum_{n=1}^{\infty}\left[n^2\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-1+\frac{1}{n}\right]=-\frac{2}{3}+\frac{\zeta(2)}{2}+\frac{\zeta(3)}{2}.$$

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These formulae are new in the literature. Their proofs are based on the calculation of several linear series involving the tail of $\zeta(2)$. Other exercises about the calculation of series, linear or quadratic, involving the tail of Riemann zeta function values, as well as open problems can be found in [3] and [4]. The main results of this paper are recorded in Theorem 1 and Lemma 4.

Theorem 1. Pearls of quadratic series.

The following identities hold:

$$\text{(a)} \sum_{n=1}^{\infty}(-1)^n\left[n^2\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-1\right] = \frac{1}{2}+\frac{\zeta(2)}{2}+\frac{5}{8}\zeta(3)-\frac{\pi^2\ln 2}{4};$$

$$\text{(b)} \sum_{n=1}^{\infty}\left[n\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-\frac{1}{n}\right]=\frac{3}{2}-\frac{\zeta(2)}{2}-\frac{3}{2}\zeta(3);$$

$$\text{(c)} \sum_{n=1}^{\infty}\left[n^2\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-1+\frac{1}{n}\right]=-\frac{2}{3}+\frac{\zeta(2)}{2}+\frac{\zeta(3)}{2}.$$

The convergence of series in Theorem 1 is based on the behavior of the sequence

$$r_n:=\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}, \qquad n\geq 1.$$

Using Cesàro-Stolz lemma, the $\frac{0}{0}$ case, one can prove that

$$\lim_{n\to\infty} n\left[n\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)-1\right]=-\frac{1}{2}.$$

This implies that, for large values of n, we have $\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\sim \frac{1}{n}-\frac{1}{2n^2}$. Thus,

$$n^2\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-1\sim-\frac{1}{n}+\frac{1}{4n^2},$$

which shows that the series in part (a) behaves like the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n}$, hence it is semiconvergent.

On the other hand,

$$n\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-\frac{1}{n}\sim-\frac{1}{n^2}+\frac{1}{4n^3},$$

which shows the series in part (b) converges.

For the general term of the series in part(c) we have

$$n^2\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-1+\frac{1}{n}\sim\frac{1}{4n^2},$$

which implies that the series converges. The series in part (c) of Theorem 1 should be viewed as the *absolute convergence version* of the series in part (a).

Before we prove Theorem 1 we collect some results we need in our analysis. Recall that, *Abel's summation formula* ([2, p. 55], [3, Lemma A.1, p. 258]) states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences of real numbers and $A_n = \sum_{k=1}^{n} a_k$, then $\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k(b_k - b_{k+1})$. We will be using the *infinite version* of this formula

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} (A_n b_{n+1}) + \sum_{k=1}^{\infty} A_k(b_k - b_{k+1}). \tag{1}$$

The Dilogarithm function $\mathrm{Li}_2(z)$ is defined, for $|z| \leq 1$, by ([5, p. 176])

$$\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\ln(1-t)}{t}\mathrm{d}t.$$

Lemma 2. The generating function of the sequence r_n.

The following equality holds

$$\sum_{n=1}^{\infty}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)x^n = \frac{\zeta(2)x - \mathrm{Li}_2(x)}{1-x}, \quad x \in [-1, 1).$$

Proof. We apply formula (1) with $a_n = x^n$ and $b_n = \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}$ and we have, since $b_n - b_{n+1} = \frac{1}{(n+1)^2}$ and $A_n = x + x^2 + \cdots + x^n = \frac{x - x^{n+1}}{1-x}$, that

$$\begin{aligned}
\sum_{n=1}^{\infty} r_n x^n &= \lim_{n\to\infty} \frac{x - x^{n+1}}{1-x}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2}\right) \\
&\quad + \sum_{n=1}^{\infty} \frac{x - x^{n+1}}{1-x} \cdot \frac{1}{(n+1)^2} \\
&= \frac{1}{1-x}\sum_{n=1}^{\infty} \frac{x - x^{n+1}}{(n+1)^2} \\
&= \frac{1}{1-x}\sum_{n=0}^{\infty} \frac{x - x^{n+1}}{(n+1)^2} \\
&= \frac{\zeta(2)x - \mathrm{Li}_2(x)}{1-x}.
\end{aligned}$$

We used that

$$\lim_{n\to\infty} \left(x - x^{n+1}\right)\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2}\right) = 0.$$

The previous limit follows since $|x - x^{n+1}| \le 2$ and

$$\lim_{n\to\infty} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2} \right) = 0.$$

$\square$

Remark 3. We mention that the generating function of the sequence

$$\left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{n^k} \right)_{n\ge 1}, \quad k > 2,$$

can be obtained similarly as in the proof of Lemma 2 (see [4, problem 3.3, C, p. 76]).

Lemma 4. Gems involving the tail of $\zeta(2)$.

The following equalities hold:

(a) $$\sum_{n=1}^{\infty} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} - \frac{1}{n} \right) = 1 - \zeta(2);$$

(b) $$\sum_{n=1}^{\infty} \frac{1}{n} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right) = \zeta(3);$$

(c) $$\sum_{n=1}^{\infty} (-1)^n \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right) = -\frac{\zeta(2)}{4};$$

(d) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right) = \zeta(3) - \frac{\pi^2}{4} \ln 2;$$

(e) $$\sum_{n=1}^{\infty} \left[n \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right) - 1 + \frac{1}{2n} \right] = \frac{1}{4}.$$

Proof. (a) Use Abel's summation formula with $a_n = 1$ and $b_n = \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} - \frac{1}{n}$, see [4, problem 2.46, p. 50].

(b) Use Abel's summation formula with $a_n = \frac{1}{n}$ and $b_n = \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}$, see [3, problem 3.20, p. 142].

(c) We give four distinct proofs of part (c) of Lemma 4.

The first proof. One proof follows directly from Lemma 2, with $x = -1$, since $\mathrm{Li}_2(-1) = -\frac{\zeta(2)}{2}$.

The second proof. The second proof, which is left as an exercise to the interested reader, is based on an application of formula (1) with $a_n = (-1)^n$ and $b_n = \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}$.

The third proof (Shifting the index of summation).
We have

$$\begin{aligned}
S =& \sum_{n=1}^{\infty}(-1)^n \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)\\
=& -(\zeta(2)-1) + \sum_{n=2}^{\infty}(-1)^n \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)\\
=& 1-\zeta(2) + \sum_{m=1}^{\infty}(-1)^{m+1} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(m+1)^2}\right)\\
=& 1-\zeta(2) - \sum_{m=1}^{\infty}(-1)^{m} \left[\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{m^2}\right) - \frac{1}{(m+1)^2}\right]\\
=& 1-\zeta(2) - S + \sum_{m=1}^{\infty}\frac{(-1)^m}{(m+1)^2}\\
=& -\zeta(2) - S + \sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)^2}\\
=& -\frac{\zeta(2)}{2} - S,
\end{aligned}$$

and the result follows. We used that $\sum\limits_{m=0}^{\infty}\frac{(-1)^m}{(m+1)^2} = \frac{\zeta(2)}{2}$.

The fourth proof. This proof is based on a direct computational technique. One can check using integration by parts that $\int_0^1 x^k \ln x \, \mathrm{d}x = -\frac{1}{(k+1)^2}$, $k \geq 0$. It follows that

$$\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} = -\int_0^1 \frac{x^n}{1-x}\ln x \, \mathrm{d}x. \tag{2}$$

We have, based on formula (2), that

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^n \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) &= -\sum_{n=1}^{\infty}(-1)^n \int_0^1 \frac{x^n \ln x}{1-x}\mathrm{d}x\\
&= -\int_0^1 \frac{\ln x}{1-x}\left(\sum_{n=1}^{\infty}(-x)^n\right)\mathrm{d}x\\
&= \int_0^1 \frac{x\ln x}{1-x^2}\mathrm{d}x\\
&\overset{x^2=t}{=} \frac{1}{4}\int_0^1 \frac{\ln t}{1-t}\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\int_0^1 \ln t\left(\sum_{n=0}^{\infty} t^n\right) \mathrm{d}t \\
&= \frac{1}{4}\sum_{n=0}^{\infty}\int_0^1 t^n \ln t\, \mathrm{d}t \\
&= -\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{(n+1)^2} \\
&= -\frac{\zeta(2)}{4}.
\end{aligned}
$$

(d) We use that $\frac{1}{k} = \int_0^1 x^{k-1}\mathrm{d}x$ and Lemma 2 and we have that

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) \\
&\quad = \sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{n-1}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)\mathrm{d}x \\
&\quad = \int_0^1 \frac{1}{x}\left[\sum_{n=1}^{\infty}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)(-x)^n\right]\mathrm{d}x \\
&\quad \overset{\text{Lemma 2}}{=} -\int_0^1 \frac{x\zeta(2) + \mathrm{Li}_2(-x)}{x(1+x)}\mathrm{d}x \\
&\quad = \int_0^1 \frac{x\zeta(2) + \mathrm{Li}_2(-x)}{1+x}\mathrm{d}x - \zeta(2) - \int_0^1 \frac{\mathrm{Li}_2(-x)}{x}\mathrm{d}x.
\end{aligned} \tag{3}
$$

On the other hand

$$
\int_0^1 \frac{\mathrm{Li}_2(-x)}{x}\mathrm{d}x = \int_0^1\left(\sum_{n=1}^{\infty}(-1)^n\frac{x^{n-1}}{n^2}\right)\mathrm{d}x = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} = -\frac{3}{4}\zeta(3). \tag{4}
$$

We calculate the first integral in (3) by parts, with $f(x) = x\zeta(2) + \mathrm{Li}_2(-x)$, $f'(x) = \zeta(2) - \frac{\ln(1+x)}{x}$, $g'(x) = \frac{1}{1+x}$, and $g(x) = \ln(1+x)$, and we have that

$$
\begin{aligned}
\int_0^1 \frac{x\zeta(2)+\mathrm{Li}_2(-x)}{1+x}\mathrm{d}x &= [x\zeta(2)+\mathrm{Li}_2(-x)]\ln(1+x)\Big|_0^1\\
&\quad - \int_0^1 \left(\zeta(2)-\frac{\ln(1+x)}{x}\right)\ln(1+x)\mathrm{d}x\\
&= [\zeta(2)+\mathrm{Li}_2(-1)]\ln 2 - \zeta(2)(2\ln 2-1)\\
&\quad + \int_0^1 \frac{\ln^2(1+x)}{x}\mathrm{d}x\\
&= -\frac{3}{2}\zeta(2)\ln 2+\zeta(2)+\frac{\zeta(3)}{4}. \qquad (5)
\end{aligned}
$$

We used that $\mathrm{Li}_2(-1)=-\frac{\zeta(2)}{2}$ and Ramanujan's integral $\int_0^1 \frac{\ln^2(1+x)}{x}\mathrm{d}x=\frac{\zeta(3)}{4}$ (see [1, pp. 291–292]). Combining (3), (4) and (5) we have that part (d) of Lemma 4 is proved.

(e) We apply formula (1), with $a_n=n$ and $b_n=\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}-\frac{1}{n}+\frac{1}{2n^2}$, and we have, since $b_n-b_{n+1}=\frac{1}{2n^2(n+1)^2}$, that

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left[n\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)-1+\frac{1}{2n}\right]\\
&\quad=\sum_{n=1}^{\infty} n\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}-\frac{1}{n}+\frac{1}{2n^2}\right)\\
&\quad=\lim_{n\to\infty}\frac{n(n+1)}{2}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}-\frac{1}{n+1}+\frac{1}{2(n+1)^2}\right)\\
&\qquad+\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\\
&\quad=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\\
&\quad=\frac{1}{4},
\end{aligned}
$$

since

$$
\lim_{n\to\infty} n(n+1)\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}-\frac{1}{n+1}+\frac{1}{2(n+1)^2}\right)=0.
$$

The lemma is proved. □

Remark 5. Using the generating function of the sequence

$$
\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}\right)_{n\geq 1},
$$

one can calculate similarly as in the proof of part (d) of Lemma 4 the following alternating series involving the tail of $\zeta(3)$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}\right)=-\frac{7}{4}\zeta(3)\ln 2+\frac{19}{16}\zeta(4).$$

Now we are ready to prove Theorem 1.

Proof. (a) We need the following equality which can be proved by mathematical induction

$$(-1)^1 1^2+(-1)^2 2^2+\cdots+(-1)^n n^2=(-1)^n\frac{n(n+1)}{2},\quad n\geq 1. \tag{6}$$

Apply formula (1) with $a_n=(-1)^n n^2$ and $b_n=\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-\frac{1}{n^2}$. We have, based on formula (6), that $A_n=(-1)^n\frac{n(n+1)}{2}$ and a calculation shows

$$b_n-b_{n+1}=-\frac{2n+1}{n^2(n+1)^2}+\frac{1}{(n+1)^4}+\frac{2}{(n+1)^2}\cdot r_{n+1}.$$

We have

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^n\left[n^2\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-1\right]\\
&\quad=\sum_{n=1}^{\infty}(-1)^n n^2\left[\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-\frac{1}{n^2}\right]\\
&\quad=\lim_{n\to\infty}(-1)^n\frac{n(n+1)}{2}\left[\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}\right)^2-\frac{1}{(n+1)^2}\right]\\
&\qquad+\sum_{n=1}^{\infty}(-1)^n\frac{n(n+1)}{2}\left[-\frac{2n+1}{n^2(n+1)^2}+\frac{1}{(n+1)^4}+\frac{2}{(n+1)^2}\cdot r_{n+1}\right]\\
&\quad=-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{2n+1}{n(n+1)}+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{n}{(n+1)^3}\\
&\qquad+\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}\right).
\end{aligned}\tag{7}$$

We used that

$$\lim_{n\to\infty}n(n+1)\left[\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}\right)^2-\frac{1}{(n+1)^2}\right]=0. \tag{8}$$

The preceding limit follows since

$$\begin{aligned}
&\lim_{n\to\infty} n(n+1)\left[\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}\right)^2-\frac{1}{(n+1)^2}\right]\\
&\quad=\lim_{n\to\infty} n\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}-\frac{1}{n+1}\right)\\
&\qquad\cdot\lim_{n\to\infty}(n+1)\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}+\frac{1}{n+1}\right)\\
&\quad=0\cdot 2\\
&\quad=0,
\end{aligned}$$

since $\lim\limits_{n\to\infty} n\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)=1$.

An easy calculation shows that

$$\sum_{n=1}^{\infty}(-1)^n\frac{2n+1}{n(n+1)}=-1 \quad\text{and}\quad \sum_{n=1}^{\infty}(-1)^n\frac{n}{(n+1)^3}=\frac{\pi^2}{12}-\frac{3}{4}\zeta(3). \tag{9}$$

On the other hand,

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}\right)\\
&\quad=\sum_{n=1}^{\infty}(-1)^n\left(1-\frac{1}{n+1}\right)\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}\right)\\
&\quad=\sum_{m=2}^{\infty}(-1)^{m-1}\left(1-\frac{1}{m}\right)\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{m^2}\right)\\
&\quad=\sum_{m=1}^{\infty}(-1)^{m-1}\left(1-\frac{1}{m}\right)\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{m^2}\right) \qquad (10)\\
&\quad=\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{m^2}\right)\\
&\qquad-\sum_{m=1}^{\infty}(-1)^m\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{m^2}\right)\\
&\quad\overset{\text{Lemma 4, (c), (d)}}{=}\zeta(3)-\frac{\pi^2}{4}\ln 2+\frac{\zeta(2)}{4}.
\end{aligned}$$

Combining (7), (9) and (10), part (a) of Theorem 1 is proved.

(b) Exactly as in the proof of part (a) of the theorem, we apply formula (1) with $a_n=n$ and $b_n=\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-\frac{1}{n^2}$ and we have, since

$$b_n-b_{n+1}=-\frac{2n+1}{n^2(n+1)^2}+\frac{1}{(n+1)^4}+\frac{2}{(n+1)^2}\cdot r_{n+1},$$

that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left[n\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-\frac{1}{n}\right]\\
&\quad=\sum_{n=1}^{\infty} n\left[\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-\frac{1}{n^2}\right]\\
&\quad=\lim_{n\to\infty}\frac{n(n+1)}{2}\left[\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}\right)^2-\frac{1}{(n+1)^2}\right]\\
&\qquad+\sum_{n=1}^{\infty}\frac{n(n+1)}{2}\left[-\frac{2n+1}{n^2(n+1)^2}+\frac{1}{(n+1)^4}+\frac{2}{(n+1)^2}\cdot r_{n+1}\right]\\
&\quad\overset{(8)}{=}\sum_{n=1}^{\infty}\frac{n(n+1)}{2}\left[-\frac{2n+1}{n^2(n+1)^2}+\frac{1}{(n+1)^4}+\frac{2}{(n+1)^2}\cdot r_{n+1}\right]\\
&\quad=\sum_{n=1}^{\infty}\left[-\frac{2n+1}{2n(n+1)}+\frac{n}{2(n+1)^3}+\frac{n}{n+1}\cdot r_{n+1}\right]\\
&\quad=-\sum_{n=1}^{\infty}\frac{1}{2n(n+1)}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{n}{(n+1)^3}+\sum_{n=1}^{\infty}\left(r_{n+1}-\frac{1}{n+1}\right)\\
&\qquad-\sum_{n=1}^{\infty}\frac{1}{n+1}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}\right).
\end{aligned}$$

It follows, based on parts (a) and (b) of Lemma 4, that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left[n\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-\frac{1}{n}\right]\\
&=-\frac{1}{2}+\frac{1}{2}(\zeta(2)-\zeta(3))+[1-\zeta(2)-(\zeta(2)-2)]-[\zeta(3)-(\zeta(2)-1)]\\
&=\frac{3}{2}-\frac{\zeta(2)}{2}-\frac{3}{2}\zeta(3).
\end{aligned}$$

(c) Let S be the sum of the series. We apply formula (1), with

$$a_n=1 \quad\text{and}\quad b_n=n^2\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2-1+\frac{1}{n}$$

and we have, since

$$b_n-b_{n+1}=-(2n+1)r_n^2+2r_n+\frac{1}{n(n+1)^2},$$

that

$$
\begin{aligned}
S &= \sum_{n=1}^{\infty} \left[n^2 \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right)^2 - 1 + \frac{1}{n} \right] \\
&= \lim_{n\to\infty} n \left[(n+1)^2 r_{n+1}^2 - 1 + \frac{1}{n+1} \right] + \sum_{n=1}^{\infty} \left[-(2n^2+n) r_n^2 + 2n r_n \right] \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \\
&= \sum_{n=1}^{\infty} \left[-(2n^2+n) r_n^2 + 2n r_n \right] + \zeta(2) - 1,
\end{aligned} \tag{11}
$$

since $\lim\limits_{n\to\infty} n \left[(n+1)^2 \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2} \right)^2 - 1 + \frac{1}{n+1} \right] = 0.$

An easy calculation shows that

$$
\begin{aligned}
&-(2n^2+n) \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right)^2 + 2n \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right) \\
&= -2 \left[n^2 \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right)^2 - 1 + \frac{1}{n} \right] \\
&\quad - \left[n \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right)^2 - \frac{1}{n} \right] \\
&\quad + 2 \left[n \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right) - 1 + \frac{1}{2n} \right].
\end{aligned} \tag{12}
$$

It follows, from (11) and (12), that

$$
\begin{aligned}
S &= -2 \sum_{n=1}^{\infty} \left[n^2 \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right)^2 - 1 + \frac{1}{n} \right] \\
&\quad - \sum_{n=1}^{\infty} \left[n \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right)^2 - \frac{1}{n} \right] \\
&\quad + 2 \sum_{n=1}^{\infty} \left[n \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right) - 1 + \frac{1}{2n} \right] + \zeta(2) - 1 \\
&= -2S - \sum_{n=1}^{\infty} \left[n \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right)^2 - \frac{1}{n} \right] \\
&\quad + 2 \sum_{n=1}^{\infty} \left[n \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right) - 1 + \frac{1}{2n} \right] + \zeta(2) - 1.
\end{aligned}
$$

Using part (b) of Theorem 1 and part (e) of Lemma 4 we have that

$$3S = -\left(\frac{3}{2} - \frac{\zeta(2)}{2} - \frac{3}{2}\zeta(3)\right) + \frac{1}{2} + \zeta(2) - 1 = -2 + \frac{3}{2}\zeta(2) + \frac{3}{2}\zeta(3),$$

and this implies that $S = -\frac{2}{3} + \frac{\zeta(2)}{2} + \frac{\zeta(3)}{2}$. Part (c) of Theorem 1 is thus proved. □

Corollary 6. A quadratic series and Apéry's constant.

The following equality holds

$$\sum_{n=1}^{\infty}\left[(n^2+n)\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2 - 1\right] = \frac{5}{6} - \zeta(3).$$

Proof. Add the series formulae in parts (b) and (c) of Theorem 1. □

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The Hölder continuity of $x^a \sin \frac{1}{x}$

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Abstract. For $a \in (0,2]$, we prove that $x^a \sin \frac{1}{x}$ is a Hölder continuous function of order $\frac{a}{2}$.

Keywords: Hölder continuous function, Lipschitz function.

MSC: Primary 26A16; Secondary 26A09.

1. Introduction

Let I be a nonempty subset of $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be *Hölder continuous* of order $\alpha \in (0,1]$ if there exists $M \geq 0$ such that

$$|f(x) - f(y)| \leq M \cdot |x-y|^\alpha, \quad \text{for every } x, y \in I. \tag{1}$$

In the particular case $\alpha = 1$, the function f is called a *Lipschitz function.*

The number α from (1) is called *a Hölder exponent* and the number M satisfying condition (1) is called *a Hölder coefficient.* The smallest Hölder coefficient is denoted $|f|_\alpha$. Let us remark that for a Hölder continuous function f of order α, the Hölder coefficient

$$|f|_\alpha = \sup_{\substack{x,y \in I \\ x \neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

is finite.

This notion is important in many areas of mathematics like Approximation Theory, Fourier series, Differential Equations and Potential Theory. There are two names associated with this notion. Many mathematicians who worked in the field of Approximation Theory and Fourier series (see for example [8, p. 9] and [9, p. 42]), called a Hölder continuous function of order α a "Lipschitz continuous function of order α", after R. Lipschitz who used this notion in a paper [5] from 1864 concerning Fourier series. In this case, the space of all Lipschitz continuous functions of order α is denoted $\mathrm{Lip}_\alpha(I)$ (see [7, p. 159]) and simply $\mathrm{Lip}(I)$, for all Lipschitz functions on I. Nowadays, the functions satisfying (1) are named after O. Hölder, who used the condition (1) in his doctoral dissertation [3] presented to the University of Tübingen in 1882 entitled "Contributions to potential theory". The space of Hölder continuous functions of order α on I is denoted $C^{0,\alpha}(I)$ (see for example [2]). For an integer $k \geq 1$, the symbol $C^{k,\alpha}(I)$ denotes the space of functions whose first kth derivatives are Hölder continuous of order α.

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The aim of this article is to study the Hölder continuity of the function $s_a : [0,\infty) \to \mathbb{R}$ defined by

$$s_a(x) = \begin{cases} x^a \cdot \sin \dfrac{1}{x}, & x > 0, \\ 0, & x = 0, \end{cases}$$

where $a \in (0,2]$. This function is a rich source of examples and counterexamples of many results in analysis (see [4]). We will add to the many interesting properties of this function its Hölder continuity. We will prove that for $a \in (0,2]$ the function s_a is Hölder continuous of order $\frac{a}{2}$ on $[0,\infty)$. For $a = 2$ the function s_2 is a Lipschitz function. Indeed, using the Mean Value Theorem and inequality $\sin t \le t$ $(t \ge 0)$ we have for every $x, y \ge 0$

$$|s_2(x) - s_2(y)| = |x-y| \cdot |s_2'(c)| \le \left| 2c \sin \frac{1}{c} - \cos \frac{1}{c} \right| \cdot |x-y| \le 3 \cdot |x-y|.$$

The particular case $a = 1$ is also known (see [1] and also [6] and Example 1.1.8 from [2]). In [6] it is proved that

$$|s_1(x) - s_1(y)| \le \sqrt{2|x-y|}, \quad x, y \ge 0.$$

Before presenting the proof of our result, let us remark that s_a cannot be Hölder continuous with a greater exponent than $\frac{a}{2}$. Suppose s_a is Hölder continuous with the exponent $b > \frac{a}{2}$. Let $x_n = \frac{1}{n\pi}$ and $y_n = \frac{1}{n\pi + \frac{\pi}{2}}$. Then

$$\frac{|s_a(x_n) - s_a(y_n)|}{|x_n - y_n|^b} = \frac{\frac{1}{\left(n\pi + \frac{\pi}{2}\right)^a}}{\left(\frac{1}{n\pi} - \frac{1}{n\pi + \frac{\pi}{2}}\right)^b} = 2^b \pi^{b-a} \left(1 + \frac{1}{2n}\right)^{b-a} \cdot n^{2b-a}$$

is unbounded, showing that the Hölder coefficient $|s_a|_b$ cannot be finite.

The main result of this paper is Theorem 4, where we show that the Hölder coefficient of the function s_a, $a \in (0,2)$, is bounded by the constant $(a+1)(3\pi/2)^{(2-a)/2}$. To prove this, we first find the monotonicity intervals of the function s_a. The properties of s_a related to these intervals are included in the section Auxiliary results.

2. Auxiliary results

Lemma 1. *For $a \in (0,2]$, the equation $s_a'\left(\frac{1}{x}\right) = 0$ has a unique solution in the interval $(n\pi, n\pi + \pi)$, for every integer $n \ge 1$, which will be denoted by a_n. In addition, $a \tan a_n = a_n$ and a_n is located in the interval $\left(n\pi, n\pi + \frac{\pi}{2}\right)$.*

Proof. For the integers $n \ge 1$, the real numbers $\frac{1}{n\pi}$ are the only positive roots of the equation $s_a(x) = 0$. By Rolle's Theorem, the derivative $s_a'(x)$ has at least one zero in the interval $\left(\frac{1}{n\pi}, \frac{1}{n\pi + \pi}\right)$, $n \ge 1$. We will prove that it is unique. The expression of the derivative of s_a is

$$s_a'(x) = x^{a-1}\left(a\sin\frac{1}{x} - \frac{1}{x}\cdot\cos\frac{1}{x}\right).$$

The equation $s_a'(1/x) = 0$ has only one solution in $(n\pi, n\pi+\pi)$ if and only if the equation

$$a\sin t - t\cos t = 0$$

has only one solution in the interval $(n\pi, n\pi+\pi)$. With $t = u + n\pi$, this is true if and only if the function defined on $(0,\pi)$ by

$$f(u) = a\sin u - (u+n\pi)\cos u$$

has only one root. For $u \in \left(\frac{\pi}{2}, \pi\right)$, the function f is positive, so it cannot be zero in this subinterval. For $u \in \left(0, \frac{\pi}{2}\right)$, f is convex, because $f''(u) = (2-a)\sin u + (u+n\pi)\cos u > 0$. The derivative is

$$f'(u) = (a-1)\cos u + (u+n\pi)\sin u.$$

We have two cases depending on the value of a. If $a \geq 1$, then $f'(u) > 0$ and because at the endpoints of the interval the function f has opposite signs ($f(0) = -n\pi < 0$ and $f\left(\frac{\pi}{2}\right) = a > 0$), there is a unique $u_0 \in \left(0, \frac{\pi}{2}\right)$ such that $f(u_0) = 0$. If $a \in (0,1)$, then $f'(0) = a - 1 < 0$ and because $f'\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + n\pi > 0$, we deduce from the convexity of f that $f'(u)$ is negative on some interval $(0, u_1)$ and positive on $\left(u_1, \frac{\pi}{2}\right)$, for some $u_1 \in \left(0, \frac{\pi}{2}\right)$. This means that f decreases on $(0, u_1)$ and because $f(0) = -n\pi$ we know that f is negative on $(0, u_1)$. In the interval $\left(u_1, \frac{\pi}{2}\right)$, the function f increases from negative values to positive values (because $f(u_1) < 0$ and $f\left(\frac{\pi}{2}\right) = a > 0$) and, since it is continuous, it has exactly one zero in the interval $\left(u_1, \frac{\pi}{2}\right)$ denoted by u_0.

We have proved that $f(u) = 0$ has only one solution in the interval $\left(0, \frac{\pi}{2}\right)$, denoted by u_0. With $a_n = u_0 + n\pi \in \left(n\pi, n\pi + \frac{\pi}{2}\right)$ we have proved that a_n is the only solution of the equation $s_a'(1/x) = 0$ in the interval $(n\pi, n\pi+\pi)$ and that it verifies $a\sin a_n - a_n\cos a_n = 0$ which is equivalent with $a\tan a_n = a_n$. □

The solutions of the equation $s_a'\left(\frac{1}{x}\right) = 0$ form a strictly increasing sequence of positive numbers denoted (a_n) (see Lemma 1). We consider the intervals $I_n = \left[\frac{1}{a_{n+1}}, \frac{1}{a_n}\right]$, for $n \geq 1$, and $I_0 = \left[\frac{1}{a_1}, \infty\right)$.

Lemma 2. *We have*

$$(0,\infty) = \bigcup_{n=0}^{\infty} I_n, \quad \text{and} \quad s_a(I_0) \supset s_a(I_1) \supset s_a(I_2) \supset \cdots.$$

Proof. To prove the inclusions $s_a(I_{n+1}) \subset s_a(I_n)$, for $n \geq 1$, let us evaluate the extrema of s_a. Consider $b_n = n\pi + \frac{\pi}{2} - a_n \in \left(0, \frac{\pi}{2}\right)$. Using the relation

$a \tan a_n = a_n$ we get

$$\tan b_n = \tan\left(\frac{\pi}{2} - a_n\right) = \frac{1}{\tan a_n} = \frac{a}{a_n}.$$

Now,

$$s_a\left(\frac{1}{a_n}\right) = \frac{1}{a_n^a}\sin a_n = \frac{1}{a_n^a}\sin\left(n\pi + \frac{\pi}{2} - b_n\right) = \frac{(-1)^n \cos b_n}{a_n^a}.$$

Using the formula $\cos x = \frac{1}{\sqrt{1+\tan^2 x}}$, for $x = b_n$, we obtain

$$s_a\left(\frac{1}{a_n}\right) = \frac{(-1)^n}{a_n^a\sqrt{1+\frac{a^2}{a_n^2}}} = \frac{(-1)^n}{\sqrt{a_n^{2a} + a^2 a_n^{2a-2}}}.$$

The function s_a is monotonic on each interval I_n and changes the monotonicity on consecutive intervals. It remains to prove that $|s_a(1/a_{n+2})| < |s_a(1/a_n)|$, for every $n \geq 1$. This holds true if the sequence $\left(a_n^{2a} + a^2 a_n^{2a-2}\right)_{n\geq 1}$ is increasing. This is indeed true because the derivative of the function $g(t) = t^{2a} + a^2 t^{2a-2}$ is positive on $(a_1, \infty) \subset (\pi, \infty)$, since

$$g'(t) = 2at^{2a-3}\left[t^2 + a(a-1)\right] > 0, \quad t > \pi.$$

Now, to prove that $s_a(I_1) \subset s_a(I_0)$ it is sufficient to prove that $s_a\left(\frac{1}{a_2}\right) < s_a\left(\frac{2}{\pi}\right)$. But this is equivalent to $g(a_2) > \left(\frac{\pi}{2}\right)^{2a}$, which is true since g is increasing and $a_2 > 2\pi$. □

Lemma 3. *Let $0 \leq x < y$. Then, there exist $\ell \geq 0$ and v, w such that $v, w \in I_\ell$ and $x \leq v \leq w \leq y$ with the property that*

$$s_a(v) = s_a(x) \quad \text{and} \quad s_a(w) = s_a(y).$$

Proof. Using Lemma 2, we know that for $y \in (0, \infty)$ there is some integer $j \geq 0$ such that $y \in I_j$. If $0 < x < y$, then there is $k \geq j$ such that $x \in I_k$. Because $s_a(I_k) \subset s_a(I_j)$, there is some $z \in I_j$ such that $s_a(z) = s_a(x)$. Let us remark that if $j = 0$ then $z < \frac{2}{\pi}$. If $x = 0$, we can choose $z = \frac{1}{(j+1)\pi} \in I_j$ which has the property that $s_a(z) = s_a(x) = 0$.

If $z \leq y$, we choose $v = z$, $w = y$ and $\ell = j$ and the Lemma is proved.

If $z > y$, we have two cases. The function s_a is either increasing on I_j or decreases on I_j (if $j = 0$ we can take instead of I_0 the subinterval $\left[\frac{1}{a_1}, \frac{2}{\pi}\right] \subset I_0$ where z and y are located and where the function s_a is increasing). Suppose first that s_a is increasing on I_j. We have $s_a(z) > s_a(y)$. Because $s_a(I_{j+1}) \subset s_a(I_j)$, there are $v, w \in I_{j+1}$ such that $s_a(v) = s_a(z)$ and $s_a(w) = s_a(y)$. We deduce that $s_a(v) > s_a(w)$. On the interval I_{j+1} the function s_a is decreasing. So we must have $v < w$. We choose $\ell = j + 1$ and the Lemma is proved. If s_a is decreasing on I_j, then $s_a(z) < s_a(y)$. Because $s_a(I_{j+1}) \subset s_a(I_j)$, there are $v, w \in I_{j+1}$ such that $s_a(v) = s_a(z)$ and $s_a(w) = s_a(y)$. On I_{j+1} the function

s_a is increasing and so again we must have $v < w$. Choosing $\ell = j + 1$ and the Lemma is proved. □

3. MAIN RESULT

Theorem 4. *Let $a \in (0, 2)$. The inequality*

$$|s_a(x) - s_a(y)| \leq (a+1) \left(\frac{3\pi}{2}\right)^{\frac{2-a}{2}} \cdot |x-y|^{\frac{a}{2}} \tag{2}$$

holds true for every $x, y \geq 0$.

Proof. If $x = y$, then (2) is true. Let $y > x \geq 0$. According to Lemma 3, there exist $\ell \geq 0$ and v, w such that $v, w \in I_\ell$ and $x \leq v \leq w \leq y$ with the property that

$$s_a(v) = s_a(x) \quad \text{and} \quad s_a(w) = s_a(y).$$

If we prove that (2) is true for all $v, w \in I_\ell$ then

$$\begin{aligned} |s_a(y) - s_a(x)| = |s_a(w) - s_a(v)| &\leq (a+1) \left(\frac{3\pi}{2}\right)^{\frac{2-a}{2}} \cdot |w-v|^{\frac{a}{2}} \\ &\leq (a+1) \left(\frac{3\pi}{2}\right)^{\frac{2-a}{2}} \cdot |y-x|^{\frac{a}{2}} \end{aligned}$$

and we are done.

Let $w, v \in I_\ell$. We can write

$$|s_a(w) - s_a(v)| = \left| \int_v^w s_a'(t)\, \mathrm{d}t \right| \leq \int_v^w 1 \cdot \left| s_a'(t) \right| \, \mathrm{d}t.$$

We use Hölder inequality

$$\int_v^w |f(x)| \cdot |g(x)| \, \mathrm{d}x \leq \left(\int_v^w |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \cdot \left(\int_v^w |g(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}},$$

which is true for two integrable functions f, g on the interval $[v, w]$ and two real numbers $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We apply this inequality for $p = 2/a > 1$ and $q = 2/(2-a) > 1$. We have

$$|s_a(w) - s_a(v)| \leq \left(\int_v^w \mathrm{d}t \right)^{\frac{a}{2}} \cdot \left(\int_v^w \left| s_a'(t) \right|^{\frac{2}{2-a}} \, \mathrm{d}t \right)^{\frac{2-a}{2}}.$$

With the variable change $u = \frac{1}{t}$ this is equivalent with

$$|s_a(w) - s_a(v)| \leq |w-v|^{\frac{a}{2}} \cdot \left(\int_{\frac{1}{w}}^{\frac{1}{v}} \frac{\left| s_a'\left(\frac{1}{u}\right) \right|^{\frac{2}{2-a}}}{u^2} \, \mathrm{d}u \right)^{\frac{2-a}{2}}.$$

It remains to prove that

$$\left|\int_{\frac{1}{w}}^{\frac{1}{v}} \frac{\left|s_a'\left(\frac{1}{u}\right)\right|^{\frac{2}{2-a}}}{u^2}\,\mathrm{d}u\right|^{\frac{2-a}{2}} \leq (a+1)\left(\frac{3\pi}{2}\right)^{\frac{2-a}{2}}. \tag{3}$$

Now, if $\ell \geq 1$ and $v, w \in I_\ell$, then

$$\frac{1}{v} - \frac{1}{w} \leq a_{\ell+1} - a_\ell < \frac{3\pi}{2}.$$

If $v, w \in I_0$, then $\frac{1}{v}, \frac{1}{w} \in (0, a_1)$ and we get the same upper bound

$$\frac{1}{v} - \frac{1}{w} \leq a_1 < \frac{3\pi}{2}.$$

On the other hand, $s_a'\left(\frac{1}{u}\right) = u^{2-a}\left(a\frac{\sin u}{u} - \cos u\right)$. Using the inequality

$$\left|a\frac{\sin u}{u} - \cos u\right| \leq \left|a\frac{\sin u}{u}\right| + |\cos u| \leq a+1, \tag{4}$$

we deduce that

$$\begin{aligned}\left|\int_{\frac{1}{w}}^{\frac{1}{v}} \frac{\left|s_a'\left(\frac{1}{u}\right)\right|^{\frac{2}{2-a}}}{u^2}\,\mathrm{d}u\right|^{\frac{2-a}{2}} &\leq \left|\int_{\frac{1}{w}}^{\frac{1}{v}} \left|a\frac{\sin u}{u} - \cos u\right|^{\frac{2}{2-a}}\,\mathrm{d}u\right|^{\frac{2-a}{2}} \\ &\leq (a+1)\left|\frac{1}{v} - \frac{1}{w}\right|^{\frac{2-a}{2}} \leq (a+1)\left(\frac{3\pi}{2}\right)^{\frac{2-a}{2}},\end{aligned}$$

which proves (3). □

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An implicitly defined sequence

Dumitru Popa[1)]

Abstract. We prove that if $A \subset \mathbb{R}^k$ is a compact, connected, Jordan measurable set with $\lambda_k(A) > 0$, $g : A \to [0, \infty)$ a continuous function with the property that there exists $x \in \operatorname{int}(A)$ such that $g(x) > 0$, $M \geq \int_A g(x)\,\mathrm{d}x$ and $(a_n)_{n\in\mathbb{N}} \subset (0,\infty)$ is such that $\lim\limits_{n\to\infty} a_n = \infty$, then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the equation $a_n \int_A \ln\left(1 + \frac{g(x)+u}{a_n}\right)\mathrm{d}x = M$ has a unique solution in the interval $[0,\infty)$, denoted by u_n, and moreover, $\lim\limits_{n\to\infty} u_n = \frac{M-\int_A g(x)\mathrm{d}x}{\lambda_k(A)}$, $\lim\limits_{n\to\infty} a_n^2(u_n - s) = \frac{\int_A [g(x)]^2\mathrm{d}x}{2\lambda_k(A)} + \frac{s\int_A g(x)\mathrm{d}x}{\lambda_k(A)} + \frac{s^2}{2}$, where $s = \frac{M-\int_A g(x)\mathrm{d}x}{\lambda_k(A)}$. Some applications are given.

Keywords: Multiple Riemann integral, limit of sequences of integrals, implicitly defined sequence, asymptotic evaluation.

MSC: Primary 26B15, 40A05; Secondary 28A35.

1. Introduction

The main purpose of this paper is to prove the result stated in the Abstract, see Theorem 7. We will use the multiple Riemann integral. For details regarding the multiple Riemann integral we recommend the reader the excellent treatment of this concept in [1]. If $a \in \mathbb{R}^k$ and $\delta > 0$, $\overline{B}(a,\delta) := \left\{x \in \mathbb{R}^k \mid d(x,a) \leq \delta\right\}$, where $d(x,a) = \sqrt{(x_1-a_1)^2 + \cdots + (x_k-a_k)^2}$ is the Euclidean distance; $x = (x_1, ..., x_k)$, $a = (a_1, ..., a_k)$. For a set $A \subset \mathbb{R}^k$ we write $\operatorname{int}(A)$ to denote the interior of A; if A is Jordan measurable, $\lambda_k(A)$ denote its Jordan measure, see [1]; we recall that $\lambda_k([a_1,b_1] \times \cdots \times [a_k,b_k]) = (b_1-a_1)\cdots(b_k-a_k)$, for $a_i \leq b_i$, $i = 1,\ldots,k$. The notation and notions used in this paper are standard.

2. Preliminary results

For the sake of the completeness we include the proofs of some results regarding the multiple Riemann integral needed in the paper.

Proposition 1. *Let $A \subset \mathbb{R}^k$ be a Jordan measurable set and $f : A \to [0,\infty)$ a continuous function. If $\int_A f(x)\,\mathrm{d}x = 0$ then $f(x) = 0$, $\forall x \in \operatorname{int}(A)$.*

Proof. Let us suppose that there exists $a \in \operatorname{int}(A)$ such that $f(a) \neq 0$, hence $f(a) > 0$. Since f is continuous, for $\varepsilon = \frac{f(a)}{2} > 0$ there exists $\delta_1 > 0$ such that $\forall x \in A \cap \overline{B}(a,\delta_1)$ we have $|f(x) - f(a)| \leq \frac{f(a)}{2}$, $-\frac{f(a)}{2} \leq f(x) - f(a) \leq \frac{f(a)}{2}$ and thus $f(x) \geq \frac{f(a)}{2}$. From the definition of the interior there exists $\delta_2 > 0$

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such that $\overline{B}(a,\delta_2) \subset A$ $(a \in \operatorname{int}(A))$. Then for $\delta = \min(\delta_1,\delta_2) > 0$ we have $\overline{B}(a,\delta) \subset \overline{B}(a,\delta_1) \cap \overline{B}(a,\delta_2) \subset A$ and hence

$$f(x) \geq \frac{f(a)}{2},\ \forall x \in \overline{B}(a,\delta). \tag{1}$$

Since f takes its values in $[0,\infty)$,

$$\int_{\overline{B}(a,\delta)} f(x)\,\mathrm{d}x \leq \int_A f(x)\,\mathrm{d}x = 0.$$

From (1) by integration we have

$$\frac{f(a)}{2}\lambda_k\left(\overline{B}(a,\delta)\right) = \int_{\overline{B}(a,\delta)} \frac{f(a)}{2}\mathrm{d}x \leq \int_{\overline{B}(a,\delta)} f(x)\,\mathrm{d}x.$$

We get $\frac{f(a)}{2}\lambda_k\left(\overline{B}(a,\delta)\right) \leq 0$, $\lambda_k\left(\overline{B}(a,\delta)\right) = 0$. If $a = (a_1,\ldots,a_k)$, from the inclusion $\left[a_1 - \frac{\delta}{\sqrt{k}}, a_1 + \frac{\delta}{\sqrt{k}}\right] \times \cdots \times \left[a_k - \frac{\delta}{\sqrt{k}}, a_k + \frac{\delta}{\sqrt{k}}\right] \subset \overline{B}(a,\delta)$ we deduce that

$$\lambda_k\left(\left[a_1 - \frac{\delta}{\sqrt{k}}, a_1 + \frac{\delta}{\sqrt{k}}\right] \times \cdots \times \left[a_k - \frac{\delta}{\sqrt{k}}, a_k + \frac{\delta}{\sqrt{k}}\right]\right) \leq \lambda_k\left(\overline{B}(a,\delta)\right) = 0$$

that is $\left(\frac{2\delta}{\sqrt{k}}\right)^k \leq 0$, which is false. □

Proposition 2. *Let $A \subset \mathbb{R}^k$ be a connected set, $f : A \to \mathbb{R}$ a continuous function, a, $b \in A$ and $f(a) \leq c \leq f(b)$. Then there exists $\xi \in A$ such that $c = f(\xi)$.*

Proof. Since A is connected and f is continuous, by a well-known result $f(A) \subset \mathbb{R}$ is connected, and thus $f(A)$ is an interval. Since $f(a), f(b) \in f(A)$, it follows that $[f(a), f(b)] \subset f(A)$ and from $c \in [f(a), f(b)]$ we deduce that $c \in f(A)$, that is, there exists $\xi \in A$ such that $c = f(\xi)$. □

We need the mean value theorem for the multiple Riemann integral, see [2, problem 6, page 190], or [3, page 167].

Theorem 3. *Let $A \subset \mathbb{R}^k$ be a Jordan measurable, compact and connected set, $f : A \to \mathbb{R}$ a continuous function. Then there exists $\xi \in A$ such that $\int_A f(x)\,\mathrm{d}x = f(\xi)\,\lambda_k(A)$.*

Proof. If $\lambda_k(A) = 0$ then, as is well known, $\int_A f(x)\,\mathrm{d}x = 0$ and we can take any $\xi \in A$. Let us suppose that $\lambda_k(A) > 0$. Since f is continuous, A compact, from the Weierstrass theorem, there are $a, b \in A$ such that $f(a) = m = \inf\limits_{x\in A} f(x)$, $f(b) = M = \sup\limits_{x\in A} f(x)$. From the inequalities $m \leq f(x) \leq M$, $\forall x \in A$, by integration we get $m\lambda_k(A) \leq \int_A f(x)\,\mathrm{d}x \leq M\lambda_k(A)$, or

$$f(a) \leq \frac{\int_A f(x)\,\mathrm{d}x}{\lambda_k(A)} \leq f(b).$$

From Proposition 2 there exists $\xi \in A$ such that $\frac{\int_A f(x)\mathrm{d}x}{\lambda_k(A)} = f(\xi)$. $\square$

The next proposition indicates a natural way to obtain asymptotic evaluations for multiple Riemann integrals. Its proof is modeled on the solution to problem 3.28 in [5].

Proposition 4. *Let $c > 0$ be a real number, $f, g : [-c, c] \to \mathbb{R}$ two continuous functions with the properties: $f(0) = 0$; $g(t) = 0$ if and only if $t = 0$, and*

$$\lim_{t\to 0} \frac{f(t)}{g(t)} = 0.$$

Let $A \subset \mathbb{R}^k$ be a Jordan measurable set with $\operatorname{int}(A) \neq \varnothing$. If $h_n : A \to \mathbb{R}$ is a sequence of continuous functions with the property that there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ there exists $x \in \operatorname{int}(A)$ such that $h_n(x) \neq 0$ and $\lim_{n\to\infty} h_n = 0$ uniformly on A, then

$$\lim_{n\to\infty} \frac{\int_A f(h_n(x))\,\mathrm{d}x}{\int_A |g(h_n(x))|\,\mathrm{d}x} = 0.$$

Proof. Let $\varepsilon > 0$. From $\lim_{t\to 0} \frac{f(t)}{g(t)} = 0$ there exists $\eta_\varepsilon > 0$ such that $\forall t \in [-c, c]$ with $0 < |t| < \eta_\varepsilon$ we have $\left|\frac{f(t)}{g(t)}\right| \leq \varepsilon$, that is, $|f(t)| \leq \varepsilon |g(t)|$. Since for $t = 0$, $f(0) = g(0) = 0$, we deduce that $\forall t \in [-c, c]$ with $|t| < \eta_\varepsilon$ we have

$$|f(t)| \leq \varepsilon |g(t)|. \tag{2}$$

Let $\delta_\varepsilon = \min(\eta_\varepsilon, c) > 0$. Then by (2) we have

$$|f(t)| \leq \varepsilon |g(t)|, \ \forall |t| < \delta_\varepsilon. \tag{3}$$

Since $\lim_{n\to\infty} h_n = 0$ uniformly on A, for $\delta_\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$|h_n(x)| < \delta_\varepsilon, \forall n \geq n_\varepsilon \text{ and } \forall x \in A. \tag{4}$$

Let $n \geq \max(n_\varepsilon, n_0)$. From (3) and (4) it follows that

$$|f(h_n(x))| \leq \varepsilon |g(h_n(x))|, \quad \forall x \in A$$

and, by integration,

$$\int_A |f(h_n(x))|\,\mathrm{d}x \leq \varepsilon \int_A |g(h_n(x))|\,\mathrm{d}x.$$

Since $\left|\int_A f(h_n(x))\,\mathrm{d}x\right| \leq \int_A |f(h_n(x))|\,\mathrm{d}x$, we get

$$\left|\int_A f(h_n(x))\,\mathrm{d}x\right| \leq \varepsilon \int_A |g(h_n(x))|\,\mathrm{d}x. \tag{5}$$

If $\int_A |g(h_n(x))|\,\mathrm{d}x = 0$, then we have by Proposition 1, $g(h_n(x)) = 0$, $\forall x \in \operatorname{int}(A)$, and by the property of g, $h_n(x) = 0$, $\forall x \in \operatorname{int}(A)$, which contradicts the hypothesis. Hence, $\int_A |g(h_n(x))| \, \mathrm{d}x > 0$ and from (5) we deduce that

$$\frac{\left|\int_A f(h_n(x)) \, \mathrm{d}x\right|}{\int_A |g(h_n(x))| \, \mathrm{d}x} \leq \varepsilon,$$

which ends the proof. □

We need later the following particular case of Proposition 4.

Corollary 5. *Let $A \subset \mathbb{R}^k$ be a Jordan measurable set with $\operatorname{int}(A) \neq \varnothing$, $c > 0$ a real number and $\varphi : [-c, c] \to \mathbb{R}$ a twice differentiable function at 0.*

(i) *If $h_n : A \to \mathbb{R}$ is a sequence of continuous functions with the property that there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ there exists $x \in \operatorname{int}(A)$ such that $h_n(x) \neq 0$ and $\lim\limits_{n\to\infty} h_n = 0$ uniformly on A, then*

$$\lim_{n\to\infty} \frac{\int_A \varphi(h_n(x)) \, \mathrm{d}x - \varphi(0)\lambda_k(A) - \varphi'(0)\int_A h_n(x) \, \mathrm{d}x}{\int_A (h_n(x))^2 \, \mathrm{d}x} = \frac{\varphi''(0)}{2}.$$

(ii) *If A is compact, $(a_n)_{n\in\mathbb{N}} \subset (0, \infty)$ is such that $\lim\limits_{n\to\infty} a_n = \infty$, then for every continuous function $f : A \to \mathbb{R}$ with the property that there exists $x \in \operatorname{int}(A)$ such that $f(x) \neq 0$, the following equality holds*

$$\lim_{n\to\infty} a_n \left[\int_A a_n \varphi\left(\frac{f(x)}{a_n}\right) \mathrm{d}x - a_n \varphi(0)\lambda_k(A) - \varphi'(0)\int_A f(x) \, \mathrm{d}x\right]$$
$$= \frac{\varphi''(0)}{2}\int_A [f(x)]^2 \, \mathrm{d}x.$$

Proof. (i) Since φ is twice differentiable at 0, $\lim\limits_{t\to 0} \frac{\varphi(t)-\varphi(0)-t\varphi'(0)-\frac{\varphi''(0)t^2}{2}}{t^2} = 0$ and by Proposition 4

$$\lim_{n\to\infty} \frac{\int_A \varphi(h_n(x)) \, \mathrm{d}x - \varphi(0)\int_A 1\mathrm{d}x - \varphi'(0)\int_A h_n(x) \, \mathrm{d}x - \frac{\varphi''(0)\int_A (h_n(x))^2 \mathrm{d}x}{2}}{\int_A (h_n(x))^2 \, \mathrm{d}x}$$
$$= 0.$$

This is obviously equivalent to the desired statement.

(ii) We take in (i) $h_n(x) = \frac{f(x)}{a_n}$. Let us note that since f is continuous and A compact, by the Weierstrass theorem there exists $M \geq 0$ such that $|f(x)| \leq M$, $\forall x \in A$. Then $|h_n(x)| \leq \frac{M}{a_n}$, $\forall x \in A$. Since $\lim\limits_{n\to\infty} \frac{1}{a_n} = 0$, it follows that $\lim\limits_{n\to\infty} h_n = 0$ uniformly on A. □

We also need the following theorem of Pólya; for a proof, see [4, problem 127, page 81], or [5, problem 4.23(ii), page 176].

Theorem 6. *Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of increasing functions and $f : [a, b] \to \mathbb{R}$ a continuous function. If $\lim\limits_{n\to\infty} f_n(x) = f(x)$, $\forall x \in [a, b]$, then $\lim\limits_{n\to\infty} f_n = f$ uniformly on $[a, b]$.*

3. The main result

Now we are ready to state and prove the main result of this paper.

Theorem 7. *Let $k \in \mathbb{N}$ and $A \subset \mathbb{R}^k$ be a compact, connected, Jordan measurable set with $\lambda_k(A) > 0$, $g : A \to [0,\infty)$ a continuous function with the property that there exists $x \in \operatorname{int}(A)$ such that $g(x) > 0$ and $M \geq \int_A g(x)\,\mathrm{d}x$. Let also $(a_n)_{n\in\mathbb{N}} \subset (0,\infty)$ be such that $\lim_{n\to\infty} a_n = \infty$. Then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the equation*

$$a_n \int_A \ln\left(1 + \frac{g(x)+u}{a_n}\right)\mathrm{d}x = M$$

has a unique solution in the interval $[0,\infty)$, denoted by u_n, and

$$\lim_{n\to\infty} u_n = \frac{M - \int_A g(x)\,\mathrm{d}x}{\lambda_k(A)},$$

$$\lim_{n\to\infty} a_n(u_n - s) = \frac{\int_A [g(x)]^2\,\mathrm{d}x}{2\lambda_k(A)} + \frac{s\int_A g(x)\,\mathrm{d}x}{\lambda_k(A)} + \frac{s^2}{2}$$

where $s = \frac{M - \int_A g(x)\mathrm{d}x}{\lambda_k(A)}$.

Proof. For every $n \in \mathbb{N}$ let $\psi_n : [0,\infty) \to \mathbb{R}$,

$$\psi_n(u) = a_n \int_A \ln\left(1 + \frac{g(x)+u}{a_n}\right)\mathrm{d}x - M,$$

and note that ψ_n is continuous (as an integral with a parameter). We have

$$\psi_n(0) = a_n \int_A \ln\left(1 + \frac{g(x)}{a_n}\right)\mathrm{d}x - M \leq \int_A g(x)\,\mathrm{d}x - M \leq 0$$

(we have used that $\ln(1+t) \leq t$, $\forall t \geq 0$, and the hypothesis). Let $u \geq 0$. From the inequalities $t - \frac{t^2}{2} \leq \ln(1+t) \leq t$, $\forall t \geq 0$, we deduce that

$$\int_A (g(x)+u)\,\mathrm{d}x - \frac{1}{2a_n}\int_A (g(x)+u)^2\,\mathrm{d}x \leq a_n \int_A \ln\left(1 + \frac{g(x)+u}{a_n}\right)\mathrm{d}x$$

$$\leq \int_A (g(x)+u)\,\mathrm{d}x,\ \forall n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} \frac{1}{a_n} = 0$, from the squeeze theorem we deduce that

$$\lim_{n\to\infty} \psi_n(u) = \int_A (g(x)+u)\,\mathrm{d}x - M = \psi(u), \tag{6}$$

where $\psi : [0,\infty) \to \mathbb{R}$ is defined by $\psi(u) = u\lambda_k(A) + \int_A g(x)\,\mathrm{d}x - M$. Let us define $v = \frac{M - \int_A g(x)\mathrm{d}x}{\lambda_k(A)} + 1 > 0$ and note that $\psi(v) = \lambda_k(A) > 0$. Then, since $\lim_{n\to\infty} \psi_n(v) = \psi(v) > 0$, there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ we have $\psi_n(v) > 0$. Let $u_1, u_2 \in [0,\infty)$ be such that $\psi_n(u_1) = \psi_n(u_2)$, that is

$$\int_A \left[\ln\left(1 + \frac{g(x)+u_1}{n}\right) - \ln\left(1 + \frac{g(x)+u_2}{n}\right)\right] \mathrm{d}x = 0.$$

From the mean value theorem 3, there exists $\xi \in A$ such that

$$\ln\left(1 + \frac{g(\xi)+u_1}{n}\right) - \ln\left(1 + \frac{g(\xi)+u_2}{n}\right) = 0$$

and hence $u_1 = u_2$. Thus ψ_n is continuous, injective and $\psi_n(0) \leq \psi_n(v)$. It follows that the equation $\psi_n(u) = 0$ has a unique solution in $[0,\infty)$, denoted by u_n. Moreover, $u_n \in [0,v]$, $\forall n \geq n_0$. Remembering that each solution verifies the equation, we get

$$\psi_n(u_n) = 0, \forall n \geq n_0. \tag{7}$$

Since by (6) $\lim_{n\to\infty} \psi_n(u) = \psi(u)$, $\forall u \in [0,v]$, all ψ_n are increasing (this is obvious) and ψ is continuous, from Pólya's theorem 6 it follows that $\lim_{n\to\infty} \psi_n = \psi$ uniformly on $[0,v]$. Hence $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$ and $\forall u \in [0,v]$ we have $|\psi_n(u) - \psi(u)| < \varepsilon$. Then for every $n \geq \max(n_\varepsilon, n_0)$ we have $|\psi_n(u_n) - \psi(u_n)| < \varepsilon$. By (7) and the definition of the function ψ we get $\left|u_n\lambda_k(A) + \int_A g(x)\,\mathrm{d}x - M\right| < \varepsilon$. This means that

$$\lim_{n\to\infty} u_n = \frac{M - \int_A g(x)\,\mathrm{d}x}{\lambda_k(A)} = s.$$

Let $n \geq n_0$. From the Lagrange formula, $\psi_n(u_n) - \psi_n(s) = (u_n - s)\,\psi_n'(\xi_n)$ for some ξ_n between u_n and s, which, by (7), becomes

$$\psi_n(s) = -(u_n - s)\,\psi_n'(\xi_n). \tag{8}$$

Then $|\xi_n - s| \leq |u_n - s|$ and since $\lim_{n\to\infty} u_n = s$, by the squeeze theorem

$$\lim_{n\to\infty} \xi_n = s. \tag{9}$$

Differentiating under the integral we get $\psi_n'(u) = \int_A \frac{1}{1+\frac{g(x)+u}{a_n}}\mathrm{d}x$,

$$\begin{aligned}
\left|\psi_n'(\xi_n) - \lambda_k(A)\right| &= \left|\int_A \left(\frac{1}{1+\frac{g(x)+\xi_n}{a_n}} - 1\right)\mathrm{d}x\right| = \int_A \frac{g(x)+\xi_n}{a_n + g(x) + \xi_n}\mathrm{d}x \\
&\leq \frac{1}{a_n}\int_A (g(x)+\xi_n)\,\mathrm{d}x = \frac{1}{a_n}\int_A g(x)\,\mathrm{d}x + \frac{\xi_n}{a_n}\lambda_k(A),
\end{aligned}$$

whence, by (9), $\lim_{n\to\infty} \frac{1}{a_n} = 0$, and the squeeze theorem,

$$\lim_{n\to\infty} \psi_n'(\xi_n) = \lambda_k(A) > 0. \tag{10}$$

Hence there exists $n_1 \in \mathbb{N}$ such that

$$\psi_n'(\xi_n) > 0, \forall n \geq n_1. \tag{11}$$

By the definition of s we have

$$\begin{aligned}\psi_n(s) &= a_n \int_A \ln\left(1 + \frac{g(x)+s}{a_n}\right) \mathrm{d}x - M \\ &= a_n \int_A \ln\left(1 + \frac{g(x)+s}{a_n}\right) \mathrm{d}x - \int_A (g(x)+s)\,\mathrm{d}x.\end{aligned}$$

From Corollary 5 (ii) applied for $\varphi : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}$, $\varphi(t) = \ln(1+t)$ it follows that

$$\begin{aligned}\lim_{n\to\infty} \quad & a_n \left[\int_A a_n \ln\left(1 + \frac{g(x)+s}{a_n}\right) \mathrm{d}x - \int_A (g(x)+s)\,\mathrm{d}x\right] \\ &= -\frac{1}{2} \int_A [g(x)+s]^2\,\mathrm{d}x,\end{aligned}$$

that is

$$\lim_{n\to\infty} a_n \psi_n(s) = -\frac{1}{2} \int_A [g(x)+s]^2\,\mathrm{d}x. \tag{12}$$

From (8) and (11) we deduce that

$$a_n(u_n - s) = -\frac{a_n \psi_n(s)}{\psi_n'(\xi_n)}, \forall n \geq n_1,$$

and hence passing to the limit and using (10) and (12) we get

$$\lim_{n\to\infty} a_n(u_n - s) = \frac{\int_A [g(x)+s]^2\,\mathrm{d}x}{2\lambda_k(A)} = \frac{\int_A [g(x)]^2\,\mathrm{d}x}{2\lambda_k(A)} + \frac{s\int_A g(x)\,\mathrm{d}x}{\lambda_k(A)} + \frac{s^2}{2}.$$

□

Corollary 8. *Let $k \in \mathbb{N}$ and $A \subset \mathbb{R}^k$ be a compact, connected, Jordan measurable set with $\lambda_k(A) > 0$ and $g : A \to [0, \infty)$ a continuous function with the property that there exists $x \in \operatorname{int}(A)$ such that $g(x) > 0$. Let also $(a_n)_{n\in\mathbb{N}} \subset (0, \infty)$ be such that $\lim\limits_{n\to\infty} a_n = \infty$ and $M \geq \int_A g(x)\,\mathrm{d}x$. Then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the equation*

$$a_n \int_A \ln\left(t + \frac{g(x)}{a_n}\right) \mathrm{d}x = M$$

has a unique solution in the interval $[1, \infty)$, denoted by t_n, and

$$\lim_{n\to\infty} a_n(t_n - 1) = s,$$

$$\lim_{n\to\infty} a_n[a_n(t_n - 1) - s] = \frac{\int_A [g(x)]^2\,\mathrm{d}x}{2\lambda_k(A)} + \frac{s\int_A g(x)\,\mathrm{d}x}{\lambda_k(A)} + \frac{s^2}{2},$$

where $s = \frac{M - \int_A g(x)\mathrm{d}x}{\lambda_k(A)}$.

Proof. Let us note that for every $n \in \mathbb{N}$ the function $\omega_n : [1,\infty) \to \mathbb{R}$, $\omega_n(t) = a_n \int_A \ln\left(t + \frac{g(x)}{a_n}\right) \mathrm{d}x$ is injective (use the mean value theorem) and thus if the equation $\omega_n(t) = M$ has a solution, this is unique. Now, the relation (7) says that there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, u_n is the solution of the equation $\psi_n(u) = 0$, or equivalently $\omega_n\left(1 + \frac{u_n}{a_n}\right) = M$. Hence, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $1 + \frac{u_n}{a_n}$ is a solution of the equation $\omega_n(t) = M$ and so, the equation from the statement has a unique solution, $t_n = 1 + \frac{u_n}{a_n}$, or $a_n(t_n - 1) = u_n$, $\forall n \geq n_0$. Then, by Theorem 7,

$$\begin{aligned}
\lim_{n\to\infty} a_n(t_n - 1) &= \lim_{n\to\infty} u_n = s, \\
\lim_{n\to\infty} a_n\left[a_n(t_n - 1) - s\right] &= \lim_{n\to\infty} a_n(u_n - s) \\
&= \frac{\int_A [g(x)]^2 \mathrm{d}x}{2\lambda_k(A)} + \frac{s\int_A g(x)\mathrm{d}x}{\lambda_k(A)} + \frac{s^2}{2}.
\end{aligned}$$

□

4. Some applications

In the sequel we give some applications of Corollary 8.

Corollary 9. (i) *Let $(a_n)_{n\in\mathbb{N}} \subset (0,\infty)$ be such that $\lim\limits_{n\to\infty} a_n = \infty$, $k \geq 2$ a natural number and $M \geq \frac{k}{2}$. Then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the equation*

$$a_n \int_{[0,1]^k} \ln\left(t + \frac{x_1 + \cdots + x_k}{a_n}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_k = M$$

has a unique solution in $[1,\infty)$, denoted by t_n, and

$$\lim_{n\to\infty} a_n(t_n - 1) = M - \frac{k}{2},$$

$$\lim_{n\to\infty} a_n\left[a_n(t_n - 1) - M + \frac{k}{2}\right] = \frac{3k^2 + k}{24} + \frac{k}{2}\left(M - \frac{k}{2}\right) + \frac{1}{2}\left(M - \frac{k}{2}\right)^2.$$

(ii) *Let $(a_n)_{n\in\mathbb{N}} \subset (0,\infty)$ be such that $\lim\limits_{n\to\infty} a_n = \infty$, $k \geq 2$ a natural number and $M \geq \frac{1}{2^k}$. Then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the equation*

$$a_n \int_{[0,1]^k} \ln\left(t + \frac{x_1 \cdots x_k}{a_n}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_k = M$$

has a unique solution in $[1,\infty)$, *denoted by* t_n, *and*

$$\lim_{n\to\infty} a_n(t_n-1) = M - \frac{1}{2^k},$$
$$\lim_{n\to\infty} a_n\left[a_n(t_n-1) - M + \frac{1}{2^k}\right] = \frac{1}{2\cdot 3^k} + \frac{2^k M - 1}{4^k} + \frac{1}{2}\left(M - \frac{1}{2^k}\right)^2.$$

Proof. (i) Let $g:[0,1]^k \to [0,\infty)$, $g(x_1,\ldots,x_k) = x_1+\cdots+x_k$. We have $\int_{[0,1]^k} g(x_1,\ldots,x_k)\,\mathrm{d}x_1\cdots\mathrm{d}x_k = \frac{k}{2}$, $\int_{[0,1]^k} g^2(x_1,\ldots,x_k)\,\mathrm{d}x_1\cdots\mathrm{d}x_k = \frac{3k^2+k}{12}$. From Corollary 8 we get

$$\begin{aligned}\lim_{n\to\infty} a_n\left[a_n(t_n-1) - M + \frac{k}{2}\right] &= \frac{\int_A [g(x)]^2\,\mathrm{d}x}{2\lambda_k(A)} + \frac{s\int_A g(x)\,\mathrm{d}x}{\lambda_k(A)} + \frac{s^2}{2}\\ &= \frac{3k^2+k}{24} + \frac{k}{2}\left(M-\frac{k}{2}\right) + \frac{1}{2}\left(M-\frac{k}{2}\right)^2\end{aligned}$$

(ii) Let $g:[0,1]^k \to [0,\infty)$, $g(x_1,\ldots,x_k) = x_1\cdots x_k$. We have

$$\int_{[0,1]^k} (x_1\cdots x_k)\,\mathrm{d}x_1\cdots\mathrm{d}x_k = \left(\int_0^1 x\,\mathrm{d}x\right)^k = \frac{1}{2^k},$$
$$\int_{[0,1]^k} (x_1\cdots x_k)^2\,\mathrm{d}x_1\cdots\mathrm{d}x_k = \left(\int_0^1 x^2\,\mathrm{d}x\right)^k = \frac{1}{3^k}.$$

From Corollary 8 we get

$$\lim_{n\to\infty} a_n\left[a_n(t_n-1) - M + \frac{1}{2^k}\right] = \frac{1}{2\cdot 3^k} + \frac{M-\frac{1}{2^k}}{2^k} + \frac{1}{2}\left(M - \frac{1}{2^k}\right)^2.$$

□

Corollary 10. (i) *Let* $(a_n)_{n\in\mathbb{N}} \subset (0,\infty)$ *be such that* $\lim_{n\to\infty} a_n = \infty$ *and* $M \geq \frac{1}{3}$. *Then there exists* $n_0\in\mathbb{N}$ *such that for every* $n\geq n_0$ *the equation*

$$a_n \iint_{x+y\leq 1,\ x\geq 0,\ y\geq 0} \ln\left(t + \frac{x+y}{a_n}\right)\mathrm{d}x\mathrm{d}y = M$$

has a unique solution in $[1,\infty)$, *denoted by* t_n, *and*

$$\lim_{n\to\infty} a_n(t_n-1) = 2\left(M-\frac{1}{3}\right),$$

$$\lim_{n\to\infty} a_n\left[a_n(t_n-1) - 2M + \frac{2}{3}\right] = \frac{1}{4} + \frac{2}{3}\left(M-\frac{1}{3}\right) + 2\left(M-\frac{1}{3}\right)^2.$$

(ii) *Let* $(a_n)_{n\in\mathbb{N}} \subset (0,\infty)$ *be such that* $\lim\limits_{n\to\infty} a_n = \infty$ *and* $M \geq \frac{2}{3}$. *Then there exists* $n_0 \in \mathbb{N}$ *such that for every* $n \geq n_0$ *the equation*

$$a_n \iint\limits_{x^2+y^2\leq 1,\ x\geq 0,\ y\geq 0} \ln\left(t + \frac{x+y}{a_n}\right) \mathrm{d}x\mathrm{d}y = M$$

has a unique solution in $[1,\infty)$, *denoted by* t_n, *and*

$$\lim_{n\to\infty} a_n\left(t_n - 1\right) = \frac{4(3M-2)}{3\pi},$$

$$\lim_{n\to\infty} a_n \left[a_n\left(t_n - 1\right) - \frac{4(3M-2)}{3\pi}\right] = \frac{\pi+2}{4\pi} + \frac{32(3M-2)}{9\pi^2} + \frac{8(3M-2)^2}{9\pi^2}.$$

Proof. (i) Consider the function $g : A \to [0,\infty)$ defined by $g\left(x,y\right) = x + y$ and the set $A = \left\{(x,y) \in \mathbb{R}^2 \mid x + y \leq 1,\ x, y \geq 0\right\}$. For every $k \in \mathbb{N}$ we have

$$\iint\limits_A g^k\left(x,y\right) \mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^{1-x} (x+y)^k\, \mathrm{d}y = \frac{1}{k+2}.$$

From Corollary 8 by some calculations we get the limits from the statement.

(ii) Now consider $g : A \to [0,\infty)$ given by $g\left(x,y\right) = x + y$ and the set $A = \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1,\ x, y \geq 0\right\}$. We have

$$\iint\limits_A g\left(x,y\right) \mathrm{d}x\mathrm{d}y = \iint\limits_{[0,1]\times\left[0,\frac{\pi}{2}\right]} \left(\rho\cos\theta + \rho\sin\theta\right) \rho \mathrm{d}\rho\mathrm{d}\theta = \frac{2}{3},$$

$$\iint\limits_A g^2\left(x,y\right) \mathrm{d}x\mathrm{d}y = \iint\limits_{[0,1]\times\left[0,\frac{\pi}{2}\right]} \left(\rho\cos\theta + \rho\sin\theta\right)^2 \rho \mathrm{d}\rho\mathrm{d}\theta = \frac{\pi+2}{8}.$$

The limits from the statement follow from Corollary 8 by some calculations. □

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Finding the admissible values for the parameter of a quadratic product-type inequality

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Abstract. We determine all the values of the real parameter k such that the inequality $(a_1^2+k)(a_2^2+k)\cdots(a_n^2+k) \geq (1+k)^n$ holds for all real numbers $a_1, a_2, \dots, a_n$ satisfying $a_1+a_2+\cdots+a_n = n$, where $n \geq 2$ is a given natural number. We prove that these values form an interval of the type $[k_n, \infty)$, where k_n is a root of a $(2n-3)$rd degree polynomial equation. Using the properties of the sequence $(k_n)_{n\geq 2}$, we obtain that there is no k such that the considered inequality holds for all $n \geq 2$. In particular, for $n = 3$ and $n = 4$ we derive that the corresponding values k_n are irrational and the polynomials defining them are minimal. We estimate also the values k_n for $n \leq 20$. Finally, we propose three open problems regarding the irrationality, the minimal polynomial and the explicit formula (or a recurrence relation) of k_n.

Keywords: Jensen's inequality, half convex function theorem, minimal polynomial.
MSC: 26D07

1. Introduction

Various particular cases of the following inequality

$$\prod_{i=1}^{n}(a_i^2+k) \geq (1+k)^n, \text{ for all } a_1, \dots, a_n \in \mathbb{R} \text{ such that } \sum_{i=1}^{n} a_i = n, \qquad (1)$$

have been proposed in magazines and related websites. For example, for $n = 4$ and $k = 3$ we obtain the inequality proposed by Lascu and Zvonaru [5], for $n = 3$ and $k = 11/4$ we obtain the inequality proposed in [7], and for $n = 5$ and $k = 39/25$ we obtain the inequality proposed by Trăncănău [8].

In this paper, we are interested to find all the values of k for that the inequality (1) holds, for any fixed $n \geq 2$. We denote by K_n the set of these values, i.e.

$$K_n = \left\{ k \in \mathbb{R} : \prod_{i=1}^{n}(a_i^2+k) \geq (1+k)^n \ \forall a_1, \dots, a_n \in \mathbb{R} \text{ with } \sum_{i=1}^{n} a_i = n \right\}.$$

This set is called the admissible domain or the range of the parameter k.

For $n = 2$ by replacing $a_2 = 2 - a_1$ the inequality (1) becomes

$$(a_1^2+k)\left[(2-a_1)^2+k\right] - (1+k)^2 \geq 0 \text{ for all } a_1 \in \mathbb{R},$$

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which can be rearranged as

$$(a_1-1)^2\left[(a_1-1)^2+2k-2\right]\ge 0 \text{ for all } a_1\in\mathbb{R},$$

and hence

$$K_2=[1,\infty). \tag{2}$$

If $k<0$ and n is odd, then the inequality (1) is false, since by taking $a_1=m$, $a_2=n-m$ and $a_3=\cdots=a_n=0$ we have

$$\lim_{m\to\infty}\prod_{i=1}^{n}(a_i^2+k)=\lim_{m\to\infty}k^{n-2}(m^2+k)\left[(n-m)^2+k\right]=-\infty.$$

If $k<0$ and $n\ge 4$ is even, then the inequality (1) is also false, since by taking $a_1=m$, $a_2=a_3=(n-m)/2$ and $a_4=\cdots=a_n=0$ we have

$$\lim_{m\to\infty}\prod_{i=1}^{n}(a_i^2+k)=\lim_{m\to\infty}k^{n-3}(m^2+k)\left[\frac{(n-m)^2}{4}+k\right]^2=-\infty.$$

Thus $K_n\subseteq[0,\infty)$, for all $n\ge 2$.

For any $n\ge 3$ and $k\in K_n$, by taking $a_n=1$ in (1) and dividing the both sides by $1+k>0$ it follows that $k\in K_{n-1}$. Therefore

$$K_n\subseteq K_{n-1},\ \text{ for all } n\ge 3. \tag{3}$$

From (3) and (2) we get

$$K_n\subseteq[1,\infty),\ \text{ for all } n\ge 2. \tag{4}$$

Consequently, for any $k\in K_n$ we can rewrite the inequality (1) as

$$\sum_{i=1}^{n}f(a_i)\ge nf\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right),\ \text{ for all } a_1,\ldots,a_n\in\mathbb{R} \text{ with } \sum_{i=1}^{n}a_i=n, \tag{5}$$

where

$$f(t)=\ln(t^2+k),\ \text{ for all } t\in\mathbb{R}. \tag{6}$$

We remark that the relation (5) has the form of the well-known Jensen's inequality, that is certainly valid for convex functions [4]. Unfortunately, the function f is not convex on $\mathbb{R}$.

However, after a few preparations we can apply the half convex function theorem (HCF-Theorem) due by Cîrtoaje [1, 2] to replace the inequality (5) with a much simpler equivalent inequality. This goal is achieved in Section 2 and will lead us to prove that the range K_n is an interval $[k_n,\infty)$, where k_n is defined by a polynomial equation. Also, we show that there is no k such that the inequality (1) holds for all $n\ge 2$.

We mention that the HCF-Theorem was extended by Cîrtoaje and Băieşu [2, 3]. Other properties of half convex functions were obtained by Pavić [6].

In Section 3 we discuss the cases $n=3$ and $n=4$. For these particular cases we obtain the minimal polynomials of the values k_n, and hence we derive the irrationality of these values and we compute rational approximations for them. Also, we estimate the values k_n for $n = 5, 6, \ldots, 20$. In the last section we propose three open problems about k_n for further research.

2. Main results

The following lemma ensures us that for finding the range set K_n it suffices to consider only nonnegative numbers $a_1, a_2, \ldots, a_n$ in the inequality (5) or, equivalently, in inequality (1).

Lemma 1. *For any $n \geq 2$ we have*

$$K_n = \left\{ k \geq 1 : \sum_{i=1}^{n} f(a_i) \geq n f(1) \ \forall a_1, \ldots, a_n \in [0, n] \text{ with } \sum_{i=1}^{n} a_i = n \right\}, \tag{7}$$

where the function f is defined by (6).

Proof. The inclusion "$\subseteq$" (left to right) is an obviously consequence of relations (4) and (5). Let us prove the reverse inclusion "$\supseteq$" (right to left). Let $k \geq 1$ be an arbitrary element of the set from the right-hand side of (7). It remains to prove that k verifies the inequality (5). Let $a_1, a_2, \ldots, a_n$ be real numbers such that $\sum_{i=1}^{n} a_i = n$. Then $\sum_{i=1}^{n} |a_i| = ns \geq n$, with $s \geq 1$. Setting

$$b_i = \frac{|a_i|}{s} \text{ for } i = 1, 2, \ldots, n,$$

it follows that $b_1, \ldots, b_n \geq 0$ and $\sum_{i=1}^{n} b_i = n$, so $b_1, \ldots, b_n \in [0, n]$, and hence $\sum_{i=1}^{n} f(b_i) \geq n f(1)$. We derive that

$$\sum_{i=1}^{n} f(a_i) = \sum_{i=1}^{n} \ln(a_i^2 + k) = \sum_{i=1}^{n} \ln(s^2 b_i^2 + k) \geq \sum_{i=1}^{n} \ln(b_i^2 + k) = \sum_{i=1}^{n} f(b_i).$$

Since $\sum_{i=1}^{n} f(b_i) \geq n f(1)$ we get $\sum_{i=1}^{n} f(a_i) \geq n f(1)$, so k verifies the inequality (5), and therefore $k \in K_n$. $\square$

The form (7) of the set K_n allows us to apply the HCF-Theorem. Indeed, by taking $t \in [0, n]$ and using the second-order derivative,

$$f''(t) = \frac{2(k - t^2)}{(t^2 + k)^2},$$

we obtain that the function f is convex on $[0,1]$ (since $k \geq 1$). Now, according to HCF-Theorem (see [1] or [2]) it follows that the inequality from the right-hand side of (7) is equivalent to

$$(n-1)f(t) + f\left(n-(n-1)t\right) \geq nf(1), \text{ for all } t \in \left[0, \frac{n}{n-1}\right].$$

Therefore

$$K_n = \left\{k \geq 1 \ : \ \min_{t \in \left[0, \frac{n}{n-1}\right]} g(t) = g(1)\right\}, \tag{8}$$

where the function g is defined by

$$\begin{aligned} g(t) &= (n-1)f(t) + f\left(n-(n-1)t\right) \\ &= (n-1)\ln(t^2+k) + \ln\left[(n-(n-1)t)^2 + k\right], \end{aligned}$$

for all $t \in \left[0, \frac{n}{n-1}\right]$.

We will need to study the monotonicity of the function g. Its derivative is

$$g'(t) = \frac{2n(n-1)(t-1)\left[(n-1)t^2 - nt + k\right]}{(t^2+k)\left[(n-(n-1)t)^2 + k\right]}, \text{ for all } t \in \left[0, \frac{n}{n-1}\right].$$

We distinguish two cases.

(a) If $k \geq \frac{n^2}{4(n-1)}$, then the function g has $t_1 = 1$ as unique critical point and minimizer, and hence by (8) we obtain

$$\left[\frac{n^2}{4(n-1)}, \infty\right) \subseteq K_n. \tag{9}$$

(b) If $k \in \left[1, \frac{n^2}{4(n-1)}\right)$, then $n \geq 3$ and we have two subcases.

(b_1) If $k = 1$, then the function g has two critical points, namely $t_1 = 1$ and $t_2 = \frac{1}{n-1}$, with $t_2 < 1$. Moreover, g is strictly decreasing on $[0, t_2]$ and strictly increasing on $\left[t_2, \frac{n}{n-1}\right]$, so $t_1 = 1$ is not a minimizer for g, and hence by (8) we obtain

$$1 \notin K_n, \text{ for all } n \geq 3. \tag{10}$$

(b_2) If $k \in \left(1, \frac{n^2}{4(n-1)}\right)$, then the function g has three critical points, namely $t_1 = 1$, $t_2 = \frac{n-\sqrt{n^2-4(n-1)k}}{2(n-1)}$ and $t_3 = \frac{n+\sqrt{n^2-4(n-1)k}}{2(n-1)}$, with $0 < t_2 < t_3 < 1$. Moreover, g is strictly decreasing on $[0, t_2]$, strictly increasing on $[t_2, t_3]$, strictly decreasing on $[t_3, 1]$ and strictly increasing on $\left[1, \frac{n}{n-1}\right]$, so

$t_1 = 1$ is a minimizer for g if and only if $g(t_2) \geq g(1)$. Therefore, by (8) we obtain

$$K_n \cap \left(1, \frac{n^2}{4(n-1)}\right) = \left\{k \,:\, h(k) \geq 0,\ k \in \left(1, \frac{n^2}{4(n-1)}\right)\right\}, \tag{11}$$

where the function h is defined by

$$\begin{aligned} h(k) &= g(t_2) - g(1) \\ &= (n-1)\ln\left[\left(\frac{n-\sqrt{n^2-4(n-1)k}}{2(n-1)}\right)^2 + k\right] - n\ln(1+k) \\ &\quad + \ln\left[\left(\frac{n+\sqrt{n^2-4(n-1)k}}{2}\right)^2 + k\right], \end{aligned} \tag{12}$$

for all $k \in \left[1, \frac{n^2}{4(n-1)}\right]$.

We have $h(1) = u(n)$, where

$$u(x) = (x-1)\ln\left(1 + \frac{1}{(x-1)^2}\right) + \ln[(x-1)^2+1] - x\ln 2,$$

for all $x \in [2, \infty)$. But

$$\begin{aligned} u'(x) &= \ln\left(1 + \frac{1}{(x-1)^2}\right) + \frac{2(x-2)}{(x-1)^2+1} - \ln 2, \\ u''(x) &= -\frac{2x(x-2)^2}{(x-1)\left[(x-1)^2+1\right]^2}, \end{aligned}$$

so u' is strictly decreasing on $[2, \infty)$, $u'(2) = 0$, u is strictly decreasing on $[2, \infty)$, $u(2) = 0$. Hence $u(n) < 0$, that is we have

$$h(1) < 0. \tag{13}$$

According to case (a) we have

$$h\left(\frac{n^2}{4(n-1)}\right) = g\left(\frac{n}{2(n-1)}\right) - g(1) > 0. \tag{14}$$

We study the monotonicity of the function h. If we denote

$$\frac{n-\sqrt{n^2-4(n-1)k}}{2(n-1)} = x, \text{i.e. } k = nx - (n-1)x^2, \tag{15}$$

from $k \in \left[1, \frac{n^2}{4(n-1)}\right]$ we get $x \in \left[\frac{1}{n-1}, \frac{n}{2(n-1)}\right]$, and from (12) we derive that

$$h(k) = v(x), \tag{16}$$

where the function v is defined by

$$v(x) = (n-1)\ln x + \ln[(n-(n-1)x] + n\ln[(n-(n-2)x] \\ - n\ln[1+nx-(n-1)x^2], \tag{17}$$

for all $x \in \left[\frac{1}{n-1}, \frac{n}{2(n-1)}\right]$. Since

$$v'(x) = \frac{n(n-1)(1-x)^2[n-2(n-1)x]}{x[n-(n-1)x][n-(n-2)x][1+nx-(n-1)x^2]}, \tag{18}$$

it follows that v is strictly increasing on $\left[\frac{1}{n-1}, \frac{n}{2(n-1)}\right]$, so according to (16) and (15) we can derive that

$$h \text{ is strictly increasing on } \left[1, \frac{n^2}{4(n-1)}\right]. \tag{19}$$

From (13), (14) and (19) it follows that the equation $h(k) = 0$ has a unique solution

$$k_n \in \left(1, \frac{n^2}{4(n-1)}\right), \tag{20}$$

and using (11) we conclude that

$$K_n \cap \left(1, \frac{n^2}{4(n-1)}\right) = \left[k_n, \frac{n^2}{4(n-1)}\right). \tag{21}$$

According to the above cases, from (9), (10) and (21) we obtain the following main result.

Theorem 2. *For any $n \geq 3$ we have*

$$K_n = [k_n, \infty), \tag{22}$$

where k_n is the unique solution in the interval $\left(1, \frac{n^2}{4(n-1)}\right)$ of the equation $h(k) = 0$ and h is defined by (12).

The following result is a direct consequence of Theorem 2, relation (2) and inclusion (3).

Corollary 3. *For any $n \geq 3$ we have $k_n \geq k_{n-1}$.*

Corollary 4. *We have $\lim\limits_{n\to\infty} k_n = \infty$.*

Proof. Obviously, $\frac{1}{\sqrt{n-1}} \in \left(\frac{1}{n-1}, \frac{n}{2(n-1)}\right)$, for any $n \geq 3$. According to (17), the function $v(x)$ can be rewritten as

$$v(x) = n\ln\frac{nx-(n-2)x^2}{1+nx-(n-1)x^2} + \ln\frac{n-(n-1)x}{x},$$

for all $x \in \left[\frac{1}{n-1}, \frac{n}{2(n-1)}\right]$ and $n \geq 3$. Therefore

$$v\left(\frac{1}{\sqrt{n-1}}\right) = n \ln\left(1 - \frac{n-2}{n\sqrt{n-1}}\right) + \ln(n\sqrt{n-1} - n + 1)$$

$$= \frac{n-2}{\sqrt{n-1}}\left[\frac{\ln\left(1 - \frac{n-2}{n\sqrt{n-1}}\right)}{\frac{n-2}{n\sqrt{n-1}}} + \frac{\ln(n\sqrt{n-1} - n + 1)}{\frac{n-2}{\sqrt{n-1}}}\right],$$

and hence

$$\lim_{n\to\infty} v\left(\frac{1}{\sqrt{n-1}}\right) = -\infty.$$

Consequently, there exists $n_0 \geq 3$ such that

$$v\left(\frac{1}{\sqrt{n-1}}\right) < 0, \text{ for all } n \geq n_0.$$

Using (15) and (16) it follows that

$$n\frac{1}{\sqrt{n-1}} - (n-1)\left(\frac{1}{\sqrt{n-1}}\right)^2 = \frac{n}{\sqrt{n-1}} - 1 \in \left(1, \frac{n^2}{4(n-1)}\right)$$

and

$$h\left(\frac{n}{\sqrt{n-1}} - 1\right) < 0, \text{ for all } n \geq n_0,$$

so by (14) and (19) we derive that

$$k_n \in \left(\frac{n}{\sqrt{n-1}} - 1, \frac{n^2}{4(n-1)}\right), \text{ for all } n \geq n_0.$$

Therefore we have $\lim_{n\to\infty} k_n = \infty$. □

Corollary 5. *There is no real number k such that (1) holds for all $n \geq 2$.*

Proof. From Theorem 2, relation (2), Corollaries 3 and 4 we get

$$\bigcap_{n\geq 2} K_n = \bigcap_{n\geq 2} [k_n, \infty) = \emptyset.$$

It is clear that this relation is equivalent to the statement of the corollary. □

Remark 6. According to (17), the equation $v(x) = 0$ can be rewritten in the polynomial form $p(x) = 0$, where

$$p(x) = x^{n-1}[(n - (n-1)x][(n - (n-2)x]^n - [1 + nx - (n-1)x^2]^n.$$

Obviously, the degree of p is equal to $2n$. Moreover, by (17) and (18) we get $v(1) = v'(1) = v''(1) = 0$, so the polynomial $p(x)$ is divisible by $(x-1)^3$. Denote by $q(x)$ the quotient of the division of $p(x)$ by $(x-1)^3$. Then q is an $(2n-3)$th degree polynomial and according to (15), (16) and (20) we derive that the equation $q(x) = 0$ has a unique solution x_n in the interval $\left(\frac{1}{n-1}, \frac{n}{2(n-1)}\right)$ and

$$k_n = nx_n - (n-1)x_n^2, \quad \frac{n - \sqrt{n^2 - 4(n-1)k_n}}{2(n-1)} = x_n.$$

Substituting $x = \frac{n-\sqrt{n^2-4(n-1)k}}{2(n-1)}$ in the equation $q(x) = 0$ we obtain that k_n is the unique solution in the interval $\left(1, \frac{n^2}{4(n-1)}\right)$ of an $(2n-3)$th degree polynomial equation $r(k) = 0$.

In the next section we detail this procedure for $n = 3$ and for $n = 4$.

3. Particular cases

Firstly, we consider the case of $n = 3$. According to Remark 6 we get

$$\begin{aligned} p(x) &= x^2(3-2x)(3-x)^3 - (1+3x-2x^2)^3 \\ &= 10x^6 - 57x^5 + 123x^4 - 126x^3 + 60x^2 - 9x - 1 \\ &= (x-1)^3 q(x), \end{aligned}$$

where

$$q(x) = 10x^3 - 27x^2 + 12x + 1.$$

We substitute $x = \frac{3-\sqrt{9-8k}}{4}$ in the equation $q(x) = 0$ in two steps. We can denote

$$\sqrt{9-8k} = 3 - 4x = y.$$

By replacing $x = \frac{3-y}{4}$, the equation $q(x) = 0$ becomes

$$5y^3 + 9y^2 - 93y + 31 = 0,$$

then replacing $y = \sqrt{9-8k}$ we derive that k_3 is the unique root in the interval $\left(1, \frac{9}{8}\right)$ of the polynomial

$$r(k) = 25k^3 + 42k^2 - 63k - 16.$$

The numbers of the form (divisor of 16)/(divisor of 25) are not roots of the polynomial $r(k)$, so $r(k)$ has not rational roots, and hence k_3 is irrational and $r(k)$ is even the minimal polynomial (with integer coefficients) of the value k_3. The approximation of k_3 with 4 exact decimal digits is

$$k_3 \approx 1.1099$$

$(r(1.1099) < 0 < r(1.11))$, so (22) implies that $1.11 \in K_3$ but $1.1099 \notin K_3$.

Next, we consider the case of $n = 4$. According to Remark 6, now we get

$$\begin{aligned} p(x) &= 16x^3(4-3x)(2-x)^4 - (1+4x-3x^2)^4 \\ &= -129x^8 + 880x^7 - 2420x^6 + 3408x^5 - 2550x^4 + 912x^3 - 84x^2 \\ &\quad - 16x - 1 \\ &= (x-1)^3 q(x), \end{aligned}$$

where

$$q(x) = -129x^5 + 493x^4 - 554x^3 + 138x^2 + 19x + 1.$$

We substitute $x = \frac{2-\sqrt{4-3k}}{3}$ in the equation $q(x) = 0$ in two steps. We can denote

$$\sqrt{4-3k} = 2 - 3x = y.$$

By replacing $x = \frac{2-y}{3}$, the equation $q(x) = 0$ becomes

$$43y^5 + 63y^4 - 562y^3 - 338y^2 + 2127y - 709 = 0,$$

then replacing $y = \sqrt{4-3k}$ we derive that k_4 is the unique root in the interval $\left(1, \frac{4}{3}\right)$ of the polynomial

$$r(k) = 1849k^5 + 5107k^4 + 42k^3 - 9002k^2 - 3923k - 729.$$

We have $729 = 3^6$ and $1849 = 43^2$. The numbers of the form (divisor of 3^6)/(divisor of 43^2) are not roots of the polynomial $r(k)$, so $r(k)$ has not rational roots, and hence k_4 is irrational. Moreover, $r(k)$ can not be decomposed as a product of the form $s(k)t(k)$ with $s(k)$ and $t(k)$ polynomials of degree 2 and 3, respectively, having integer coefficients. It follows again that $r(k)$ is the minimal polynomial (with integer coefficients) of the value k_4. The approximation of k_4 with 4 exact decimal digits is

$$k_4 \approx 1.2883$$

$(r(1.2883) < 0 < r(1.2884))$, so (22) implies that $1.2884 \in K_4$ but $1.2883 \notin K_4$.

Remark 7. Using the bisection method, we obtain the following approximations of k_n, $n = 5, 6, \ldots, 20$, with 4 exact decimal digits:

$$\begin{gathered} k_5 \approx 1.4789,\ k_6 \approx 1.6709,\ k_7 \approx 1.8616,\ k_8 \approx 2.0499,\ k_9 \approx 2.2357, \\ k_{10} \approx 2.4189,\ k_{11} \approx 2.5997,\ k_{12} \approx 2.7782,\ k_{13} \approx 2.9545,\ k_{14} \approx 3.1288, \\ k_{15} \approx 3.3012,\ k_{16} \approx 3.4719,\ k_{17} \approx 3.6409,\ k_{18} \approx 3.8084,\ k_{19} \approx 3.9744, \\ k_{20} \approx 4.1390. \end{gathered}$$

4. Conclusions and open problems

In this paper we showed that the range of the parameter k in inequality (1) is an interval of the form $[k_n, \infty)$, where k_n verifies a certain polynomial equation. Also, we obtained that $k_2 = 1$, k_3 and k_4 are irrational and the sequence $(k_n)_{n\geq 2}$ is increasing and tends to infinity.

At the end we propose three open problems.

Problem 1. Is the number k_n irrational for every $n \geq 3$?

Problem 2. Is the polynomial $r(k)$ defined in Remark 6 the minimal polynomial (with integer coefficients) of the value k_n for any $n \geq 3$?

Problem 3. Is there an explicit formula for the sequence $(k_n)_{n\geq 2}$, or even a recurrence relation?

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PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of May 2021**.

PROPOSED PROBLEMS

504. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with f' continuous on $[0,1]$, such that $|f'(x)| \leq 1\ \forall x \in [0,1]$. Prove that if $2\left|\int_0^1 f(x)\mathrm{d}x\right| \leq 1$ then $(n+2)\left|\int_0^1 x^n f(x)\mathrm{d}x\right| \leq 1\ \forall n \geq 1$.

Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.

505. Let $n, p, q \in \mathbb{N}$ such that $1 \leq q < p < n$. If there exists $A \in \mathcal{M}_n(\{0,1\})$ such that AA^t has all the elements on the diagonal equal to p and all the other elements equal to q, prove that:

a) $p(p-1) = q(n-1)$;
b) $AA^t = A^tA$;
c) if n is even, then $p-q$ is a perfect square.

Proposed by Vasile Pop and Mircea Rus, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

506. Let $N > 1$ be a squarefree integer. For every integer k we denote $q_N(k) = \gcd(N,k)$. Prove that there is a finite subset S of the unit circle such that for every polynomial $f = \sum_{k=0}^n a_k X^k \in \mathbb{C}[X]$ we have

$$\mu(N)\sum_{k=0}^{n} \mu(q_N(k))\phi(q_N(k))a_k = \sum_{\zeta\in S} f(\zeta).$$

(Here μ and ϕ denote the Möbius function and Euler's totient function, respectively.)

Proposed by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bârlad, Romania.

507. Calculate the integral

$$\int_1^\infty \frac{\ln x}{x^3 + x\sqrt{x} + 1}\mathrm{d}x.$$

Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.

508. Let K be a field and let $A, B \in M_n(K)$, with $n \geq 1$, such that $AB - BA = c(A - B)$ for some $c \in K \setminus \{0\}$.

(i) Prove that if $\operatorname{char} K = 0$ or $\operatorname{char} K > n$, then A and B have the same eigenvalues.

(ii) Prove that if $0 < \operatorname{char} K \leq n$ then (i) is no longer true.

Remark. The statement (i) is an extension of statement b) of problem 495, whose solution is published in the present issue of GMA.

`Proposed by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.`

509. Let m and n be positive integers and let $A_1, \dots, A_m \in \mathcal{M}_n(\mathbb{R})$. For every $i \in \{1, \dots, m\}$ denote by $\lambda_{i1}, \dots, \lambda_{in} \in \mathbb{C}$ the eigenvalues of A_i.

Prove that there exist $\varepsilon_1, \dots, \varepsilon_m \in \{-1, 1\}$ such that the eigenvalues $\mu_1, \dots, \mu_n \in \mathbb{C}$ of the matrix $\varepsilon_1 A_1 + \cdots + \varepsilon_m A_m \in \mathcal{M}_n(\mathbb{R})$ satisfy the inequality

$$\sum_{j=1}^{n} \mu_j^2 \geq \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}^2.$$

`Proposed by Vasile Pop, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

510. Prove that

$$\sum_{n=1}^{\infty} \binom{4n}{2n} \frac{1}{16^n n^2 (2n+1)} = 4 \operatorname{Li}_2\left(\frac{1-\sqrt{2}}{2}\right) + \frac{\pi^2}{3} + 4 \log\left(\frac{1+\sqrt{2}}{4}\right) - 2 \log^2\left(\frac{1+\sqrt{2}}{2}\right) - \log^2(4) + 4\left(\sqrt{2} - 1\right).$$

Here $\operatorname{Li}_2(x)$ is the dilogarithm with integral representation given by

$$\operatorname{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\, dt.$$

`Proposed by Seán M. Stewart, Bomaderry, NSW, Australia.`

511. Find the best lower and upper bounds for $\sum_{i=1}^{n} \cos(\angle A_i)$ over all convex n-gons $A_1 A_2 \dots A_n$.

`Proposed by Leonard Giugiuc, Traian National College, Drobeta Turnu Severin, and Florin Vişescu, Mihai Eminescu National College, Bucureşti, Romania.`

512. Evaluate the series
$$\sum_{n=1}^{\infty}\left(n\left(n\left(n\sum_{k=n}^{\infty}\frac{1}{k^2}-1\right)-\frac{1}{2}\right)-\frac{1}{6}\right).$$

Proposed by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bârlad, Romania.

SOLUTIONS

489. Let $m \le n$ be positive integers. For $A \in \mathcal{M}_{m,n}(\mathbb{C})$ and $B \in \mathcal{M}_{n,m}(\mathbb{C})$ define the functions
$$f_{A,B} : \mathcal{M}_n(\mathbb{C}) \longrightarrow \mathcal{M}_m(\mathbb{C}), \quad f_{A,B}(X) = A\cdot X\cdot B$$
$$f_{B,A} : \mathcal{M}_m(\mathbb{C}) \longrightarrow \mathcal{M}_n(\mathbb{C}), \quad f_{B,A}(Y) = B\cdot Y\cdot A.$$
Prove that $f_{A,B}$ is surjective (onto) if and only if $f_{B,A}$ is injective (one-to-one).

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania.

Solution by the author. We prove the equivalence of the two properties
$$P_1 : f_{A,B} \text{ is surjective}$$
$$P_2 : f_{B,A} \text{ is injective}$$
via a third equivalent property:
$$P_3 : \operatorname{rank} A = \operatorname{rank} B = m.$$

$\boxed{P_1 \Rightarrow P_3}$ Let $C \in \mathcal{M}_n(\mathbb{C})$ such that $f_{A,B}(C) = I_m$. Then
$$m = \operatorname{rank} I_m = \operatorname{rank}(A\cdot C\cdot B) \le \min\{\operatorname{rank} A, \operatorname{rank} B\},$$
hence P_3.

$\boxed{P_3 \Rightarrow P_1}$ Let $D \in \mathcal{M}_m(\mathbb{C})$. We have to show that the equation $A\cdot X\cdot B = D$ has solution $X \in \mathcal{M}_n(\mathbb{C})$.

First, consider the equation
$$A\cdot Y = D, \qquad Y \in \mathcal{M}_{n,m}(\mathbb{C}). \tag{1}$$
Writing $D = [D_1|D_2|\dots|D_m]$ (D_i is the i-th column of D), the equation (1) is equivalent to m linear systems
$$A\cdot Y_i = D_i \quad (i = 1, 2, \dots, m),$$
where $\{Y_i : i = 1, 2, \dots, m\}$ are the columns of Y. As $\operatorname{rank} A = m$, it follows that $\operatorname{rank}[A|D_i] = m$ for all $i = 1, 2, \dots, m$, hence every linear system has solution $Y_i \in \mathcal{M}_{m,1}(\mathbb{C})$. Concluding, the equation (1) has solution.

Next, for Y satisfying (1), consider the equation

$$X \cdot B = Y, \qquad X \in \mathcal{M}_n(\mathbb{C}), \tag{2}$$

which is equivalent to $B^t \cdot X^t = Y^t$. Since $\operatorname{rank} B^t = m$, it follows by the same argument applied for (1) that there exists $Z \in \mathcal{M}_n(\mathbb{C})$ such that $B^t \cdot Z = Y^t$, hence $X = Z^t$ is a solution of (2).

Concluding,

$$A \cdot X \cdot B = A \cdot Y = D.$$

$\boxed{P_3 \Rightarrow P_2}$ We have the equivalences

$$f_{B,A}(Y_1) = f_{B,A}(Y_2) \Leftrightarrow B \cdot (Y_1 - Y_2) \cdot A = O_n \Leftrightarrow B \cdot Y \cdot A = O_n, Y = Y_1 - Y_2,$$

hence $f_{B,A}$ is injective if and only if the equation

$$B \cdot Y \cdot A = O_n, \qquad Y \in \mathcal{M}_m(\mathbb{C}) \tag{3}$$

has a unique solution $Y = O_m$.

Consider first the equation

$$B \cdot U = O_n, \qquad U = [U_1|U_2|\ldots|U_n] \in \mathcal{M}_{m,n}(\mathbb{C}) \tag{4}$$

which is equivalent to n identical homogeneous linear systems

$$B \cdot U_i = O_{n,1}, \qquad U_i \in \mathcal{M}_{m,1}(\mathbb{C}),\ i = 1, 2, \ldots, m.$$

Since the number of unknowns for each of the linear systems is the same as the rank of B, it follows that the only solution is the trivial one, hence the only solution to (4) is $U = O_{m,n}$. Returning to (3), it follows that $Y \cdot A = O_{m,n}$, hence

$$A^t \cdot Z = O_{n,m}, \qquad Z = [Z_1|Z_2|\ldots|Z_m] = Y^t \in \mathcal{M}_m(\mathbb{C}). \tag{5}$$

Similarly, (5) is equivalent to m identical homogeneous linear systems

$$A^t \cdot Z_i = O_{n,1}, \qquad Z_i \in \mathcal{M}_{m,1}(\mathbb{C}),\ i = 1, 2, \ldots, m,$$

each of them having only the trivial solution, since $\operatorname{rank} A = m$, which is the number of unknowns in each of the systems. Concluding, $Z = O_m$, hence $Y = O_m$ is the only solution of (3).

$\boxed{P_2 \Rightarrow P_3}$ Assume, by contradiction, that $\operatorname{rank} A \neq m$, hence $\operatorname{rank} A^t < m$. Using the same reasoning from the previous implication, it follows that (5) has a non-trivial solution $Z \in \mathcal{M}_m(\mathbb{C})$, $Z \neq O_m$, hence there exists $Y = Z^t \in \mathcal{M}_m(\mathbb{C})$, $Y \neq O_m$ a non-trivial solution to (3):

$$B \cdot Y \cdot A = B \cdot \left(A^t \cdot Z\right)^t = B \cdot O_{m,n} = O_n.$$

This contradicts that $f_{B,A}$ is injective. Concluding, $\operatorname{rank} A = m$.

Similarly, assuming next that $\operatorname{rank} B < m$, then (4) has a non-trivial solution $U \in \mathcal{M}_{m,n}(\mathbb{C})$, $U \neq O_{m,n}$. Since $\operatorname{rank} A^t = m$, it follows (using the same argument used in $P_3 \Rightarrow P_1$) that there exists $Z \in \mathcal{M}_m(\mathbb{C})$ such that

$A^t \cdot Z = U^t$ and $Z \neq O_m$ since $U \neq O_{m,n}$. Then again $Y = Z^t$ is a non-trivial solution to (3):

$$B \cdot Y \cdot A = B \cdot \left(A^t \cdot Z\right)^t = B \cdot U = O_n$$

which contradicts that $f_{B,A}$ is injective. Concluding, $\operatorname{rank} B = m$. □

Solution by Cornel Băețica. Let $r = \operatorname{rank} A$ and $s = \operatorname{rank} B$. There exist invertible matrices $U, Q \in \mathcal{M}_m(\mathbb{C})$ and $V, P \in \mathcal{M}_n(\mathbb{C})$ such that

$$UAV = \left(\begin{array}{c|c} I_r & 0_{r,n-r} \\ \hline 0_{m-r,r} & 0_{m-r,n-r} \end{array}\right) \text{ and } PBQ = \left(\begin{array}{c|c} I_s & 0_{s,m-s} \\ \hline 0_{n-s,s} & 0_{n-s,m-s} \end{array}\right).$$

It is easily seen that $f_{A,B}$ is surjective if and only if $f_{UAV,PBQ}$ is surjective, and $f_{B,A}$ is injective if and only if $f_{PBQ,UAV}$ is injective. We have

$$f_{UAV,PBQ}(X) = \left(\begin{array}{c|c} X_{r,s} & 0_{r,m-s} \\ \hline 0_{m-r,s} & 0_{m-r,m-s} \end{array}\right),$$

where $X_{r,s}$ is the submatrix of X obtained by deleting the last $n-r$ rows and $n-s$ columns, and

$$f_{PBQ,UAV}(Y) = \left(\begin{array}{c|c} Y_{s,r} & 0_{s,n-r} \\ \hline 0_{n-s,r} & 0_{n-s,n-r} \end{array}\right),$$

where $Y_{s,r}$ is the submatrix of Y obtained by deleting the last $m-s$ rows and $m-r$ columns.

Now it is obvious that $f_{UAV,PBQ}$ is surjective if and only if $r = s = m$, and $f_{PBQ,UAV}$ is injective if and only if $r = s = m$. □

490. Let $n \in \mathbb{N}^*$. Calculate

$$\int_0^1 \left(\frac{\ln(1-x) + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n}}{x}\right)^2 \mathrm{d}x.$$

`Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

Solution by the authors. The integral equals $\frac{2}{n}(H_{2n} - H_n)$, where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the nth harmonic number.

Let I_n be the integral we want to calculate. Since $\ln(1-x)+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n}=-\sum_{i=n+1}^{\infty}\frac{x^i}{i}$, we have

$$\begin{aligned}
I_n &= \int_0^1 \frac{\left(-\sum\limits_{i=n+1}^{\infty}\frac{x^i}{i}\right)\cdot\left(-\sum\limits_{j=n+1}^{\infty}\frac{x^j}{j}\right)}{x^2}\mathrm{d}x = \sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty}\frac{1}{ij}\int_0^1 x^{i+j-2}\mathrm{d}x \\
&= \sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty}\frac{1}{ij(i+j-1)} = \sum_{i=n+1}^{\infty}\frac{1}{i}\sum_{j=n+1}^{\infty}\frac{1}{j(j+i-1)} \\
&= \sum_{i=n+1}^{\infty}\frac{1}{i}\cdot\frac{1}{i-1}\sum_{j=n+1}^{\infty}\left(\frac{1}{j}-\frac{1}{j+i-1}\right) \\
&= \sum_{i=n+1}^{\infty}\frac{1}{i(i-1)}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+i-1}\right).
\end{aligned}$$

We calculate the preceding sum by observing that the series telescopes.

We have

$$\begin{aligned}
&\frac{1}{i(i-1)}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+i-1}\right) \\
&= \frac{\frac{1}{n+1}+\cdots+\frac{1}{n+i-1}}{i-1}-\frac{\frac{1}{n+1}+\cdots+\frac{1}{n+i-1}}{i} \\
&= \frac{\frac{1}{n+1}+\cdots+\frac{1}{n+i-1}}{i-1}-\frac{\frac{1}{n+1}+\cdots+\frac{1}{n+i}}{i}+\frac{1}{i(n+i)} \\
&= \frac{\frac{1}{n+1}+\cdots+\frac{1}{n+i-1}}{i-1}-\frac{\frac{1}{n+1}+\cdots+\frac{1}{n+i}}{i}+\frac{1}{n}\left(\frac{1}{i}-\frac{1}{n+i}\right)
\end{aligned}$$

and it follows that

$$\begin{aligned}
I_n &= \sum_{i=n+1}^{\infty}\left[\frac{\frac{1}{n+1}+\cdots+\frac{1}{n+i-1}}{i-1}-\frac{\frac{1}{n+1}+\cdots+\frac{1}{n+i}}{i}+\frac{1}{n}\left(\frac{1}{i}-\frac{1}{n+i}\right)\right] \\
&= \frac{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}}{n}+\frac{1}{n}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right) \\
&= \frac{2}{n}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right) \\
&= \frac{2}{n}\left(H_{2n}-H_n\right).
\end{aligned}$$

The problem is solved. $\square$

Solution by Burghelea Zaharia, Sibiu, România. The integral equals $\frac{2\bar{H}_{2n}}{n}$, where $\bar{H}_{2n} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}$ denotes the nth alternating harmonic number. We will use the Taylor series expansion of the logarithm, $\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$, in order to write the numerator of the integrand as $\sum_{k=n+1}^{\infty} \frac{x^k}{k}$. But first we use integration by parts using $(1-\frac{1}{x})' = \frac{1}{x^2}$. We use $1-\frac{1}{x}$, rather than $-\frac{1}{x}$, in order to avoid divergence issues. More precisely, the limits both when $x \searrow 0$ and $x \nearrow 1$ of $(1-\frac{1}{x})(\ln(1-x) + \sum_{k=1}^{n} \frac{x^k}{k})^2$ are zero. This happens because $\lim_{x\searrow 0} \frac{1}{x} \ln^2(1-x) = 0$ and $\lim_{x\nearrow 1}(1-x)\ln^2(1-x) = 0$. Putting

$$I := \int_0^1 \left(\frac{\ln(1-x) + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n}}{x} \right) \mathrm{d}x,$$

we have

$$\begin{aligned}
I &= \int_0^1 \left(1-\frac{1}{x}\right)' \left(\ln(1-x) + \sum_{k=1}^{n} \frac{x^k}{k}\right)^2 \mathrm{d}x \\
&= -\int_0^1 \left(1-\frac{1}{x}\right) \cdot 2 \left(\sum_{k=n+1}^{\infty} \frac{x^k}{k}\right)' \left(\sum_{k=n+1}^{\infty} \frac{x^k}{k}\right) \mathrm{d}x \\
&= 2\int_0^1 \left(\frac{1}{x}-1\right) \left(\sum_{j=n}^{\infty} x^j\right) \left(\sum_{k=n+1}^{\infty} \frac{x^k}{k}\right) \mathrm{d}x \\
&= 2\sum_{k=n+1}^{\infty} \frac{1}{k} \sum_{j=n}^{\infty} \int_0^1 (x^{k+j-1} - x^{k+j})\mathrm{d}x \\
&= 2\sum_{k=n+1}^{\infty} \frac{1}{k} \sum_{j=n}^{\infty} \left(\frac{1}{k+j} - \frac{1}{k+j+1}\right) \\
&= 2\sum_{k=n+1}^{\infty} \frac{1}{k} \cdot \frac{1}{k+n} = \frac{2}{n} \sum_{k=n+1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+n}\right) = \frac{2}{n} \sum_{i=n+1}^{2n} \frac{1}{i} \\
&= \frac{2}{n}\left(\sum_{i=1}^{2n} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{i}\right) = \frac{2(H_{2n} - H_n)}{n} = \frac{2\bar{H}_{2n}}{n}.
\end{aligned}$$

Solution by Moti Levy, Rehovot, Israel. We use induction on n. Let I_n be the integral we want to calculate. Then

$$\begin{aligned}
I_{n+1} &= \int_0^1 \left(\frac{\ln(1-x) + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n}}{x} + \frac{x^n}{n+1} \right)^2 \mathrm{d}x \\
&= \int_0^1 \left(\frac{\ln(1-x) + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n}}{x} \right)^2 \mathrm{d}x \\
&\quad + 2\int_0^1 \frac{x^n}{n+1} \frac{\ln(1-x) + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n}}{x} \mathrm{d}x + \int_0^1 \frac{x^{2n}}{(n+1)^2} \mathrm{d}x \\
&= I_n + \frac{2}{n+1} \int_0^1 x^{n-1} \ln(1-x)\, \mathrm{d}x + \int_0^1 \frac{x^{2n}}{(n+1)^2} \mathrm{d}x \\
&\quad + \frac{2}{n+1} \int_0^1 \left(x^n + \frac{x^{n+1}}{2} + \cdots + \frac{x^{2n-1}}{n} \right) \mathrm{d}x
\end{aligned}$$

and

$$\int_0^1 x^{n-1} \ln(1-x)\, \mathrm{d}x = -\frac{H_n}{n},$$

so

$$I_{n+1} - I_n = -\frac{2H_n}{n(n+1)} + \frac{1}{(n+1)^2 (2n+1)} + \frac{2}{n+1} \sum_{k=1}^{n} \frac{1}{k(k+n)}. \tag{23}$$

But

$$\sum_{k=1}^{n} \frac{1}{k(k+n)} = \frac{1}{n} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+n} \right) = \frac{1}{n} (2H_n - H_{2n}). \tag{24}$$

Substituting (24) in (23) we get,

$$\begin{aligned}
I_{n+1} - I_n &= -\frac{2H_n}{n(n+1)} + \frac{1}{(n+1)^2 (2n+1)} + \frac{4H_n}{n(n+1)} - \frac{2H_{2n}}{n(n+1)} \\
&= \frac{2H_n}{n(n+1)} - \frac{2H_{2n}}{n(n+1)} + \frac{1}{(n+1)^2 (2n+1)} \\
&= \frac{2}{n} H_n - \frac{2}{n+1} H_n - \frac{2}{n} H_{2n} + \frac{2}{n+1} H_{2n} + \frac{1}{(n+1)^2 (2n+1)} \\
&= \frac{2}{n+1} \left(H_{2n+2} - \frac{1}{2n+1} - \frac{1}{2n+2} \right) - \frac{2}{n+1} \left(H_{n+1} - \frac{1}{n+1} \right) \\
&\quad - \frac{2}{n} H_{2n} + \frac{2}{n} H_n + \frac{1}{(n+1)^2 (2n+1)} \\
&= \frac{2}{n+1} (H_{2n+2} - H_{n+1}) - \frac{2}{n} (H_{2n} - H_n).
\end{aligned}$$

Now we evaluate $I_1 = \int_0^1 \left(\frac{\ln(1-x)+x}{x}\right)^2 \mathrm{d}x$, and verify that it is equal to $\frac{2}{1}(H_2 - H_1) = 1$:

We have the following definite integrals: $\int_0^1 \frac{\ln(1-x)}{x}\mathrm{d}x = -\zeta(2)$ and $\int_0^1 \frac{\ln^2(1-x)}{x^2}\mathrm{d}x = 2\zeta(2)$, hence

$$I_1 = \int_0^1 \left(\frac{\ln(1-x)+x}{x}\right)^2 \mathrm{d}x = \int_0^1 \left(1 + \frac{2}{x}\ln(1-x) + \frac{1}{x^2}\ln^2(1-x)\right)\mathrm{d}x$$
$$= 1.$$

We conclude by mathematical induction that

$$I_n = \frac{2}{n}(H_{2n} - H_n).$$

Note from the editor. Moti Levy uses, without reference, some results that are not very well known. The relation $\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x = -\frac{H_n}{n}$ can be deduced by partial integration. We note that $x^{n-1} = \left(\frac{x^n-1}{n}\right)'$ and $\lim_{x\nearrow 1}(x-1)\ln(1-x) = 0$, so

$$\begin{aligned}\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x &= \frac{x^n-1}{n}\ln(1-x)\Big|_0^1 - \int_0^1 \frac{x^n-1}{n}\cdot\frac{1}{x-1}\mathrm{d}x\\ &= -\frac{1}{n}\int_0^1 (1+x+\cdots+x^{n-1})\mathrm{d}x\\ &= -\frac{1}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right).\end{aligned}$$

The relation $\int_0^1 \frac{\ln(1-x)}{x}\mathrm{d}x = -\zeta(2)$ follows (after some discussion on convergence) by integrating each term of the sum $\frac{\ln(1-x)}{x} = 1 + \frac{x}{2} + \frac{x^2}{3} + \cdots$. Also $\int_0^1 \frac{\ln^2(1-x)}{x^2}\mathrm{d}x = 2\zeta(2)$ follows from $\int_0^1 \frac{\ln^2(1-x)}{x^2}\mathrm{d}x = -2\int_0^1 \frac{\ln(1-x)}{x}\mathrm{d}x$, which is proved by integration by parts. We have $\frac{1}{x^2} = \left(1-\frac{1}{x}\right)'$, $\lim_{x\searrow 0}\frac{1}{x}\ln^2(1-x) = 0$ and $\lim_{x\nearrow 1}(x-1)\ln^2(1-x) = 0$, so

$$\begin{aligned}\int_0^1 \frac{\ln^2(1-x)}{x^2}\mathrm{d}x &= \left(1-\frac{1}{x}\right)\ln^2(1-x)\Big|_0^1 - \int_0^1 \left(1-\frac{1}{x}\right)\frac{2}{x-1}\ln(1-x)\mathrm{d}x\\ &= -2\int_0^1 \frac{\ln(1-x)}{x}\mathrm{d}x.\end{aligned}$$

491. If the arithmetic mean of $a, b, c, d \geq 0$ is 1, then their quadratic mean $q = \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$ takes values in the interval $[1, 2]$.

If $q \in [1, 2]$ then we denote by $M = M_q$ the largest possible value of the geometric mean of four numbers $a, b, c, d \geq 0$ with the arithmetic mean 1 and the quadratic mean q.

Determine M in terms of q and prove that $M + q \geq 2$.

Proposed by Leonard Giugiuc, Traian National College, Drobeta Turnu Severin, Romania and Alexander Bogomolny, New Jersey, USA.

Solution by the authors. The quadratic mean q is $\geq$ than the arithmetic mean 1. For the inequality $q \leq 2$ note that if $a, b, c, d \geq 0$ with $a+b+c+d = 4$ the $a^2 + b^2 \leq a^2 + b^2 + 2ab = (a + b)^2 + 0^2$. Then $(a + b) + 0 + c + d = 4$ and $(a + b)^2 + 0^2 + c^2 + d^2 \geq a^2 + b^2 + c^2 + d^2$. Hence, by replacing a, b with $a+b, 0$, the sum $a+b+c+d = 4$ is preserved, but $a^2 + b^2 + c^2 + d^2$ increases. We repeat the procedure for the variables a and c and then for a and d. We get that $a^2 + b^2 + c^2 + d^2$, and so q, is maximal when $b = c = d = 0$ and so $a = 4$. Thus the maximal value of q is $\sqrt{\frac{4^2+0^2+0^2+0^2}{4}} = 2$.

Let $t = \sqrt{\frac{q^2-1}{3}}$, so that $q = \sqrt{1 + 3t^2}$. Then $q \in [1, 2]$ is equivalent to $t \in [0, 1]$. We have $a + b + c + d = 4$ and $a^2 + b^2 + c^2 + d^2 = 4q^2 = 4(1 + 3t^2)$. It follows that $ab + ac + ad + bc + bd + cd = \frac{1}{2}(4^2 - 4(1 + 3t^2)) = 6(1 - t^2)$.

Lemma. *With the notation above, the maximal value of abcd is* $(1 - t)^3(1 + 3t)$ *and it is attained when* (a, b, c, d) *is a permutation of* $(1 - t, 1 - t, 1 - t, 1 + 3t)$.

Proof. First note that if $a = b = c = 1 - t$ and $d = 1 + 3t$, then $\frac{a+b+c+d}{4} = 1$ and $\sqrt{\frac{a^2+b^2+c^2+d^2}{4}} = \sqrt{\frac{4+12t^2}{4}} = \sqrt{1 + 3t^2} = q$. So a, b, c, d satisfy the required properties and $abcd = (1 - t)^3(1 + 3t)$.

If $t = 1$, then $ab + ac + ad + bc + bd + cd = 6(1 - t^2) = 0$, so all products ab, ac, ad, bc, bd, cd must be 0. Hence all but one of a, b, c, d are 0. It follows that the maximal value of $abcd$ is $0 = (1 - t)^3(1 + 3t)$, as claimed.

Assume that $t < 1$. Then $(1 - t)^2(1 + 3t) > 0$, so we are looking for a positive maximal value of $abcd$, which holds when $a, b, c, d > 0$.

Let $P \in \mathbb{R}[X]$, $P(x) = (x - a)(x - b)(x - c)(x - d)$. We have

$$P(x) = x^4 - 4x^3 + 6(1-t^2) - mx + p, \text{ where } m = abc+abd+acd+bcd,\ p = abcd.$$

Let $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{P(x)}{x} = x^3 - 4x^2 + 6(1-t^2)x - m + px^{-1}$. By Rolle's theorem, since P, so f, has four positive roots, f' has at least three positive roots. (Here we count multiplicities of the roots.) But $f'(x) = \frac{g(x)}{x^2}$, where

$$g(x) = 3x^4 - 8x^3 + 6(1 - t^2)x^2 - p.$$

It follows that g has at least three positive roots. Since the product of all roots is $-p < 0$, the fourth root is negative. We denote by x_1 the negative root of g and by $x_2 \leq x_3 \leq x_4$ the positive ones. We have

$$g'(x) = 12x^3 - 24x^2 + 12(1-t^2)x = 12x(x^2-2x+1-t^2) = 12x(x-1-t)(x-1+t)$$

with the positive solutions $1 + t$ and $1 - t$. It follows that $x_1 < 0 < x_2 \leq 1 - t \leq x_3 \leq 1 + t \leq x_4$.

If we take $x = 1 - t$, then $x - x_1, x - x_2 \geq 0$ and $x - x_3, x - x_4 \leq 0$. Hence $g(x) = 3(x - x_1)(x - x_2)(x - x_3)(x - x_4) \geq 0$. We have

$$0 \leq g(1-t) = 3(1-t)^4 - 8(1-t)^3 + 6(1-t^2)(1-t)^2 - p = (1-t)^3(1+3t) - p$$

so $abcd = p \leq (1-t)^3(1+3t)$. This proves that $\max abcd = (1-t)^3(1+3t)$, with equality when $(a,b,c,d) = (1-t, 1-t, 1-t, 1+3t)$ or the permutations. $\square$

Hence for every $t \in [0,1]$ we have $M = \sqrt[4]{(1-t)^3(1+3t)}$. It follows that $M + q = h(t)$, where $h : [0,1] \to \mathbb{R}$, $h(t) = \sqrt[4]{(1-t)^3(1+3t)} + \sqrt{1+3t^2}$. Now h is continuous on $[0,1]$ and differentiable on $[0,1)$. Since $((1-t)^3(1+3t))' = -12t(1-t)^2$, we get

$$\begin{aligned} g'(t) &= \frac{-12t(1-t)^2}{4\sqrt[4]{(1-t)^9(1+3t)^3}} + \frac{6t}{2\sqrt{1+3t^2}} = \frac{-3t}{\sqrt[4]{(1-t)(1+3t)^3}} + \frac{3t}{\sqrt{1+3t^2}} \\ &= \frac{3t}{\sqrt{1+3t}\sqrt[4]{(1-t)(1+3t)^3}}\left(\sqrt[4]{(1-t)(1+3t)^3} - \sqrt{1+3t}\right). \end{aligned}$$

For every $t \in (0,1)$ the sign of $h'(t)$ coincides with that of $(1-t)(1+3t)^3 - (1+3t)^2)^2 = -4t(9t^3 - 3t - 2)$, so with that of $-(9t^3 - 3t - 2)$.

Let $Q(t) = 9t^3 - 3t - 2$. We have $Q'(t) = 3(9t^2 - 1)$. Then $Q'(t) < 0$ for $t \in (0, 1/3)$ and $Q'(t) > 0$ for $t \in (1/3, 1)$. It follows that Q is decreasing on $[0, 1/3]$ and incresing on $[1/3, 1]$. Since $Q(0) = -2 < 0$ and $Q(1) = 4 > 0$, there is a unique $t_0 \in [0,1]$ with $Q(t_0) = 0$. More precisely, $t_0 \in (1/3, 1)$ and we have $Q(t) < 0$ for $t \in (0, t_0)$ and $Q(t) > 0$ for $t \in (t_0, 1)$. Form this we conclude that $h'(t) > 0$ for $t \in (0, t_0)$ and $h'(t) > 0$ for $t \in (t_0, 1)$. It follows that h is strictly increasing on $[0, t_0]$ and strictly decreasing on $[t_0, 1]$. Then we have $\min_{t \in [0,1]} h(t) = \min\{h(0), h(1)\} = 2$. (We have $h(0) = h(1) = 2$.) Hence $M_q + q \leq 2$, with equality if and only if $t \in \{0, 1\}$, i.e., if and only if $q \in \{1, 2\}$. $\square$

492. Let V be a vector space over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ and let $f : V \to \mathbb{R} \cup \{\infty\}$ satisfying $f(x) = \infty$ iff $x = 0$ and

$$f(x+y) \geq \min\{f(x), f(y)\} \quad \forall x, y \in V.$$

For every $c \in V$ we define $g_c : V \to \mathbb{R} \cup \{\infty\}$ by $g_c(x) = f(x) + f(x+c)$.

(i) Prove that g_c satisfies the same inequality as f, viz.,

$$g_c(x+y) \geq \min\{g_c(x), g_c(y)\} \quad \forall x, y \in V.$$

Equivalently, if $x, y, z, t \in V$ with $x + y = z + t$ then

$$f(x+z) + f(x+t) \geq \min\{f(x) + f(y), f(z) + f(t)\}.$$

For any $a, b \in V$ we define $h_{a,b} : V \to \mathbb{R} \cup \{\infty\}$ by formula $h_{a,b}(x) = f(x) + f(x+a) + f(x+b)$.

(ii) If $x, y, a, b \in V$ such that $f(x) \leq f(y)$ prove that

$$h_{x,x+a+b}(y) \geq \min\{h_{a,b}(x), h_{a,b}(y)\}.$$

Let $k : V^2 \to \mathbb{R} \cup \{\infty\}$, $k(x, y) = f(x) + f(y) + f(x + y)$.

(iii) If $a, b, x, y \in V$ prove that

$$h_{a,b}(x + y) \geq \min\{h_{a,b}(x), h_{a,b}(y), k(x, y)\}$$

and

$$k(x, y) \geq \min\{h_{a,b}(x), h_{a,b}(y), h_{a,b}(x + y)\}.$$

Conclude that none of the four numbers $h_{a,b}(x)$, $h_{a,b}(y)$, $h_{a,b}(x + y)$ and $k(x, y)$ is strictly smaller than all remaining three numbers.

(iv) If $a, b, x, y, z \in V$ prove that

$$\max\{h_{y,z}(x), h_{z,x}(y), h_{x,y}(z)\} \geq \min\{h_{a,b}(x), h_{a,b}(y), h_{a,b}(z)\}.$$

`Proposed by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.`

Solution by the author. First note that $2x = 0$ $\forall x \in V$, as V is an $\mathbb{F}_2$-vector space.

We also have the following property, called the *domination principle*:

$$\text{If } f(x) < f(y) \text{ then } f(x + y) = f(x).$$

Indeed, we have $f(x + y) \geq \min\{f(x), f(y)\}$ and $x = (x + y) + y$, so $f(x) \geq \min\{f(x + y), f(y)\}$. But $f(x) < f(y)$, so the first inequality implies $f(x + y) \geq f(x)$ and the second one implies $f(x) \geq f(x+y)$. Hence $f(x+y) = f(x)$.

(i) Note that $g_c(x+c) = f(x+c)+f(x+c+c) = f(x+c)+f(x) = g_c(x)$.

Let $m = \min\{g_c(x), g_c(y)\}$. We must prove that $g_c(x+y) \geq m$. Suppose that $g_c(x + y) < m$.

Without loss of generality, we can assume that $f(x) \leq f(y)$. Then $f(x + y) \geq \min\{f(x), f(y)\} = f(x)$. Since also $f(x + y) + f(x + y + c) = g_c(x+y) < g_c(x) = f(x)+f(x+c)$, we must have $f(x+y+c) < f(x+c)$. Since $(x+y+c)+(x+c)$, by the domination principle, we get $f(x+y+c) = f(y)$. We have $g_c(x)+g_c(y) \geq 2m$, i.e., $(f(x)+f(y))+(f(x+c)+f(y+c)) \geq 2m$. It follows that at least one of the sums $f(x)+f(y)$ and $f(x+c)+f(y+c)$ is $\geq m$. If $f(x)+f(y) \geq m$ then $g_c(x+y) = f(x+y)+f(x+y+c) \geq f(x)+f(y) \geq m$ and we are done.

Assume now that $f(x+c)+f(y+c) \geq m$. Let $x' = x+c$, $y' = y+c$. We have $g_c(x') = g_c(x)$ and $g_c(y') = g_c(y)$, so $\min\{g_c(x'), g_c(y')\} = m$. Since also $f(x') + f(y') \geq m$, by the same reasoning as above, we have $g_c(x' + y') \geq m$. But $x' + y' = (x + c) + (y + c) = x + y$, so $g_c(x + y) \geq m$, as claimed.

For the equivalence with the second claim, note that we have a bijection $\alpha : V^3 \to \{(x, y, z, t) \in V^4 \mid x+y = z+t\}$ given by $(x, y, c) \mapsto (x, x+c, y, y+c)$. Its inverse is $\beta : \{(x, y, z, t) \in V^4 \mid x + y = z + t\} \to V^3$, given by $(x, y, z, t) \mapsto (x, z, x + y)$. (We have $\beta \circ \alpha(x, y, c) = (x, y, x + (x + c)) = (x, y, c)$ and $\alpha \circ \beta(x, y, z, t) = (x, x + (x + y), z, z + (x + y)) = (x, y, z, t)$, since $z+(x+y)=z+(z+t)=t$.) So $\{(x,y,z,t)\in V^4 \mid x+y=z+t\} = \{(x,x+c,y,y+c)\mid x,y,c\in V\}$ Therefore the statement $f(x+z)+f(x+t)\geq \min\{f(x)+f(y),f(z)+f(t)\}$ $\forall x,y,z,t\in V$ with $x+y=z+t$ is equivalent to $f(x+y)+f(x+(y+c))\geq \min\{f(x)+f(x+c),f(y)+f(y+c)\}$, i.e., $g_c(x+y)\geq \min\{g_c(x),g_c(y)\}$ $\forall x,y,c\in V$.

(ii) We have $h_{x,x+a+b}(y)=f(y)+f(x+y)+f(x+y+a+b)$.

Let $x'=x+a$, $y'=x+b$, $z'=y+a$, $t'=y+b$. We have $x'+y'=z'+t'$, so by (i) we have $f(x'+z')+f(x'+t')\geq \min\{f(x')+f(y'),f(z')+f(t')\}$. But $x'+z'=(x+a)+(y+a)=x+y$ and $x'+t'=(x+a)+(y+b)=x+y+a+b$. Hence

$$f(x+y)+f(x+y+a+b)\geq \min\{f(x+a)+f(x+b),f(y+a)+f(y+b)\}.$$

Thus $f(x+y)+f(x+y+a+b)\geq f(x+a)+f(x+b)$ or $f(x+y)+f(x+y+a+b)\geq f(y+a)+f(y+b)$. In the first case, since also $f(y)\geq f(x)$, we have $f(y)+f(x+y)+f(x+y+a+b)\geq f(x)+f(x+a)+f(x+b)$, i.e., $h_{x,x+a+b}(y)\geq h_{a,b}(y)$. In the second case we have $f(y)+f(x+y)+f(x+y+a+b)\geq f(y)+f(y+a)+f(y+b)$, i.e., $h_{x,x+a+b}(y)\geq h_{a,b}(x)$. In both cases we get the claimed inequality.

(iii) We denote $m=\min\{h_{a,b}(x),h_{a,b}(y),k(x,y)\}$. We must prove that $h_{a,b}(x+y)\geq m$. Suppose that $h_{a,b}(x+y)<m$.

Note that $h_{a,b}(x)=g_a(x)+f(x+b)$. By (i) we have $g_a(x+y)\geq \min\{g_a(x),g_a(y)\}$. Without loss of generality, we may assume that $g_a(x)\leq g_a(y)$, which implies $g_a(x+y)\geq g_a(x)$. But we also have $g_a(x+y)+f(x+y+b)=h_{a,b}(x+y)<m\leq h_{a,b}(x)=g_a(x)+f(x+b)$. It follows that $f(x+y+b)<f(x+b)$. By the domination principle, this implies that $f(x+y+b)=f(x+y+b+x+b)=f(y)$.

Next we see that $k(x,y)\geq m>h_{a,b}(x+y)$, i.e., $f(x)+f(y)+f(x+y)>f(x+y)+f(x+y+a)+f(x+y+b)=f(x+y)+f(x+y+a)+f(y)$. It follows that $f(x)>f(x+y+a)$. By the domination principle, we get $f(x+y+a)=f(x+y+a+x)=f(y+a)$.

In conclusion, $h_{a,b}(x+y)=f(x+y)+f(x+y+a)+f(x+y+b)=f(x+y)+f(y+a)+f(y)=g_a(y)+f(x+y)$. Now $x+y=(x+b)+(y+b)$, so $f(x+y)\geq \min\{f(x+b),f(y+b)\}$. If $f(x+y)\geq f(y+b)$, then $h_{a,b}(x+y)=g_a(y)+f(x+y)\geq g_a(y)+f(y+b)=h_{a,b}(y)\geq m$ and we are done. If $f(x+y)\geq f(x+b)$, then we also have $g_a(y)\geq g_a(x)$, so $h_{a,b}(x+y)=g_a(y)+f(x+y)\geq g_a(x)+f(x+b)=h_{a,b}(x)\geq m$ and again we are done.

For the second inequality, let $m=\min\{h_{a,b}(x),h_{a,b}(y),h_{a,b}(x+y)\}$. We must prove that $f(x)+f(y)+f(x+y)=k(x,y)\geq m$.

Without loss of generality, we may assume that $f(x)\leq f(y)$. Then $f(x+y)\geq \min\{f(x),f(y)\}=f(x)$. Since $f(x)\leq f(y)$ and $f(x)\leq f(x+y)$, by (ii) we get

$$f(y)+f(x+y)+f(x+y+a+b)=h_{x,x+a+b}(y)\geq \min\{h_{a,b}(x),h_{a,b}(y)\}\geq m$$

and

$$f(x+y)+f(y)+f(y+a+b)=h_{x,x+a+b}(x+y)\geq \min\{h_{a,b}(x),h_{a,b}(x+y)\}\geq m.$$

Now $(x+y+a+b)+(y+a+b)=x$, so that $f(x)\geq \min\{f(x+y+a+b),f(y+a+b)\}$, i.e., $f(x)\geq f(x+y+a+b)$ or $f(x)\geq f(y+a+b)$. In the first case we have $k(x,y)=f(x)+f(y)+f(x+y)\geq f(y)+f(x+y)+f(x+y+a+b)\geq m$. In the second case, $k(x,y)=f(x)+f(y)+f(x+y)\geq f(x+y)+f(y)+f(y+a+b)\geq m$. In both cases we are done.

For the last statement, we have already proved that each of $h_{a,b}(x+y)$ and $k(x,y)$ is greater than or equal to the minimum of the remaining three terms of the sequence $h_{a,b}(x)$, $h_{a,b}(y)$, $h_{a,b}(x+y)$ and $k(x,y)$ is. For $h_{a,b}(y)$ let $x'=x$, $y'=x+y$. Then $x'+y'=y$ and $k(x',y')=f(x)+f(x+y)+f(y)=k(x,y)$. Therefore the first inequality of (iii), applied to the pair (x',y'), yields

$$h_{a,b}(y)\geq \min\{h_{a,b}(x),h_{a,b}(x+y),k(x,y)\}.$$

The similar inequality for $h_{a,b}(x)$ follows by permuting x and y.

(iv) Let $m=\min\{h_{a,b}(x),h_{a,b}(y),h_{a,b}(z)\}$. We must prove that at least one of $h_{y,z}(x)$, $h_{z,x}(y)$ and $h_{x,y}(z)$ is $\geq m$.

Without loss of generality, we may assume that $f(x)\leq f(y),f(z)$. By (ii) we have

$$f(y)+f(x+y)+f(x+y+a+b)=h_{x,x+a+b}(y)\geq \min\{h_{a,b}(x),h_{a,b}(y)\}\geq m$$

and

$$f(z)+f(x+z)+f(x+z+a+b)=h_{x,x+a+b}(z)\geq \min\{h_{a,b}(x),h_{a,b}(z)\}\geq m$$

But $(x+y+a+b)+(x+z+a+b)=y+z$, so $f(y+z)\geq \min\{f(x+y+a+b),f(x+z+a+b)\}$. Then we have $f(y+z)\geq f(x+y+a+b)$, which implies $h_{z,x}(y)=f(y)+f(y+z)+f(x+y)\geq f(y)+f(x+y)+f(x+y+a+b)\geq m$, or $f(y+z)\geq f(x+z+a+b)$, which implies $h_{x,y}(z)=f(z)+f(x+z)+f(y+z)\geq f(z)+f(x+z)+f(x+z+a+b)\geq m$. In both cases we are done. □

493. (a) Calculate

$$\lim_{n\to\infty} n\int_0^\infty \frac{\sin x}{\mathrm{e}^{(n+1)x}-\mathrm{e}^{nx}}\mathrm{d}x.$$

(b) Let $k>-1$ be a real number. Calculate

$$\lim_{n\to\infty} n^{k+1}\int_0^\infty \frac{x^k\sin x}{\mathrm{e}^{(n+1)x}-\mathrm{e}^{nx}}\mathrm{d}x.$$

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Romania.

Solution by the authors. (a) The limit equals 1. We will be using in our calculations the formula

$$\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}, \quad a,b\in\mathbb{R},\ a>0, \tag{25}$$

which can be proved using integration by parts twice.

We have

$$\begin{aligned}
\int_0^\infty \frac{\sin x}{e^{(n+1)x}-e^{nx}}dx &= \int_0^\infty \frac{\sin x e^{-(n+1)x}}{1-e^{-x}}dx\\
&= \int_0^\infty \sin x\, e^{-(n+1)x}\left(\sum_{i=0}^\infty e^{-ix}\right)dx\\
&\overset{(*)}{=} \sum_{i=0}^\infty \int_0^\infty \sin x\, e^{-(n+1+i)x}dx\\
&\overset{(25)}{=} \sum_{i=0}^\infty \frac{1}{(n+1+i)^2+1}\\
&= \sum_{j=n+1}^\infty \frac{1}{j^2+1}.
\end{aligned}$$

It follows, based on Cesàro-Stolz lemma (the $\frac{0}{0}$ case), that

$$\lim_{n\to\infty} n\int_0^\infty \frac{\sin x}{e^{(n+1)x}-e^{nx}}dx = \lim_{n\to\infty}\frac{\sum\limits_{j=n+1}^\infty \frac{1}{j^2+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{-\frac{1}{(n+1)^2+1}}{\frac{1}{n+1}-\frac{1}{n}} = 1.$$

Observation. The justification of (*) is as follows:

$$\begin{aligned}
&\left|\int_0^\infty \sin x\, e^{-(n+1)x}\left(\sum_{i=0}^\infty e^{-ix}\right)dx - \int_0^\infty \sin x\, e^{-(n+1)x}\left(\sum_{i=0}^N e^{-ix}\right)dx\right|\\
&=\left|\int_0^\infty \sin x\, e^{-(n+1)x}\left(\sum_{i=N+1}^\infty e^{-ix}\right)dx\right| \le \sum_{i=N+1}^\infty \int_0^\infty |\sin x| e^{-(n+1+i)x}dx\\
&\le \sum_{i=N+1}^\infty \int_0^\infty x e^{-(n+1+i)x}dx = \sum_{i=N+1}^\infty \frac{1}{(n+1+i)^2}.
\end{aligned}$$

Passing to the limit, as $N\to\infty$, in the previous equality one has that

$$\int_0^\infty \sin x\, e^{-(n+1)x}\left(\sum_{i=0}^\infty e^{-ix}\right)dx = \sum_{i=0}^\infty \int_0^\infty \sin x\, e^{-(n+1+i)x}dx.$$

(b) The limit equals $\Gamma(k+1)$, where Γ denotes the Gamma function. We have

$$
\begin{aligned}
n^{k+1}\int_0^\infty \frac{x^k\sin x}{\mathrm{e}^{(n+1)x}-\mathrm{e}^{nx}}\mathrm{d}x &= n^{k+1}\int_0^\infty \frac{x^k\sin x\,\mathrm{e}^{-(n+1)x}}{1-\mathrm{e}^{-x}}\mathrm{d}x\\
&\overset{\mathrm{e}^{-x}=t}{=} -n^{k+1}\int_0^1 \frac{(-\ln t)^k\sin(\ln t)t^n}{1-t}\mathrm{d}t\\
&\overset{t^n=y}{=} \int_0^1 (-\ln y)^{k+1}\cdot\frac{\sin\left(\ln\sqrt[n]{y}\right)}{\ln\sqrt[n]{y}}\cdot\sqrt[n]{y}\cdot\frac{\frac{1}{n}}{1-\sqrt[n]{y}}\mathrm{d}y.
\end{aligned}
$$

Let $f_n(y)=(-\ln y)^{k+1}\cdot\frac{\sin\left(\ln\sqrt[n]{y}\right)}{\ln\sqrt[n]{y}}\cdot\sqrt[n]{y}\cdot\frac{\frac{1}{n}}{1-\sqrt[n]{y}},\quad y\in(0,1)$.

We have that $\lim\limits_{n\to\infty}f_n(y)=(-\ln y)^k$, $y\in(0,1)$, and

$$
|f_n(y)|=\left|(-\ln y)^{k+1}\cdot\frac{\sin\left(\ln\sqrt[n]{y}\right)}{\ln\sqrt[n]{y}}\cdot\sqrt[n]{y}\cdot\frac{\frac{1}{n}}{1-\sqrt[n]{y}}\right|\leq\frac{(-\ln y)^{k+1}}{1-y},
$$

since $\left|\frac{\sin x}{x}\right|\leq 1$, $\forall x\in\mathbb{R}$, and $\frac{\frac{1}{n}}{1-\sqrt[n]{y}}\leq\frac{1}{1-y}$, for all $n\in\mathbb{N}^*$. The last inequality follows from the fact that the function $g:[0,1]\to\mathbb{R}$, $g(x)=\frac{x}{1-y^x}$ is **an increasing function** when $y\in(0,1)$.

Since

$$
\begin{aligned}
\int_0^1\frac{(-\ln y)^{k+1}}{1-y}\mathrm{d}y &\overset{y=\mathrm{e}^{-t}}{=}\int_0^\infty\frac{t^{k+1}\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}\,\mathrm{d}t=\int_0^\infty t^{k+1}\mathrm{e}^{-t}\sum_{i=0}^\infty\mathrm{e}^{-it}\,\mathrm{d}t\\
&=\sum_{i=0}^\infty\int_0^\infty t^{k+1}\mathrm{e}^{-(i+1)t}\,\mathrm{d}t\\
&\overset{(i+1)t=x}{=}\sum_{i=0}^\infty\frac{1}{(i+1)^{k+2}}\int_0^\infty x^{k+1}\mathrm{e}^{-x}\,\mathrm{d}x=\Gamma(k+2)\zeta(k+2),
\end{aligned}
$$

we have that the positive function $y\to\frac{(-\ln y)^{k+1}}{1-y}$ is **integrable over** the interval $[0,1]$.

It follows, based on the Lebesgue Dominated Convergence Theorem, that

$$
\begin{aligned}
\lim_{n\to\infty}n^{k+1}\int_0^\infty\frac{x^k\sin x}{\mathrm{e}^{(n+1)x}-\mathrm{e}^{nx}}\mathrm{d}x &= \lim_{n\to\infty}\int_0^1(-\ln y)^{k+1}\frac{\sin\left(\ln\sqrt[n]{y}\right)}{\ln\sqrt[n]{y}}\cdot\frac{\frac{1}{n}\sqrt[n]{y}}{1-\sqrt[n]{y}}\mathrm{d}y\\
&=(-1)^k\int_0^1(\ln y)^k\mathrm{d}y\\
&\overset{y=\mathrm{e}^{-t}}{=}\int_0^\infty t^k\mathrm{e}^{-t}\mathrm{d}t\\
&=\Gamma(k+1).
\end{aligned}
$$

□

Remark. We mention that a solution of part (a) of the problem follows from the solution of part (b), however we provided another solution of part (a) which does not use the Lebesgue Dominated Convergence Theorem.

Solution by Moti Levy, Rehovot, Israel. We first solve (b). We make the change of variable $u = nx$ and we get

$$n^{k+1}\int_0^\infty \frac{x^k \sin x}{e^{(n+1)x} - e^{nx}}\mathrm{d}x = \int_0^\infty \frac{u^k}{e^u}\cdot\frac{\sin\left(\frac{u}{n}\right)}{\left(e^{\frac{u}{n}}-1\right)}\mathrm{d}u.$$

Define a sequence of functions:

$$f_n\left(u\right) := \frac{u^k}{e^u}\cdot\frac{\sin\left(\frac{u}{n}\right)}{e^{\frac{u}{n}}-1}.$$

We have $\lim_{y\to 0}\frac{\sin y}{e^y - 1} = 1$ and for every $y > 0$ we have $e^y - 1 > y > |\sin y|$, so $\left|\frac{\sin y}{e^y-1}\right| < 1$. It follows that $\lim_{n\to\infty} f_n(u) = f(u)$ uniformly on every interval $[r, R] \subset (0, \infty)$ and $|f_n(u)| < f(u)$ for every $u \in (0, \infty)$, where $f(u) = \frac{\sin u}{e^u - 1}$. It follows that

$$\begin{aligned}\lim_{n\to\infty}\int_r^R f_n(u)\mathrm{d}u &= \int_r^R f(u)\mathrm{d}u = \Gamma(k+1, R) - \Gamma(k+1, r),\\ \left|\int_0^r f_n(u)\mathrm{d}u\right| &\le \int_0^r f(u)\mathrm{d}u = \Gamma(k+1, r),\\ \left|\int_R^\infty f_n(u)\mathrm{d}u\right| &\le \int_R^\infty f(u)\mathrm{d}u = \Gamma(k+1) - \Gamma(k+1, r).\end{aligned}$$

Here $\Gamma(k+1, t)$ is the incomplete Gamma function, $\Gamma(k+1,t) = \int_0^t u^k e^{-u}\mathrm{d}u$.

Let $\varepsilon > 0$. Since $\lim_{r\searrow 0}\Gamma(k+1, r) = 0$ and $\lim_{R\to\infty}\Gamma(k+1, R) = \Gamma(k+1)$, there are $r_\varepsilon, R_\varepsilon \in (0, \infty)$, with $r_\varepsilon < R_\varepsilon$, such that $\Gamma(k+1, r_\varepsilon) < \frac{\varepsilon}{8}$ and $\Gamma(k+1) - \Gamma(k+1, R_\varepsilon) < \frac{\varepsilon}{8}$.

Since $\lim_{n\to\infty}\int_{r_\varepsilon}^{R_\varepsilon} f_n(u)\mathrm{d}u = \Gamma(k+1, R_\varepsilon) - \Gamma(k+1, r_\varepsilon)$, there is some integer $N_\varepsilon > 0$ such that for every $n \ge N_\varepsilon$ we have

$$\left|\int_{r_\varepsilon}^{R_\varepsilon} f_n(u)\mathrm{d}u - \left(\Gamma(k+1, R_\varepsilon) - \Gamma(k+1, r_\varepsilon)\right)\right| < \frac{\varepsilon}{2}.$$

Then for every $n \geq N_\varepsilon$ we have

$$\begin{aligned}\left|\int_0^\infty f_n(u)\mathrm{d}u - \Gamma(k+1)\right| &= \Bigg|\left(\int_0^{r_\varepsilon} + \int_{r_\varepsilon}^{R_\varepsilon} + \int_{R_\varepsilon}^\infty\right) f_n(u)\mathrm{d}u - (\Gamma(k+1, R_\varepsilon) \\ &\quad - \Gamma(k+1, r_\varepsilon)) - \Gamma(k+1, r_\varepsilon) - (\Gamma(k+1) \\ &\quad - \Gamma(k+1, R_\varepsilon))\Bigg| \\ &\leq \left|\int_{r_\varepsilon}^{R_\varepsilon} f_n(u)\mathrm{d}u - (\Gamma(k+1, R_\varepsilon) - \Gamma(k+1, r_\varepsilon))\right| \\ &\quad + \left|\int_0^{r_\varepsilon} f_n(u)\mathrm{d}u\right| + \left|\int_{R_\varepsilon}^\infty f_n(u)\mathrm{d}u\right| \\ &\quad + \Gamma(k+1, r_\varepsilon) + (\Gamma(k+1) - \Gamma(k+1, R_\varepsilon)).\end{aligned}$$

Since $\left|\int_{r_\varepsilon}^{R_\varepsilon} f_n(u)\mathrm{d}u - (\Gamma(k+1, R_\varepsilon) - \Gamma(k+1, r_\varepsilon))\right| < \frac{\varepsilon}{2}$, $\left|\int_0^{r_\varepsilon} f_n(u)\mathrm{d}u\right| \leq \Gamma(k+1, r_\varepsilon)$ and $\left|\int_{R_\varepsilon}^\infty f_n(u)\mathrm{d}u\right| \leq \Gamma(k+1) - \Gamma(k+1, R_\varepsilon)$, we get

$$\begin{aligned}\left|\int_0^\infty f_n(u)\mathrm{d}u - \Gamma(k+1)\right| &< \frac{\varepsilon}{2} + 2\Gamma(k+1, r_\varepsilon) + 2(\Gamma(k+1) - \Gamma(k+1, R_\varepsilon)) \\ &< \frac{\varepsilon}{2} + 2\cdot\frac{\varepsilon}{8} + 2\cdot\frac{\varepsilon}{8} = \varepsilon.\end{aligned}$$

In conclusion, our limit is $\Gamma(k+1)$.

When we take $k = 0$ in (b), we obviously get that the limit from (a) is $\Gamma(0+1) = 1$. □

Notes from the Editor.

1. The solution we received from Moti Levy was somewhat incomplete in the sense that the improperness of the integral at infinity and (if $-1 < k < 0$) at 0 was not addressed rigorously. Therefore we had to make some adjustments.

2. We received a solution for part (a) from Daniel Văcaru, from Piteşti, Romania. He wrote the integrand as a sum of simple fractions, $\frac{\sin x}{e^{(n+1)x} - e^{nx}} = -\sum_{k=1}^n \frac{\sin x}{e^{kx}} + \frac{\sin x}{e^x - 1}$. By using formulas 3.893 1 and 3.911 2 in I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products. Transl. from the Russian by Scripta Technica, Inc.* 5th ed. Boston, MA: Academic Press, Inc. (1994), he concludes that

$$\int_0^\infty \frac{\sin x}{e^{(n+1)x} - e^{nx}}\mathrm{d}x = -\sum_{k=1}^n \frac{1}{k^2+1} + \frac{\pi}{2}\coth\pi - \frac{1}{2},$$

whose limit as $n \to \infty$ is 0. (See Gradshteyn and Ryzhik 1421 4 or 1.445 2.) Then, by Cesàro-Stolz, he concludes that the limit is equal to

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)^2+1}}{\frac{1}{n} - \frac{1}{n+1}} = 1.$$

3. We also received a lengthy solution from Neil Greuber, from Newport News, Virginia, USA, but only in the case when k is an integer. His proof involved the Laplace transformation and polygamma functions.

494. Let $n \geq 3$ and let $a_1, \ldots, a_n$ be nonnegative real numbers such that $a_1^2 + \cdots + a_n^2 = n - 1$.

(i) Prove that $a_1 + \cdots + a_n - a_1 \cdots a_n \leq n - 1$.

(ii) Prove that if $k < 1$ then the inequality $a_1 + \cdots + a_n - ka_1 \cdots a_n \leq n - 1$ is not always true.

Proposed by Leonard Giugiuc, Traian National College, Drobeta Turnu Severin, Romania and Alexander Bogomolny, New Jersey, USA.

Solution by the authors. For (i) we prove a stronger result, where the condition that $a_1, \ldots, a_n \geq 0$ from the hypothesis is removed.

Moreover, if we renounce the hypothesis that $a_1, \ldots, a_n \geq 0$, then (ii) can be replaced by the following stronger result:

(ii') Prove that if $k \neq 1$ then the inequality $a_1 + \cdots + a_n - ka_1 \cdots a_n \leq n - 1$ is not always true.

We may assume, without loss of generality, that $a_1 \geq \cdots \geq a_n$. We have two cases.

Case 1. $a_n - a_1 \cdots a_n \leq 0$. It follows that $a_1 + \cdots + a_n - a_1 \cdots a_n \leq a_1 + \cdots + a_{n-1}$. But

$$\left(\frac{a_1 + \cdots + a_{n-1}}{n-1}\right)^2 \leq \frac{a_1^2 + \cdots + a_{n-1}^2}{n-1} \leq \frac{a_1^2 + \cdots + a_n^2}{n-1} = 1.$$

It follows that $|a_1 + \cdots + a_{n-1}| \leq n - 1$, so that $a_1 + \cdots + a_{n-1} \leq n - 1$ and we are done.

In order that the equality holds, one needs that $a_1^2 + \cdots + a_{n-1}^2 = a_1^2 + \cdots + a_n^2$, that is, $a_n = 0$ and $a_1 = \cdots = a_{n-1} = 1$. So we have equality when and only when, up to a permutation, $(a_1, \ldots, a_n) = (1, \ldots, 1, 0)$.

Case 2. $a_n - a_1 \cdots a_n > 0$. We have

$$\sqrt[n-1]{a_1^2 \cdots a_{n-1}^2} \leq \frac{a_1^2 + \cdots + a_{n-1}^2}{n-1} \leq \frac{a_1^2 + \cdots + a_n^2}{n-1} = 1,$$

so that $|a_1 \cdots a_{n-1}| \leq 1$, which in turn implies that $1 - a_1 \cdots a_{n-1} \geq 0$. Since $a_n(1 - a_1 \cdots a_{n-1}) > 0$, we must have $1 - a_1 \cdots a_{n-1} > 0$ and so $a_n > 0$. We conclude that it holds $a_1, \ldots, a_n > 0$.

We have $\left(\frac{a_1+\cdots+a_n}{n}\right)^2 \leq \frac{a_1^2+\cdots+a_n^2}{n} = \frac{n-1}{n}$, so $a_1+\cdots+a_n \leq \sqrt{n(n-1)}$, with equality if and only if $a_1=\cdots=a_n=\sqrt{\frac{n-1}{n}}$.

If $a_1+\cdots+a_n \leq n-1$, then we have the strict inequality $a_1+\cdots+a_n - a_1\cdots a_n < n-1$ and we are done.

So from now on we restrict ourselves to the case when $a_1+\cdots+a_n \in \left(n-1, \sqrt{n(n-1)}\,\right]$.

We consider first the case when $a_1=\cdots=a_{n-1}=x$ and $a_n=y$, with $x \geq y > 0$. The relation $a_1^2+\cdots+a_n^2=n-1$ writes as $(n-1)x^2+y^2=n-1$, so $y=\sqrt{(n-1)(1-x^2)}$. We have $0<y^2\leq x^2$, so that from $(n-1)x^2<(n-1)x^2+y^2=n-1\leq nx^2$ we deduce $x\in\left[\sqrt{\frac{n-1}{n}},1\right)$. We also have $a_1+\cdots+a_n=(n-1)x+y=f(x)$, where $f:[\sqrt{\frac{n-1}{n}},1)\to\mathbb{R}$, $f(x)=(n-1)x+\sqrt{(n-1)(1-x^2)}$.

We prove that f is decreasing. For every $x\in(\sqrt{\frac{n-1}{n}},1)$ we have $f'(x)=n-1-\frac{x\sqrt{n-1}}{\sqrt{1-x^2}}$. Then the relation $f'(x)<0$ is equivalent to $\sqrt{(n-1)(1-x^2)}<x$, i.e., $(n-1)(1-x^2)<x^2$, or $x^2>\frac{n-1}{n}$, which follows from $x>\sqrt{\frac{n-1}{n}}$.

Since $f(\sqrt{\frac{n-1}{n}})=\sqrt{n(n-1)}$ and $f(1)=n-1$ and f is decreasing, we have a bijection $f:\left[\sqrt{\frac{n-1}{n}},1\right)\to\left(n-1,\sqrt{n(n-1)}\,\right]$.

Now we come back to the general case, where $a_1\geq\cdots\geq a_n>0$ and $a_1+\cdots+a_n\in(n-1,\sqrt{n(n-1)}]$. Then there is a unique $x\in[\sqrt{\frac{n-1}{n}},1)$ with $a_1+\cdots+a_n=f(x)=(n-1)x+y$, with $x\geq y>0$ such that $(n-1)x^2+y^2=(n-1)$, i.e., $y=\sqrt{(n-1)(1-x^2)}$. From $a_1+\cdots+a_n=(n-1)x+y$ and $a_1^2+\cdots+a_n^2=(n-1)x^2+y^2$ we get

$$\sum_{i<j}a_ia_j=\frac{1}{2}(((n-1)x+y)^2-((n-1)x^2+y^2))=\frac{(n-2)(n-1)}{2}x^2+(n-1)xy.$$

Let $P(t)=(t-a_1)\cdots(t-a_n)$. We have

$$\begin{aligned}P(t)=&t^n-((n-1)x+y)t^{n-1}+\left(\frac{(n-2)(n-1)}{2}x^2+(n-1)xy\right)t^{n-2}\\&+Q(t)+(-1)^np,\end{aligned}$$

where $Q(t)$ is a polynomial of degree $n-3$ divisible by t and $p=a_1\cdots a_n$.

The function $g:(0,\infty)\to\mathbb{R}$, $g(t)=\frac{P(t)}{t}$, has n roots (counting multiplicities), so $g^{(n-3)}$ has at least 3 roots. Since $\frac{Q(t)}{t}$ is a polynomial of degree

$n-4$, its derivative of order $n-3$ is 0. We get

$$\begin{aligned} g^{(n-3)}(t) &= \frac{(n-1)!}{2}t^2 - (n-2)!((n-1)x+y)t \\ &\qquad + (n-3)!\left(\frac{(n-2)(n-1)}{2}x^2 + (n-1)xy\right) - \frac{(n-3)!}{t^{n-2}}p \\ &= \frac{(n-3)!}{2t^{n-2}}h(t), \end{aligned}$$

where $h : (0,\infty) \to \mathbb{R}$ is defined by

$$\begin{aligned} h(t) =&(n-1)(n-2)t^n - 2(n-2)((n-1)x+y)t^{n-1} \\ &+ ((n-2)(n-1)x^2 + 2(n-1)xy)t^{n-2} - 2p. \end{aligned}$$

We have

$$\begin{aligned} \frac{h'(t)}{(n-1)(n-2)t^{n-3}} &= nt^2 - 2\big((n-1)x+y\big)t + (n-2)x^2 + 2xy \\ &= (t-x)(nt-(n-2)x-2y). \end{aligned}$$

The last expression has the roots x and $x' = \frac{(n-2)x+y}{n}$. Since $x \geq y$, we have $x \geq x'$. We have $h'(t) > 0$ for $t \in (0,x')$ or (x',∞) and $h'(t) < 0$ for $t \in (x',x)$. Hence h is increasing on $(0,x']$ and $[x,\infty)$ and decreasing on $[x',x]$. If $t_1 \leq t_2 \leq t_3$ are three roots of h then, by Rolle's theorem, $t_1 \leq x' \leq t_2 \leq x \leq t_3$. Since $t_2 \in [x',x]$ and h is increasing on $[x',x]$, we have $0 = h(t_2) \geq h(x)$. By computation we find that $h(x) = 2x^{n-1}y - 2p$, so that $p \geq x^{n-1}y$. The equality holds when $(a_1,\dots,a_n) = (x,\dots,x,y)$.

Consequently, the maximum of $a_1+\cdots+a_n - a_1\cdots a_n$ where $a_1 \geq \cdots \geq a_n > 0$, $a_1^2+\cdots+a_n = n-1$ and $a_1+\cdots+a_n = (n-1)x+y$ is $(n-1)x+y-x^{n-1}y$ and it is reached for $(a_1,\dots,a_n) = (x,\dots,x,y)$.

Recall that $x \in [\sqrt{\frac{n-1}{n}},1)$. We prove that $(n-1)x+y-x^{n-1}y = (n-1)x + (1-x^{n-1})\sqrt{(n-1)(1-x^2)} < n-1$. This writes as

$$(1-x^{n-1})\sqrt{(n-1)(1-x^2)} < (n-1)(1-x).$$

By dividing by $\sqrt{n-1}(1-x) > 0$, the last inequality becomes

$$(1+x+\cdots+x^{n-2})\sqrt{1-x^2} < \sqrt{n-1}.$$

As $\sqrt{\frac{n-1}{n}} \leq x < 1$, we find $1+x+\cdots+x^{n-2} < 1+\cdots+1 = n-1$ and $\sqrt{1-x^2} \leq \sqrt{\frac{1}{n}}$, whence

$$(1+x+\cdots+x^{n-2})\sqrt{1-x^2} < \frac{n-1}{\sqrt{n}} < \sqrt{n-1}.$$

Hence in all cases $a_1+\cdots+a_n - a_1\cdots a_n \leq n-1$ and the equality holds only in **Case 1**, namely, iff $(a_1,\dots,a_n)$ is a permutation of $(1,\dots,1,0)$.

(ii) Let $k < 1$. We first take, as before, $a_1 = \cdots = a_{n-1} = x$ and $a_n = \sqrt{(n-1)(1-x^2)}$ for some $x \in (0,1)$ and we prove that, for x close enough to 1, $a_1 + \cdots + a_n - ka_1 \cdots a_n > n-1$, i.e., that

$$(n-1)x + \sqrt{(n-1)(1-x^2)} - kx^{n-1}\sqrt{(n-1)(1-x^2)} > n-1.$$

This is equivalent to

$$\frac{(n-1)x + \sqrt{(n-1)(1-x^2)} - (n-1)}{x^{n-1}\sqrt{(n-1)(1-x^2)}} > k.$$

It suffices to prove that

$$\lim_{x\nearrow 1} \frac{(n-1)x + \sqrt{(n-1)(1-x^2)} - (n-1)}{x^{n-1}\sqrt{(n-1)(1-x^2)}} = 1.$$

This follows from

$$\begin{aligned}\frac{(n-1)x + \sqrt{(n-1)(1-x^2)} - (n-1)}{x^{n-1}\sqrt{(n-1)(1-x^2)}} &= \frac{1}{x^{n-1}} - \frac{(n-1)(1-x)}{\sqrt{(n-1)(1-x^2)}}\\ &= \frac{1}{x^{n-1}} - \frac{\sqrt{n-1}}{x^{n-1}}\sqrt{\frac{1-x}{1+x}}\end{aligned}$$

and $\lim_{x\nearrow 1}\left(\frac{1}{x^{n-1}} - \frac{\sqrt{n-1}}{x^{n-1}}\sqrt{\frac{1-x}{1+x}}\right) = 1$.

For the proof of (ii’), when we renounce the hypothesis that $a_1, \ldots, a_n \geq 0$, the case when $k < 1$ was handled above. Assume now that $k > 0$. We take $a_1 = \cdots = a_{n-1} = x$ and $a_n = -\sqrt{(n-1)(1-x^2)}$ and we prove that $a_1 + \cdots + a_n + ka_1 \cdots a_n > n-1$ if x is close enough to 1. The inequality writes as $(n-1)x - \sqrt{(n-1)(1-x^2)} + kx^{n-1}\sqrt{(n-1)(1-x^2)} > n-1$ and it is equivalent to $\frac{(n-1)+\sqrt{(n-1)(1-x^2)}-(n-1)x}{x^{n-1}\sqrt{(n-1)(1-x^2)}} < k$. Again, this will follow from $\lim_{x\nearrow 1} \frac{(n-1)+\sqrt{(n-1)(1-x^2)}-(n-1)x}{x^{n-1}\sqrt{(n-1)(1-x^2)}} = 1$, which is clear by observing that

$$\frac{(n-1) + \sqrt{(n-1)(1-x^2)} - (n-1)x}{x^{n-1}\sqrt{(n-1)(1-x^2)}} = \frac{1}{x^{n-1}} + \frac{\sqrt{n-1}}{x^{n-1}}\sqrt{\frac{1-x}{1+x}}$$

and $\lim_{x\nearrow 1}\left(\frac{1}{x^{n-1}} + \frac{\sqrt{n-1}}{x^{n-1}}\sqrt{\frac{1-x}{1+x}}\right) = 1$. □

Notes from the editor.

1. The fact that x is a root of $\frac{h'(t)}{(n-1)(n-2)t^{n-3}}$, which is a key part of the proof, is not by accident. One arrives from $P(t)$ to $\frac{h'(t)}{(n-1)(n-2)t^{n-3}}$ via $n-2$ derivations and some multiplications and divisions by monomials. In the process, the coefficients of $Q(t)$ and p vanish, so the outcome depends only on the first two coefficients of $P(t)$, i.e., on $\sum_i a_i$ and $\sum_{i<j} a_i a_j$ or, equivalently, on $\sum_i a_i$ and $\sum_i a_i^2$. We have $\sum_i a_i = (n-1)x + y$ and $\sum_i a_i^2 = (n-1)x^2 + y^2$.

An obvious choice of $a_1, \dots, a_n$ with these properties is $a_1 = \cdots = a_{n-1} = x$, $a_n = y$, in which case we have $P(t) = (t-x)^{n-1}(t-y)$. Since x is a root of $P(t)$ of multiplicity $n-1$, after $n-2$ derivations and multiplications and some divisions by monomials it will still be a root off multiplicity 1. This is why x is a root of $\frac{h'(t)}{(n-1)(n-2)t^{n-3}}$.

2. We also received a solution from Costel Sava. He, too, proves the stronger version of the result, where $a_, \dots, a_n$ are not necessarily nonnegative. The proof goes on similar lines as the authors', except that he considers the cases when $a_1 \cdots a_n \le 0$ (which is contained in the authors' **Case 1**) and $a_1 \cdots a_n \ge 0$. In the difficult case when $a_1, \dots, a_n \ge 0$ Costel Sava uses a result by Vasile Cîrtoaje, called Equal Variables Theorem (see V. Cîrtoaje, The equal variables method, *J. Inequal. Pure Appl. Math.* **8**, No. 1, Art. 15, 2007). If $a_1 \ge \cdots \ge a_n \ge 0$, $a_1 + \cdots + a_n$ is fixed and $a_1^2 + \cdots + a_n^2 = n-1$, then Cîrtoaje's theorem implies that the product $a_1 \cdots a_n$ is minimal when either $a_n = 0$ or $a_1 = \cdots = a_{n-1} \ge a_n$.

Cîrtoaje's result is more general and involves convexity methods. The authors' proof is a nontrivial generalization of the solution in the case $n = 3$. In this proof one goes from P to $\frac{h'(t)}{(n-1)(n-2)t^{n-3}}$ by dividing $P(t)$ by t, then one takes the derivative $n-3$ times, then one multiplies by $\frac{2t^{n-2}}{(n-3)!}$, then one takes the derivative once more and finally one divides by $(n-1)(n-2)t^{n-3}$. If $n = 3$ then $\frac{h'(t)}{(n-1)(n-2)t^{n-3}}$ is simply $P'(t)$.

495. Let $n \ge 2$ and $A, B \in \mathcal{M}_n(\mathbb{C})$ such that

$$A \cdot B - B \cdot A = c(A - B) \tag{1}$$

for some $c \in \mathbb{C}^*$.

a) For $n = 2$, give an example of distinct matrices A and B that satisfy the above condition.

b) Prove that A and B have the same eigenvalues.

`Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania, and Mihai Opincariu, Avram Iancu National College, Brad, Romania.`

Solution by the authors. a) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & c \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$. Then

$$A \cdot B - B \cdot A = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix},$$

$$A - B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$A \cdot B - B \cdot A = c(A - B).$$

b) Let x be a variable. We have the equivalences

$$(1) \Leftrightarrow (A - xI_n)(B - xI_n) - (B - xI_n)(A - xI_n) = c((A - xI_n) - (B - xI_n))$$

$$\Leftrightarrow (A - xI_n)(B - xI_n - cI_n) = (B - xI_n)(A - xI_n - cI_n) \qquad (2)$$

By taking determinants in (2), we get

$$f_A(x) \cdot f_B(x + c) = f_B(x) \cdot f_A(x + c),$$

where by f_X we denote the characteristic polynomial of the matrix M. We prove that $f_A = f_B$, which is enough to conclude the proof.

More generally, we claim that if P and Q are *unitary* polynomials of the same degree ($m \geq 0$) that satisfy

$$P(x) \cdot Q(x + c) = Q(x) \cdot P(x + c) \quad \text{for all } x \in \mathbb{C}, \qquad (3)$$

then $P = Q$. We prove this by induction on m. The statement is obviously true for $m = 0$ ($P = Q = 1$).

For the inductive step, we assume it to be true for $m - 1$ and prove it for m. Let P and Q be unitary polynomials of degree $m \geq 0$ that satisfy (3) and let $x_0 \in \mathbb{C}$ be a root of P such that $x_0 + c$ is not a root of P (otherwise, if $x_0 + c$ is a root of P for every root x_0 of P, then by induction $x_0 + Nc$ is a root of P for all nonnegative integers N, meaning that $P = 0$ by having an infinite number of roots, which contradicts the assumptions on P).

Using (3) for $x := x_0$, it follows that

$$0 = P(x_0) \cdot Q(x_0 + c) = Q(x_0) \cdot \underbrace{P(x_0 + c)}_{\neq 0},$$

hence x_0 is a root of Q. Writing

$$P(x) = (x - x_0)P_1(x),$$
$$Q(x) = (x - x_0)Q_1(x),$$

it follows by (3) that for all $x \in \mathbb{C}$ it holds

$$(x - x_0)(x + c - x_0) \cdot P_1(x) \cdot Q_1(x + c) = (x - x_0)(x + c - x_0) \cdot Q_1(x) \cdot P_1(x + c).$$

It follows that

$$P_1(x) \cdot Q_1(x + c) = Q_1(x) \cdot P_1(x + c) = 0 \quad \text{for all } x \in \mathbb{C},$$

which is the same as (3) but for the unitary polynomials P_1 and Q_1 of degree $m - 1$. By the hypothesis of the induction, it follows that $P_1 = Q_1$, hence $P = Q$, which concludes the argument and the proof.

We also received a solution from Daniel Văcaru, from Piteşti, Romania. The example he produced for a) is

$$A = \begin{pmatrix} c & c + 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} c - \frac{c+1}{c}a & c + 1 - \frac{(c+1)^2}{c}a \\ a & \frac{c+1}{c}a \end{pmatrix}.$$

For b) he denotes by P and Q the characteristic polynomials of A and B and obtains the same relation $P(X)Q(X+c) = P(X+c)Q(X)$ from the authors' solution, which also writes as $\frac{P(X+c)}{Q(X+c)} = \frac{P(X)}{Q(X)}$. Since P and Q are monic of the same degree, if $P \neq Q$, then we have a partial fraction decomposition

$$\frac{P(X)}{Q(X)} = 1 + \sum_{i=1}^{r} \frac{\alpha_i}{(X-a_i)^{\beta_i}}.$$

Since the translation $X \mapsto X + c$ leaves the fraction $\frac{P}{Q}$ invariant, the set of poles $\mathcal{P} = \{a_i\}_{i\in\{\overline{1,r}\}}$ is invariant by this translation. But this is impossible, as $\mathcal{P}$ is a finite set. Hence $P = Q$.

Solution by Cornel Băeţica. b) Let $X = A - B$. Then (1) is equivalent to

$$XA - AX = cX. \tag{4}$$

In the following we show that A and X are *simultaneously triangularizable*. In order to do this, let us first notice that

$$X^iA - AX^i = ciX^i \tag{5}$$

for all integers $i \geq 1$. The proof goes by induction on i. If we assume that $X^{i-1}A - AX^{i-1} = c(i-1)X^{i-1}$, then, by multiplying with X on the left we get $X^iA - XAX^{i-1} = c(i-1)X^i$. On the other side, if we multiply (4) by X^{i-1} on the right we get $XAX^{i-1} - AX^i = cX^i$. By adding these two relations we obtain $X^iA - AX^i = ciX^i$.

Since $\mathrm{Tr}(X^iA) = \mathrm{Tr}(AX^i)$, from (5) we have that $\mathrm{Tr}(X^i) = 0$ for all $i \geq 1$, and by a well known result we conclude that X is nilpotent.

Now let V be the vector space over $\mathbb{C}$ generated by the matrices: $A, X, \ldots, X^{n-1}$. With respect to the usual bracket we have $[A, A] = 0$, $[X^i, A] = ciX^i$, and $[X^i, X^j] = 0$. This shows that V is a finitely dimensional Lie algebra. Moreover, $[V, V] \subseteq \mathbb{C}[X]$ and if $V_1 = [V, V]$ we get $[V_1, V_1] = 0$. It follows that V is solvable, and by Lie's theorem A and X are simultaneously triangularizable, hence A and B are simultaneously triangularizable, and from (1) we get that A and B have the same eigenvalues. □

Remarks. (i) Let f be a polynomial with complex coefficients. By a similar reasoning one can show that two matrices $A, X \in \mathcal{M}_n(\mathbb{C})$ that satisfy

$$XA - AX = f(X)$$

are simultaneously triangularizable.

(ii) The above proof relies on two results:
1) $\mathrm{Tr}(X^i) = 0$ for $1 \leq i \leq n$ implies that X is nilpotent, and
2) Lie's theorem.

Since 1) and 2) hold for algebraically closed fields of characteristic 0 or $p > n$, we get that A and X are simultaneously triangularizable over algebraically closed fields of characteristic 0 or $p > n$. In particular, the conclusion of our problem holds for fields of characteristic 0 or $p > n$.