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A new proof of the quadratic series of Au-Yeung

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Abstract. In this paper we give a new proof of the following remarkable series formula

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\frac{17}{4}\,\zeta(4),$$

where $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$ denotes the nth harmonic number. The proof is based on evaluating a special harmonic series by two different methods.

Keywords: Abel's summation formula, harmonic numbers, quadratic harmonic series, Riemann zeta function.

MSC: 40A05, 40C10.

1. Introduction and the main results

In this paper we give a new proof of the following remarkable series formula

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\frac{17}{4}\,\zeta(4), \tag{1}$$

where $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$ denotes the nth harmonic number.

This formula has an interesting history. It was discovered numerically by Enrico Au-Yeung, an undergraduate student in the Faculty of Mathematics in Waterloo, and proved rigorously by David Borwein and Jonathan Borwein in [2], who used Parseval's theorem to prove it. Formula (1) was rediscovered by Freitas as Proposition A.1 in the appendix section of [3]. Freitas proved it by calculating a double integral involving a logarithmic function. This formula is revived and brought into light by Vălean and Furdui [5], who

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proved it by calculating a special integral involving a quadratic logarithmic function. The series also appears as a problem in [4, Problem 3.70, p. 150] and [7, Problem 2.6.1. p. 110]. It is clear that this remarkable quadratic harmonic series has attracted lots of attention lately and has become a classic in the theory of nonlinear harmonic series.

In this paper we prove formula (1) by calculating the series

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(2\zeta(3) - \frac{H_1}{1^2} - \frac{H_2}{2^2} - \cdots - \frac{H_n}{n^2}\right) = \frac{\pi^4}{30}$$

in two different ways. Our method is new and as elementary as possible. We record the results we prove in the next theorem.

Theorem 1. (a) **A special harmonic sum.**

The following identity holds

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(2\zeta(3) - \frac{H_1}{1^2} - \frac{H_2}{2^2} - \cdots - \frac{H_n}{n^2}\right) = \frac{\pi^4}{30}.$$

(b) **The quadratic series of Au–Yeung.**

The following identity holds

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 = \frac{17}{4}\zeta(4).$$

Proof. (a) We have, since $\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$ ([4, Problem 3.55, p. 148]), that

$$E := \sum_{n=1}^{\infty} \frac{1}{n}\left(2\zeta(3) - \frac{H_1}{1^2} - \frac{H_2}{2^2} - \cdots - \frac{H_n}{n^2}\right) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{H_{n+m}}{n(n+m)^2}.$$

It follows, based on symmetry reasons, that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{H_{n+m}}{n(n+m)^2} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{H_{n+m}}{m(n+m)^2},$$

which implies

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{H_{n+m}}{n(n+m)^2} &= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{H_{n+m}}{m(n+m)^2} + \frac{H_{n+m}}{n(n+m)^2}\right)\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{H_{n+m}}{nm(n+m)}.\end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(2\zeta(3) - \frac{H_1}{1^2} - \frac{H_2}{2^2} - \cdots - \frac{H_n}{n^2}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{H_{n+m}}{nm(n+m)}. \tag{2}$$

One may check, by using partial fractions, that the following identity holds

$$\sum_{k=1}^{\infty} \frac{1}{k(k+n)} = \frac{H_n}{n}. \tag{3}$$

Combining (2) and (3) we have that

$$\begin{aligned}
E &= \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{nmk(k+n+m)} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{nmk} \int_0^1 x^{n+m+k-1} \mathrm{d}x \\
&= \frac{1}{2} \int_0^1 \left(\sum_{k=1}^{\infty} \frac{x^{k-1}}{k} \sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{m=1}^{\infty} \frac{x^m}{m} \right) \mathrm{d}x \\
&= -\frac{1}{2} \int_0^1 \frac{\ln^3(1-x)}{x} \mathrm{d}x \\
&= -\frac{1}{2} \int_0^1 \frac{\ln^3 x}{1-x} \mathrm{d}x \\
&= -\frac{1}{2} \int_0^1 \ln^3 x \left(\sum_{i=0}^{\infty} x^i \right) \mathrm{d}x \\
&= -\frac{1}{2} \sum_{i=0}^{\infty} \int_0^1 x^i \ln^3 x \, \mathrm{d}x \\
&= 3 \sum_{i=0}^{\infty} \frac{1}{(i+1)^4} \\
&= \frac{\pi^4}{30},
\end{aligned}$$

and part (a) of the theorem is proved.

(b) Before we prove this part of the theorem we collect a formula that we need in our analysis. Recall that *Abel's summation by parts formula* ([1, p. 55], [4, Lemma A.1, p. 258]) states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences of real numbers and $A_n = \sum_{k=1}^{n} a_k$, then $\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1})$. We will be using the *infinite version* of this formula

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} (A_n b_{n+1}) + \sum_{k=1}^{\infty} A_k (b_k - b_{k+1}). \tag{4}$$

We calculate the series in part (a) by using formula (4), with

$$a_n = \frac{1}{n} \quad \text{and} \quad b_n = 2\zeta(3) - \frac{H_1}{1^2} - \frac{H_2}{2^2} - \cdots - \frac{H_n}{n^2},$$

and we have that

$$\begin{aligned} E &= \lim_{n\to\infty} H_n \left(2\zeta(3) - \frac{H_1}{1^2} - \frac{H_2}{2^2} - \cdots - \frac{H_{n+1}}{(n+1)^2}\right) \quad + \sum_{n=1}^{\infty} H_n \frac{H_{n+1}}{(n+1)^2} \\ &= \sum_{n=1}^{\infty} \frac{\left(H_{n+1} - \frac{1}{n+1}\right) H_{n+1}}{(n+1)^2} \\ &= \sum_{n=1}^{\infty} \left(\frac{H_{n+1}}{n+1}\right)^2 - \sum_{n=1}^{\infty} \frac{H_{n+1}}{(n+1)^3} \\ &= \sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 - \sum_{n=1}^{\infty} \frac{H_n}{n^3} \\ &= \sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 - \frac{\pi^4}{72}, \end{aligned}$$

since $\lim_{n\to\infty} H_n \left(2\zeta(3) - \frac{H_1}{1^2} - \frac{H_2}{2^2} - \cdots - \frac{H_{n+1}}{(n+1)^2}\right) = 0$ and $\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{\pi^4}{72}$.

It follows that

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 = \frac{\pi^4}{30} + \frac{\pi^4}{72} = \frac{17}{4}\,\zeta(4),$$

and the theorem is proved.

A proof of the series $\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{\pi^4}{72}$, which is a special linear Euler sum, is given in [4, Problem 3.58, pp. 207–208] and it also appears in literature as a problem proposed by M.S. Klamkin [6]. Another proof of the same series formula is also given in [8, Chapter 3, pp. 81–82]. For the sake of completeness we give below an elementary proof of this formula.

We have

$$S = \sum_{n=1}^{\infty} \frac{H_n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{\infty} \frac{1}{k(n+k)} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n^2 k(n+k)}.$$

We also have, based on symmetry reasons, that $S=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{nk^2(n+k)}$ and it follows that

$$\begin{aligned}2S&=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{1}{n^2k(n+k)}+\frac{1}{nk^2(n+k)}\right)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^2k^2}\\&=\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^4}{36},\end{aligned}$$

and the series is calculated. □

Corollary 2. *The following equality holds*

$$\sum_{n=1}^{\infty}H_n\left(\frac{\pi^4}{72}-\frac{H_1}{1^3}-\frac{H_2}{2^3}-\cdots-\frac{H_n}{n^3}\right)=\frac{17}{4}\zeta(4)-2\zeta(3). \tag{5}$$

Proof. We apply Abel's summation formula (4), with $a_n=H_n$ and $b_n=\frac{\pi^4}{72}-\frac{H_1}{1^3}-\frac{H_2}{2^3}-\cdots-\frac{H_n}{n^3}$ and we have, since $\sum_{k=1}^{n}H_k=(n+1)H_{n+1}-(n+1)$, that

$$\begin{aligned}&\sum_{n=1}^{\infty}H_n\left(\frac{\pi^4}{72}-\frac{H_1}{1^3}-\frac{H_2}{2^3}-\cdots-\frac{H_n}{n^3}\right)\\&=\lim_{n\to\infty}[(n+1)H_{n+1}-(n+1)]\left(\frac{\pi^4}{72}-\frac{H_1}{1^3}-\frac{H_2}{2^3}-\cdots-\frac{H_{n+1}}{(n+1)^3}\right)\\&\quad+\sum_{n=1}^{\infty}[(n+1)H_{n+1}-(n+1)]\frac{H_{n+1}}{(n+1)^3}\\&=\sum_{n=1}^{\infty}\left[\frac{H_{n+1}^2}{(n+1)^2}-\frac{H_{n+1}}{(n+1)^2}\right]\\&=\frac{17}{4}\zeta(4)-2\zeta(3).\end{aligned}$$

The corollary is proved. □

We leave to the interested reader, as an open problem, the calculation of the following series of which the second is the alternating version of the series (5).

Open problem. Calculate:

(a) $\sum_{n=1}^{\infty}(-1)^nH_n\left(2\zeta(3)-\frac{H_1}{1^2}-\frac{H_2}{2^2}-\cdots-\frac{H_n}{n^2}\right)$;

(b) $\sum_{n=1}^{\infty}(-1)^nH_n\left(\frac{\pi^4}{72}-\frac{H_1}{1^3}-\frac{H_2}{2^3}-\cdots-\frac{H_n}{n^3}\right)$.

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The asymptotic evaluation of the sum $\sum_{k=1}^{n} f\left(\sqrt[n]{k}\right)$

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Abstract. Let $\eta > 0$ and $f : (1-\eta, 1+\eta) \to \mathbb{R}$ be a function. We prove the following asymptotic evaluations:
$\sum_{k=1}^{n} f\left(\sqrt[n]{k}\right) = f(1)\,n + o(n)$ for f continuous;
$\sum_{k=1}^{n} f\left(\sqrt[n]{k}\right) = f(1)\,n + f'(1)\ln n + o(\ln n)$ for f differentiable, and
$\sum_{k=1}^{n} f\left(\sqrt[n]{k}\right) = f(1)\,n + f'(1)\ln n - f'(1) + \frac{f'(1)+f''(1)}{2}\cdot\frac{\ln^2 n}{n} + o\left(\frac{\ln^2 n}{n}\right)$
for f twice differentiable.
Some applications are given.

Keywords: Convergence and divergence of series and sequences, the Euler-Maclaurin summation formula, orders of infinity.

MSC: Primary 26A12. Secondary 40A05, 40A25.

1. Introduction

One of the central problems in mathematical analysis is to find the asymptotic evaluation of various sums. Let us mention here only the Euler result $1+\frac{1}{2}+\cdots+\frac{1}{n} = \ln n + \gamma + \frac{1}{2n} + o\left(\frac{1}{n}\right)$, see [1, 2]. The main purpose of this paper is to find the asymptotic evaluations for the sum $\sum_{k=1}^{n} f\left(\sqrt[n]{k}\right)$ for the

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case when f is continuous, Proposition 1, f is differentiable, Proposition 3 and f is twice differentiable, Proposition 5. For various different asymptotic evaluations we recommend the reader the book [4]. Our notation and notion are standard. We recall that, if $(b_n)_{n\in\mathbb{N}}$ is a sequence of real numbers such that there exists $n_0 \in \mathbb{N}$ with $b_n \neq 0$, $\forall n \geq n_0$, and $(a_n)_{n\in\mathbb{N}}$ is another sequence of real numbers, the notation $a_n = o(b_n)$ means $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$; if $c \in \mathbb{R}^*$ the notation $a_n \backsim cb_n$ means that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$; in particular, $a_n \backsim b_n$ means $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$. Also the notation $x_n = a_n + o(b_n)$ means $x_n - a_n = o(b_n)$; in particular, $a_n \backsim b_n$ is equivalent to $a_n = b_n + o(b_n)$.

2. The main results

We begin by proving the evaluation for continuous functions.

Proposition 1. *Let $\eta > 0$ and $f : (1-\eta, 1+\eta) \to \mathbb{R}$ be a continuous function. Then*

$$f\left(\sqrt[n]{1}\right) + f\left(\sqrt[n]{2}\right) + \cdots + f\left(\sqrt[n]{n}\right) = f(1)\,n + o(n).$$

Proof. Let $\varepsilon > 0$. Since f is continuous at 1, there exists $\delta_\varepsilon > 0$ such that $\forall x \in (1-\eta, 1+\eta)$ with $|x-1| < \delta_\varepsilon$ it follows that $|f(x) - f(1)| < \varepsilon$. From $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, for $\nu_\varepsilon = \min(\eta, \delta_\varepsilon) > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$ we have $\sqrt[n]{n} - 1 < \nu_\varepsilon$. Let $n \geq n_\varepsilon$. For every $k = 1, \ldots, n$ we have $0 \leq \sqrt[n]{k} - 1 \leq \sqrt[n]{n} - 1 < \nu_\varepsilon$ and thus $\left|f\left(\sqrt[n]{k}\right) - f(1)\right| \leq \varepsilon$. Then

$$\left|\sum_{k=1}^{n} f\left(\sqrt[n]{k}\right) - f(1)\,n\right| \leq \sum_{k=1}^{n} \left|f\left(\sqrt[n]{k}\right) - f(1)\right| \leq \varepsilon n,$$

or $\left|\frac{\sum_{k=1}^{n} f\left(\sqrt[n]{k}\right)}{n} - f(1)\right| \leq \varepsilon$, which ends the proof. □

To obtain the asymptotic evaluation for differentiable and twice differentiable functions we need the next result.

Proposition 2. *For every $\alpha > 0$ we have* $\lim\limits_{n\to\infty} \frac{\sum_{k=1}^{n} \left(\sqrt[n]{k} - 1\right)^\alpha}{\frac{(\ln n)^\alpha}{n^{\alpha-1}}} = 1.$

Proof. Let $\varepsilon > 0$. Since $\alpha > 0$, $\lim\limits_{x\to 0, x>0} \left(\frac{e^x - 1}{x}\right)^\alpha = 1$. It follows that $\exists \delta_\varepsilon > 0$ such that $\forall\, 0 < x < \delta_\varepsilon$ we have $\left|\frac{(e^x-1)^\alpha}{x^\alpha} - 1\right| < \varepsilon$, or

$$|(e^x - 1)^\alpha - x^\alpha| \leq \varepsilon x^\alpha,\ \forall 0 \leq x < \delta_\varepsilon. \tag{1}$$

Since $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$, for $\delta_\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$ we have $0 < \frac{\ln n}{n} < \delta_\varepsilon$. Let $n \geq n_\varepsilon$. For every $k = 1, \ldots, n$ we have $0 \leq \frac{\ln k}{n} \leq \frac{\ln n}{n} < \delta_\varepsilon$

and by (1) we deduce $\left|\left(e^{\frac{\ln k}{n}}-1\right)^{\alpha}-\left(\frac{\ln k}{n}\right)^{\alpha}\right|\leq\varepsilon\left(\frac{\ln k}{n}\right)^{\alpha}$. Then

$$\left|\sum_{k=1}^{n}\left(\sqrt[n]{k}-1\right)^{\alpha}-\sum_{k=1}^{n}\left(\frac{\ln k}{n}\right)^{\alpha}\right|\leq\sum_{k=1}^{n}\left|\left(\sqrt[n]{k}-1\right)^{\alpha}-\left(\frac{\ln k}{n}\right)^{\alpha}\right|$$
$$\leq\varepsilon\sum_{k=1}^{n}\left(\frac{\ln k}{n}\right)^{\alpha}$$

or equivalently $\left|\frac{\sum_{k=1}^{n}\left(\sqrt[n]{k}-1\right)^{\alpha}}{\sum_{k=1}^{n}\left(\frac{\ln k}{n}\right)^{\alpha}}-1\right|\leq\varepsilon$. Thus, $\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\left(\sqrt[n]{k}-1\right)^{\alpha}}{\sum_{k=1}^{n}\left(\frac{\ln k}{n}\right)^{\alpha}}=1$, i.e., $\sum_{k=1}^{n}\left(\sqrt[n]{k}-1\right)^{\alpha}\backsim\sum_{k=1}^{n}\left(\frac{\ln k}{n}\right)^{\alpha}$. Since, by the Stolz-Cesàro lemma, the case $\left[\frac{\cdot}{\infty}\right]$, or [3], [4, Capitolul V], it holds $\sum_{k=1}^{n}(\ln k)^{\alpha}\backsim n(\ln n)^{\alpha}$, it follows that $\sum_{k=1}^{n}\left(\frac{\ln k}{n}\right)^{\alpha}\backsim\frac{(\ln n)^{\alpha}}{n^{\alpha-1}}$, and hence $\sum_{k=1}^{n}\left(\sqrt[n]{n}-1\right)^{\alpha}\backsim\frac{(\ln n)^{\alpha}}{n^{\alpha-1}}$, which ends the proof of the proposition. □

Proposition 3. *Let $\eta>0$ and $f:(1-\eta,1+\eta)\to\mathbb{R}$ be a differentiable function. Then*

$$f\left(\sqrt[n]{1}\right)+f\left(\sqrt[n]{2}\right)+\cdots+f\left(\sqrt[n]{n}\right)=f(1)\,n+f'(1)\ln n+o(\ln n).$$

Proof. Let $\varepsilon>0$. Since f is differentiable at 1, $\lim_{x\to1}\frac{f(x)-f(1)}{x-1}=f'(1)$, thus there exists $\delta_{\varepsilon}>0$ such that $\forall x\in(1-\eta,1+\eta)$ with the property that $|x-1|<\delta_{\varepsilon}$, $x\neq1$, it follows that $\left|\frac{f(x)-f(1)}{x-1}-f'(1)\right|<\varepsilon$, or

$$\left|f(x)-f(1)-f'(1)(x-1)\right|\leq\varepsilon|x-1|,\ \forall\ |x-1|<\delta_{\varepsilon}. \tag{2}$$

Since $\lim_{n\to\infty}\sqrt[n]{n}=1$, for $\nu_{\varepsilon}=\min(\eta,\delta_{\varepsilon})>0$ there exists $n_{\varepsilon}\in\mathbb{N}$ such that $\forall n\geq n_{\varepsilon}$ we have $\sqrt[n]{n}-1<\nu_{\varepsilon}$. Let $n\geq n_{\varepsilon}$. For every $k=1,\ldots,n$ we have $0\leq\sqrt[n]{k}-1\leq\sqrt[n]{n}-1<\nu_{\varepsilon}$ and, by (2),

$$\left|f\left(\sqrt[n]{k}\right)-f(1)-f'(1)\left(\sqrt[n]{k}-1\right)\right|\leq\varepsilon\left(\sqrt[n]{k}-1\right).$$

Then we have

$$\left|\sum_{k=1}^{n}f\left(\sqrt[n]{k}\right)-f(1)\,n-f'(1)\sum_{k=1}^{n}\left(\sqrt[n]{k}-1\right)\right|$$
$$\leq\ \sum_{k=1}^{n}\left|f\left(\sqrt[n]{k}\right)-f(1)-f'(1)\left(\sqrt[n]{k}-1\right)\right|\leq\varepsilon\sum_{k=1}^{n}\left(\sqrt[n]{k}-1\right)$$

or $\left|\frac{\sum\limits_{k=1}^{n} f\left(\sqrt[n]{k}\right)-f(1)n}{\sum\limits_{k=1}^{n}\left(\sqrt[n]{k}-1\right)} - f'(1)\right| \le \varepsilon$. Thus $\lim\limits_{n\to\infty} \frac{\sum\limits_{k=1}^{n} f\left(\sqrt[n]{k}\right)-f(1)n}{\sum\limits_{k=1}^{n}\left(\sqrt[n]{k}-1\right)} = f'(1)$ and, by Proposition 2, we obtain

$$\lim_{n\to\infty} \frac{f\left(\sqrt[n]{1}\right)+f\left(\sqrt[n]{2}\right)+\cdots+f\left(\sqrt[n]{n}\right)-f(1)\,n}{\ln n} = f'(1)\,.$$

□

To prove the evaluation for twice differentiable function we need the next result which uses the Stirling formula.

Proposition 4. $\sum\limits_{k=1}^{n}\left(\sqrt[n]{k}-1\right) = \ln n - 1 + \frac{\ln^2 n}{2n} + o\left(\frac{\ln^2 n}{n}\right)$.

Proof. Let $\varepsilon > 0$. Since $\lim\limits_{x\to 0}\frac{e^x-1-x}{x^2} = \frac{1}{2}$, it follows that there exists $\delta_\varepsilon > 0$ such that $\forall\, 0 < x < \delta_\varepsilon$ we have $\left|\frac{e^x-1-x}{x^2} - \frac{1}{2}\right| < \varepsilon$, or

$$\left|e^x - 1 - x - \frac{x^2}{2}\right| \le \varepsilon x^2,\ \forall\, 0 \le x < \delta_\varepsilon. \tag{3}$$

Since $\lim\limits_{n\to\infty}\frac{\ln n}{n} = 0$, for $\delta_\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\forall n \ge n_\varepsilon$ we have $0 < \frac{\ln n}{n} < \delta_\varepsilon$. Let $n \ge n_\varepsilon$. For every $k = 1, \ldots, n$ we have $0 \le \frac{\ln k}{n} \le \frac{\ln n}{n} < \delta_\varepsilon$ and, by (3), $\left|e^{\frac{\ln k}{n}} - 1 - \frac{\ln k}{n} - \frac{\ln^2 k}{2n^2}\right| \le \frac{\varepsilon \ln^2 k}{n^2}$, or

$$\left|\sqrt[n]{k} - 1 - \frac{\ln k}{n} - \frac{\ln^2 k}{2n^2}\right| \le \frac{\varepsilon \ln^2 k}{n^2}.$$

We deduce that

$$\left|\sum_{k=1}^{n}\left(\sqrt[n]{k}-1\right) - \frac{1}{n}\sum_{k=1}^{n}\ln k - \frac{1}{2n^2}\sum_{k=1}^{n}\ln^2 k\right| \le \sum_{k=1}^{n}\left|\sqrt[n]{k} - 1 - \frac{\ln k}{n} - \frac{\ln^2 k}{2n^2}\right|$$

$$\le \frac{\varepsilon}{n^2}\sum_{k=1}^{n}\ln^2 k$$

or $\left|\frac{\sum\limits_{k=1}^{n}\left(\sqrt[n]{k}-1\right)-\frac{1}{n}\sum\limits_{k=1}^{n}\ln k}{\frac{1}{n^2}\sum\limits_{k=1}^{n}\ln^2 k} - \frac{1}{2}\right| \le \varepsilon$. Hence, $\lim\limits_{n\to\infty}\frac{\sum\limits_{k=1}^{n}\left(\sqrt[n]{k}-1\right)-\frac{1}{n}\sum\limits_{k=1}^{n}\ln k}{\frac{1}{n^2}\sum\limits_{k=1}^{n}\ln^2 k} = \frac{1}{2}$. Since by the Stolz-Cesàro lemma, the case $\left[\frac{\cdot}{\infty}\right]$, it holds $\sum\limits_{k=1}^{n}\ln^2 k \backsim n\ln^2 n$, we obtain $\lim\limits_{n\to\infty}\frac{\sum\limits_{k=1}^{n}\left(\sqrt[n]{k}-1\right)-\frac{1}{n}\sum\limits_{k=1}^{n}\ln k}{\frac{\ln^2 n}{n}} = \frac{1}{2}$, that is

$$\sum_{k=1}^{n}\left(\sqrt[n]{k}-1\right) = \frac{1}{n}\sum_{k=1}^{n}\ln k + \frac{\ln^2 n}{2n} + o\left(\frac{\ln^2 n}{n}\right).$$

Since, by Stirling's evaluation, $\sum\limits_{k=1}^{n} \ln k = n\ln n - n + \frac{\ln n}{2} + \ln\sqrt{2\pi} + \frac{1}{12n} + o\left(\frac{1}{n}\right)$, (see [1, 2, 5]), we deduce

$$\sum_{k=1}^{n}\left(\sqrt[n]{k}-1\right) = \ln n - 1 + \frac{\ln n}{2n} + \frac{\ln\sqrt{2\pi}}{n} + \frac{1}{12n^2} + o\left(\frac{1}{n^2}\right) + \frac{\ln^2 n}{2n} + o\left(\frac{\ln^2 n}{n}\right) = \ln n - 1 + \frac{\ln^2 n}{2n} + o\left(\frac{\ln^2 n}{n}\right),$$

because $\frac{\ln n}{2n}, \frac{1}{n}, \frac{1}{n^2} = o\left(\frac{\ln^2 n}{n}\right)$. □

Now we prove the evaluation for the case of twice differentiable functions.

Proposition 5. *Let $\eta > 0$ and $f : (1-\eta, 1+\eta) \to \mathbb{R}$ be a twice differentiable function. Then*

$$\sum_{k=1}^{n} f\left(\sqrt[n]{k}\right) = f(1)\, n + f'(1)\ln n - f'(1) + \frac{f'(1) + f''(1)}{2}\cdot\frac{\ln^2 n}{n} + o\left(\frac{\ln^2 n}{n}\right).$$

Proof. Let $\varepsilon > 0$. Since f is twice differentiable at 1, we have

$$\lim_{x\to 1}\frac{f(x) - f(1) - f'(1)(x-1)}{(x-1)^2} = \frac{f''(1)}{2},$$

thus there exists $\delta_\varepsilon > 0$ such that $\forall x \in (1-\eta, 1+\eta)$ with $|x-1| < \delta_\varepsilon$, $x \neq 1$, we have $\left|\frac{f(x)-f(1)-f'(1)(x-1)}{(x-1)^2} - \frac{f''(1)}{2}\right| < \varepsilon$, or $\forall x \in (1-\eta, 1+\eta)$, $|x-1| < \delta_\varepsilon$ the following relation holds

$$\left|f(x) - f(1) - \alpha(x-1) - \beta(x-1)^2\right| \leq \varepsilon(x-1)^2, \tag{4}$$

where $\alpha = f'(1)$, $\beta = \frac{f''(1)}{2}$. Since $\lim\limits_{n\to\infty}\sqrt[n]{n} = 1$, for $\nu_\varepsilon = \min(\eta, \delta_\varepsilon) > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$ we have $\sqrt[n]{n} - 1 < \nu_\varepsilon$. Let $n \geq n_\varepsilon$. For every $k = 1, \ldots, n$ we have $0 \leq \sqrt[n]{k} - 1 \leq \sqrt[n]{n} - 1 < \nu_\varepsilon$ and by (4),

$$\left|f\left(\sqrt[n]{k}\right) - f(1) - \alpha\left(\sqrt[n]{k}-1\right) - \beta\left(\sqrt[n]{k}-1\right)^2\right| \leq \varepsilon\left(\sqrt[n]{k}-1\right)^2.$$

We have

$$\left|\sum_{k=1}^{n} f\left(\sqrt[n]{k}\right) - f(1)\, n - \alpha\sum_{k=1}^{n}\left(\sqrt[n]{k}-1\right) - \beta\sum_{k=1}^{n}\left(\sqrt[n]{k}-1\right)^2\right|$$
$$\leq \sum_{k=1}^{n}\left|f\left(\sqrt[n]{k}\right) - f(1) - \alpha\left(\sqrt[n]{k}-1\right) - \beta\left(\sqrt[n]{k}-1\right)^2\right| \leq \varepsilon\sum_{k=1}^{n}\left(\sqrt[n]{k}-1\right)^2$$

or

$$\left|\frac{\sum\limits_{k=1}^{n} f\left(\sqrt[n]{k}\right)-f(1)\,n-\alpha\sum\limits_{k=1}^{n}\left(\sqrt[n]{k}-1\right)}{\sum\limits_{k=1}^{n}\left(\sqrt[n]{k}-1\right)^2}-\beta\right|\le\varepsilon.$$

Thus

$$\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n} f\left(\sqrt[n]{k}\right)-f(1)\,n-\alpha\sum\limits_{k=1}^{n}\left(\sqrt[n]{k}-1\right)}{\sum\limits_{k=1}^{n}\left(\sqrt[n]{k}-1\right)^2}=\beta$$

and, by Proposition 2, we deduce that

$$\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n} f\left(\sqrt[n]{k}\right)-f(1)\,n-\alpha\sum\limits_{k=1}^{n}\left(\sqrt[n]{k}-1\right)}{\frac{\ln^2 n}{n}}=\beta.$$

This is equivalent to

$$\sum_{k=1}^{n} f\left(\sqrt[n]{k}\right)=f(1)\,n+\alpha\sum_{k=1}^{n}\left(\sqrt[n]{k}-1\right)+\frac{\beta\ln^2 n}{n}+o\left(\frac{\ln^2 n}{n}\right),$$

which by Proposition 4 gives us

$$\sum_{k=1}^{n} f\left(\sqrt[n]{k}\right)=f(1)\,n+\alpha\ln n-\alpha+\left(\frac{\alpha}{2}+\beta\right)\frac{\ln^2 n}{n}+o\left(\frac{\ln^2 n}{n}\right)$$

$$=f(1)\,n+f'(1)\ln n-f'(1)+\frac{f'(1)+f''(1)}{2}\cdot\frac{\ln^2 n}{n}+o\left(\frac{\ln^2 n}{n}\right).$$

□

Now we give some examples.

Corollary 6. *Let $a>0$ and $\varphi:(0,1+a)\to\mathbb{R}$ be a twice differentiable function. Then for every natural number p*

$$\varphi\left(\sqrt[pn]{1}\right)+\varphi\left(\sqrt[pn]{2}\right)+\cdots+\varphi\left(\sqrt[pn]{n}\right)=n\varphi(1)+\frac{\varphi'(1)}{p}\cdot\ln n-\frac{\varphi'(1)}{p}$$

$$+\frac{\varphi'(1)+\varphi''(1)}{2p^2}\cdot\frac{\ln^2 n}{n}+o\left(\frac{\ln^2 n}{n}\right).$$

Proof. In Proposition 5 we take $f:(0,1+a)\to\mathbb{R}$, $f(x)=\varphi\left(\sqrt[p]{x}\right)$. We have $f'(x)=\frac{x^{\frac{1}{p}-1}\varphi'\left(\sqrt[p]{x}\right)}{p}$, $f''(x)=\frac{1}{p}\left(\frac{x^{\frac{2}{p}-2}\varphi''\left(\sqrt[p]{x}\right)}{p}+\left(\frac{1}{p}-1\right)x^{\frac{1}{p}-2}\varphi'\left(\sqrt[p]{x}\right)\right)$ and thus $f'(1)+f''(1)=\frac{\varphi''(1)+\varphi'(1)}{p^2}$. □

Corollary 7. (i) *Let p be a natural number and $\alpha \geq 0$. Then*

$$\sum_{k=1}^{n} \frac{1}{\sqrt[pn]{k}+\alpha} = \frac{n}{\alpha+1} - \frac{\ln n}{p(\alpha+1)^2} + \frac{1}{p(\alpha+1)^2} + \frac{1-\alpha}{2p^2(\alpha+1)^3} \cdot \frac{\ln^2 n}{n} + o\left(\frac{\ln^2 n}{n}\right).$$

(ii) *Let p be a natural number and $\alpha \in \mathbb{R}^*$. Then*

$$e^{\alpha \sqrt[pn]{1}} + e^{\alpha \sqrt[pn]{2}} + \cdots + e^{\alpha \sqrt[pn]{n}} = ne^{\alpha} + \frac{\alpha e^{\alpha}}{p} \ln n - \frac{\alpha e^{\alpha}}{p} + \frac{\alpha(\alpha+1)}{2p^2} \cdot e^{\alpha} \cdot \frac{\ln^2 n}{n} + o\left(\frac{\ln^2 n}{n}\right).$$

Proof. (i) We take in Corollary 6, $\varphi : (0,2) \to \mathbb{R}$, $\varphi(x) = \frac{1}{x+\alpha}$, so that $\varphi'(x) = -\frac{1}{(x+\alpha)^2}$, $\varphi''(x) = \frac{2}{(x+\alpha)^3}$.

(ii) We take in Corollary 6, $\varphi : (0,2) \to \mathbb{R}$, $\varphi(x) = e^{\alpha x}$, for which one computes $\varphi'(x) = \alpha e^{\alpha x}$, $\varphi''(x) = \alpha^2 e^{\alpha x}$. □

Corollary 8. *Let $\eta > 0$ and $f : (1-\eta, 1+\eta) \to (0,\infty)$ be a twice differentiable function. Then for every natural number p*

$$\prod_{k=1}^{n} f\left(\sqrt[pn]{k}\right) \backsim e^{-\frac{f'(1)}{pf(1)}} [f(1)]^n \cdot n^{\frac{f'(1)}{pf(1)}}.$$

Proof. Let $\varphi : (1-\eta, 1+\eta) \to \mathbb{R}$, $\varphi(x) = \ln f(x)$. Then f is twice differentiable and by Corollary 6

$$\sum_{k=1}^{n} \ln f\left(\sqrt[pn]{k}\right) = n \ln f(1) + \frac{f'(1)}{pf(1)} \cdot \ln n - \frac{f'(1)}{pf(1)} + \beta_n,$$

where $\beta_n = \frac{\frac{f'(1)}{f(1)} + \frac{f''(1)f(1) - [f'(1)]^2}{[f(1)]^2}}{2p^2} \cdot \frac{\ln^2 n}{n} + o\left(\frac{\ln^2 n}{n}\right) \to 0$ as $n \to \infty$. Then

$$\frac{\prod_{k=1}^{n} f\left(\sqrt[pn]{k}\right)}{[f(1)]^n \cdot n^{\frac{f'(1)}{pf(1)}}} = e^{-\frac{f'(1)}{pf(1)}} \cdot e^{\beta_n} \to e^{-\frac{f'(1)}{pf(1)}} \text{ as } n \to \infty.$$

□

Corollary 9. *Let p be a natural number and $a > 0$. Then*

$$\prod_{k=1}^{n} \left(1 + a \sqrt[pn]{k}\right) \backsim e^{-\frac{a}{p(a+1)}} (1+a)^n \cdot n^{\frac{a}{p(a+1)}}$$

and

$$\prod_{k=1}^{n} \ln\left(1 + a \sqrt[pn]{k}\right) \backsim e^{-\frac{a}{p(a+1)\ln(a+1)}} \ln^n(1+a) \cdot n^{\frac{a}{p(a+1)\ln(a+1)}}.$$

Proof. Take in Corollary 8, $f : (0,2) \to (0,\infty)$, $f(x) = 1 + ax$ (respectively $f(x) = \ln(1+ax)$). □

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Density of the space of bounded Lipschitz functions in the space of continuous functions

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Abstract. We provide a proof for the fact that a continuous function $f : T \to X$, where (T,d) is a compact metric space and X is a Hilbert space, can be approximated by bounded Lipschitz functions $g : T \to X$.

Keywords: Lipschitz functions, continuous functions, approximation, density of Lipschitz functions in continuous functions

MSC: 46C05, 46C07, 41A30, 26A16

1. Introduction

The Lipschitz functions form an important class of continuous functions, playing an "intermediate role" between general continuous functions and differentiable functions. In this respect, we can think at the famous Rademacher theorem, which asserts that a lipschitzian function is almost everywhere differentiable.

An important problem in general Functional Analysis is the problem of approximating elements of a normed space X with elements of a subspace Y of X, the elements of Y being more "accessible" or easier to handle. It is the case of X, the space of continuous functions, and its subspace Y, the subspace of lipschitzian functions.

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Within this framework, the problem has been attacked by many mathematicians (see e.g. [1], [3], [4]).

The present paper offers a contribution in this direction.

2. Preliminary facts

Let K be either $\mathbb{R}$ or $\mathbb{C}$, $\mathbb{N}^* = \{1, 2, \ldots\}$, and X a K-vector space (usually, X is a normed space). For any non empty set T and any normed space $(X, ||\cdot||)$, we can consider the Banach space

$$\mathcal{B}(T, X) = \{f : T \to X \mid f \text{ is bounded}\}$$

equipped with the norm

$$f \longmapsto \|f\|_\infty = \sup\left\{ \|f(t)\| \;\middle|\; t \in T \right\}$$

(the norm of uniform convergence). We will work in the particular situation when (T, d) is a compact metric space (T having at least two elements). Then we have $\mathcal{C}(T, X) \subset \mathcal{B}(T, X)$, where

$$\mathcal{C}(T, X) = \left\{ f : T \to X \;\middle|\; f \text{ is continuous} \right\}$$

is a Banach space when equipped with the induced norm $||\cdot||_\infty$.

Let (T, d) and (X, ρ) be two metric spaces, T having at least two elements, and let $f : T \to X$. The Lipschitz constant of f is defined by the formula

$$\|f\|_L = \sup\left\{ \frac{\rho\big(f(x), f(y)\big)}{d(x, y)} \;\middle|\; x,\ y \in T,\ x \neq y \right\}.$$

In case $\|f\|_L < \infty$, we say f is a Lipschitz map. In this case, we have $\rho(f(x), f(y)) \leqslant \|f\|_L \, d(x, y)$ for any $x, y \in T$. The set of all lipschitzian functions $f : T \to X$ will be denoted by $\mathrm{Lip}(T, X)$. When X is a normed space, it follows that $\mathrm{Lip}(T, X)$ is a seminormed vector space with the seminorm $f \longmapsto \|f\|_L$. When (T, d) is a compact metric space, $\mathrm{Lip}(T, X) = \mathcal{B}L(T, X)$ (bounded Lipschitz), $\mathrm{Lip}(T, X) = \mathcal{B}L(T, X) \subset \mathcal{C}(T, X) \subset \mathcal{B}(T, X)$, and $\mathrm{Lip}(T, X)$ is a normed space with the norm

$$f \longmapsto \|f\|_{\mathcal{B}L} \stackrel{\text{def}}{=} \|f\|_\infty + \|f\|_L \,.$$

Density results similar to the one we present in this paper can be found in [1], [3], [4], [5].

For general Functional Analysis, see [2].

3. The Result

Theorem 1. *Let (T, d) be a compact metric space and X a Hilbert space. Then $\mathcal{B}L(T, X)$ is dense in $\mathcal{C}(T, X)$, if $\mathcal{C}(T, X)$ is endowed with the natural norm $||.||_\infty$. In particular, if X is a separable Hilbert space, then $\mathcal{C}(T, X)$ is also separable.*

Proof. **First step.** The case $X = K$.

In this case, the proof can be obtained using the Stone-Weierstrass theorem, as a direct application of the fact that the space $\mathcal{B}L(T,K)$ is an algebra of functions with all the required properties. Moreover, there exists a sequence $(f^m)_{m\geqslant 1} \subset \mathcal{B}L(T,K)$ such that the set

$$A = \left\{ f^m \;\middle|\; m \in \mathbb{N}^\star \right\}$$

is dense in $\mathcal{C}(T,K)$. For details, see [5].

Second step. The case $X = K^n$.

According to the previous step, there exists a countable set

$$A = \left\{ g^m \;\middle|\; m \in \mathbb{N}^\star \right\} \subset \mathcal{B}L(T,K)$$

that is dense in $\mathcal{C}(T,K)$. Then the set $A^n = \underbrace{A \times A \times \ldots \times A}_{n \text{ times}} \subset \mathcal{B}L\left(T, K^n\right)$ is countable and dense in $\mathcal{C}\left(T, K^n\right)$. Indeed, if $f \in A^n$ let $f_1, \ldots, f_n \in A$ be such that $f = (f_1, \ldots, f_n)$. We have (all norms on K^n are equivalent):

$$\left\| f\left(t'\right) - f\left(t''\right) \right\|_{K^n} \leqslant \sum_{j=1}^{n} \left| f_j\left(t'\right) - f_j\left(t''\right) \right| \leqslant \sum_{j=1}^{n} \|f_j\|_{\mathcal{B}L}\, d\left(t',\ t''\right), \forall\, t', t'' \in T.$$

Then $f \in \mathcal{B}L\left(T, K^n\right)$, so the inclusion $A^n \subset \mathcal{B}L\left(T, K^n\right)$ holds. The countability of A^n is obvious. Finally, let $f \in \mathcal{C}\left(T, K^n\right)$, $f = (f_1, \ldots, f_n)$. As for every $1 \leqslant j \leqslant n$ there exists a sequence $\left(g_j^m\right)_{m\geqslant 1} \subset A$ such that

$$\left\| g_j^m - f_j \right\|_{\mathcal{B}L(T,K)} \xrightarrow[m\to\infty]{} 0,$$

we get that for $g^m := (g_1^m, \ldots, g_n^m) \in A^n$ we have

$$\|g^m - f\|_{\mathcal{B}L(T,\ K^n)} \leqslant \sum_{j=1}^{n} \left\| g_j^m - f_j \right\|_{\mathcal{B}L(T,\ K)} \xrightarrow[m\to\infty]{} 0.$$

Third step. The case when X is a finite dimensional Hilbert space.

Let $\dim_K X = n \in \mathbb{N}^\star$. We know that in this case X is isometrically isomorphic with K^n. Let $\varphi : K^n \to X$ be such an isometric isomorphism. We denote by $\Phi : \mathcal{C}\left(T,\ K^n\right) \to \mathcal{C}\left(T, X\right)$ the isometric isomorphism of Banach spaces induced by φ. Then $\Phi\left(\mathcal{B}L\left(T,\ K^n\right)\right) = \mathcal{B}L(T,X)$ and, as $\mathcal{B}L\left(T,\ K^n\right)$ is dense in $\mathcal{C}\left(T,\ K^n\right)$, we deduce that $\mathcal{B}L(T,X)$ is dense in $\mathcal{C}(T,X)$.

Fourth step. The case when X is a separable Hilbert space.

In this case there exists an orthonormal basis $(e_i)_{i\in\mathbb{N}^*}$ for X, hence there exists a canonical isomorphism $\varphi : l^2 \to X$.

Let us denote

$$\Phi : \mathcal{C}\left(T, l^2\right) \to \mathcal{C}(T,X)$$

the isometric isomorphism of Banach spaces induced by φ.

As $\Phi\left(\mathcal{B}L\left(T, l^2\right)\right) = \mathcal{B}L(T,X)$, it will be enough to prove that $\mathcal{B}L\left(T, l^2\right)$ is dense in $\mathcal{C}\left(T, l^2\right)$.

For $n \geqslant 1$ we denote by

$$\mathcal{C}^{(n)}\left(T,\ l^2\right) := \left\{ f \in \mathcal{C}\left(T,\ l^2\right) \,\middle|\, \langle f(t),\ e_k\rangle = 0,\ \forall\, t \in T,\ \forall\, k \geqslant n+1 \right\},$$

and

$$\mathcal{B}L^{(n)}\left(T,\ l^2\right) = \mathcal{B}L\left(T,\ l^2\right) \cap \mathcal{C}^{(n)}\left(T,\ l^2\right).$$

Also, we will use the notations:

$$\mathcal{C}^0\left(T,\ l^2\right) = \bigcup_{n\geqslant 1} \mathcal{C}^{(n)}\left(T,\ l^2\right),$$

$$\mathcal{B}L^0\left(T,\ l^2\right) = \bigcup_{n\geqslant 1} \mathcal{B}L^{(n)}\left(T,\ l^2\right).$$

As $\mathcal{C}^{(n)}\left(T,\ l^2\right)$ is isometrically isomorphic with $\mathcal{C}\left(T, K^n\right)$, and $\mathcal{B}L\left(T,\ K^n\right)$ is dense in $\mathcal{C}\left(T, K^n\right)$, we deduce that ($\overline{H}^{||\cdot||_\infty}$ is the closure of $H \subset \mathcal{C}(T, X)$ for the $||\cdot||_\infty$ norm):

$$\overline{\mathcal{B}L^{(n)}\left(T,\ l^2\right)}^{||\cdot||_\infty} \supset \mathcal{C}^{(n)}\left(T,\ l^2\right),\ \forall\, n \geqslant 1.$$

Let $f \in \mathcal{C}\left(T, l^2\right)$ and $\varepsilon > 0$. We have

$$f(t) = \sum_{k\geqslant 1} f_k(t) e_k,\ \forall\, t \in T,$$

where $f_k(t) := \langle f(t),\ e_k\rangle$. We know that

$$\|f(t)\|^2 = \sum_{k\geqslant 1} |f_k(t)|^2,\, t \in T,$$

the convergence being uniform on T (according to Dini's theorem).

As a consequence, there exists $n_\varepsilon \geqslant 1$ such that

$$\|f(t) - f_\varepsilon(t)\|^2 = \sum_{k=n_\varepsilon+1}^{\infty} |f_k(t)|^2 < \frac{\varepsilon^2}{4},\ \forall\, t \in T,$$

where

$$f_\varepsilon(t) = \sum_{k=1}^{n_\varepsilon} f_k(t) e_k,\ \ t \in T.$$

As we can consider that $f_\varepsilon \in \mathcal{C}\left(T, K^{n_\varepsilon}\right)$, it results that there exists $g_\varepsilon \in \mathcal{B}L\left(T, K^{n_\varepsilon}\right)$ (which can be identified with $\mathcal{B}L^{(n_\varepsilon)}\left(T,\ l^2\right)$) such that

$$\|f_\varepsilon - g_\varepsilon\|_\infty < \frac{\varepsilon}{2} \iff \|f_\varepsilon(t) - g_\varepsilon(t)\|_{l^2} < \frac{\varepsilon}{2},\ \forall t \in T.$$

Then

$$\|f(t) - g_\varepsilon(t)\|_{l^2}^2 = \|f(t) - f_\varepsilon(t)\|_{l^2}^2 + \|f_\varepsilon(t) - g_\varepsilon(t)\|_{l^2}^2 < \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{4} = \frac{\varepsilon^2}{2},\ t \in T.$$

As a consequence,

$$\|f(t) - g_\varepsilon(t)\|_{l^2} < \frac{\varepsilon}{\sqrt{2}},\ \forall t \in T \implies \|f - g_\varepsilon\|_\infty < \varepsilon.$$

Then $\mathcal{BL}^0\left(T,\ l^2\right)$ is dense in $\mathcal{C}\left(T,\ l^2\right)$. As $\mathcal{BL}\left(T,\ l^2\right) \supset \mathcal{BL}^0\left(T,\ l^2\right)$, we will have even more that $\mathcal{BL}\left(T,\ l^2\right)$ is dense in $\mathcal{C}\left(T,\ l^2\right)$.

The space $\mathcal{BL}^0\left(T,\ l^2\right)$ is separable, because in every topological space we have the implication:

$$\left(A_n \subset B_n \text{ and } \overline{A_n} \supset B_n,\ \forall n \in \mathbb{N}\right) \text{ implies } \left(\overline{\bigcup_n A_n} \supset \bigcup_n B_n\right).$$

Then $\mathcal{C}\left(T, l^2\right)$ is separable.

Fifth step. The case when X is a nonseparable Hilbert space.

It means that there exists an orthonormal basis $(e_i)_{i\in I}$ for X, with $\mathrm{card}I > \aleph_0$. Let $(t_k)_{k\geqslant 1}$ be a sequence in T dense in T. Let $f: T \to X$ be a continuous function.

We know that for every $k \geqslant 1\ \exists\ J_k \subset I$ at most countable such that

$$\langle f\left(t_k\right), e_i\rangle = 0, \text{ for } \forall\, i \in I \setminus J_k$$

and

$$f\left(t_k\right) = \sum_{i\in J_k} \langle f\left(t_k\right),\ e_i\rangle\, e_i.$$

If we denote by $J = \bigcup_{k\geqslant 1} J_k$, then J is a countable subset of I and $\langle f\left(t_k\right), e_i\rangle = 0,\ \forall\, k \geqslant 1,\ \forall i \in I \setminus J$.

The natural conclusion is that $\langle f(t), e_i\rangle = 0,\ \forall\, i \in I \setminus J$.

Then $f(t) = \sum_{i\in J} \langle f(t), e_i\rangle\, e_i,\ \forall\, t \in T$.

Moreover, if we denote by X_J the closed subspace of X generated by the countable family $(e_i)_{i\in J}$, we have that $f\left(T\right) \subset X_J$. Let $\Pi_J: X \to X_J$ be the orthogonal projection of X onto X_J. Then $f_J := \Pi_J \circ f\ \in \mathcal{C}\left(T,\ X_J\right)$. As X_J is a separable Hilbert space, according to the Fourth Step, $\exists\ \widetilde{g_\varepsilon} \in \mathcal{BL}\left(T,\ X_J\right)$ such that

$$\left\|f_J - \widetilde{g_\varepsilon}\right\| < \varepsilon \iff \left\|f_J(t) - \widetilde{g_\varepsilon}\left(t\right)\right\|_{X_J} < \varepsilon,\ \forall\ t \in T.$$

If we denote by $i_J: X_J \to X$ the canonical embedding of X_J into X, which is obviously an isometry, and we remark that $f = i_J \circ f_J$, we infer that

$$\left\|f - g_\varepsilon\right\|_\infty < \varepsilon, \text{ where } g_\varepsilon = i_J \circ \widetilde{g_\varepsilon}.$$

It is clear that $g_\varepsilon \in \mathcal{BL}(T, X)$. □

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Olimpiada de matematică a studenţilor din sud-estul Europei, SEEMOUS 2019[1)]

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Abstract. The 13th South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2019, was held on 12–17 March 2019, in Devin, Bulgaria. We present the competition problems and their solutions as given by the corresponding authors. Solutions provided by some of the competing students are also included here.

Keywords: Diagonalizable matrix, rank, trace, change of variable, integrals, series

MSC: Primary 15A03; Secondary 15A21, 26D15.

Introduction

SEEMOUS (South Eastern European Mathematical Olympiad for University Students) este o competiţie anuală de matematică, adresată studenţilor din anii I şi II ai universităţilor din sud-estul Europei. A 13-a ediţie a acestei competiţii a avut loc între 12 şi 17 martie 2019 şi a fost organizată de către Universitatea de Arhitectură, Inginerie Civilă şi Geodezie din Sofia, Bulgaria. Concursul s-a desfaşurat în localitatea Devin din Bulgaria, la acesta luând parte un număr de 83 de studenţi de la 19 universităţi din Bulgaria, FYR Macedonia, Grecia, România şi Turkmenistan.

A existat o singură probă de concurs constând din patru probleme iar pentru rezolvarea lor s-au acordat 5 ore. Acestea au fost selectate de juriu dintre cele 40 de probleme propuse şi au fost considerate ca având diverse grade de dificultate: Problema 1 – grad redus de dificultate, Problemele 2, 3 – dificultate medie, Problema 4 – grad ridicat de dificultate.

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Au fost acordate 11 medalii de aur, 19 medalii de argint şi 30 de medalii de bronz.

Prezentăm, în continuare, problemele de concurs şi soluţiile acestora, aşa cum au fost indicate de autorii lor. De asemenea, prezentăm şi soluţiile date de către unii studenţi, diferite de soluţiile autorilor.

Problema 1. Un şir $(x_n)_{n\geq 1}$ de numere din intervalul $[0,1]$ se numeşte *şir Devin* dacă pentru orice funcţie continuă $f:[0,1]\to\mathbb{R}$ are loc relaţia

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f(x_i)=\int_0^1 f(x)\mathrm{d}x. \tag{1}$$

Arătaţi că un şir $(x_n)_{n\geq 1}$ de numere din intervalul $[0,1]$ este şir Devin dacă şi numai dacă $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}x_i^k=\frac{1}{k+1}$ pentru orice $k\in\mathbb{N}$.

Juriul a considerat că este de aşteptat ca majoritatea concurenţilor să cunoască teorema Weierstrass-Stone şi să perceapă problema ca fiind o aplicaţie simplă a acesteia. Concurenţii nu au reacţionat însă conform cu aşteptările juriului, şi, întrucât o abordare alternativă care să nu se bazeze pe teoreme de aproximare a funcţiilor continue prin polinoame a fost dificil de identificat, mai puţin de 20% *dintre ei au reuşit să rezolve problema.*

Soluţie. Implicaţia directă se obţine luând în definiţia şirului Devin funcţiile particulare $f_k:[0,1]\to\mathbb{R}$, $f_k(x)=x^k$ $(k\in\mathbb{N})$.

Pentru implicaţia inversă, notăm cu $\mathcal{P}$ proprietatea $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}f(x_i)=\int_0^1 f(x)\mathrm{d}x$ referitoare la funcţii integrabile pe $[0,1]$.

Fie $\varepsilon>0$ şi $g:[0,1]\to\mathbb{R}$ o funcţie continuă. Conform teoremei Weierstrass-Stone, există un polinom $P_{g,\varepsilon}\in\mathbb{R}[X]$ astfel încât

$$|g(x)-P_{g,\varepsilon}(x)|<\frac{\varepsilon}{3},\quad \forall\, x\in[0,1]. \tag{2}$$

Din ipoteză, $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}x_i^k=\frac{1}{k+1}$, deci funcţiile $f_k:[0,1]\to\mathbb{R}$, $f_k(x)=x^k$ $(k\in\mathbb{N})$ au proprietatea $\mathcal{P}$.

Cum ambii membri ai relaţiei (1) sunt $\mathbb{R}$-liniari ca funcţii de f, obţinem faptul că orice funcţie polinomială $P:[0,1]\to\mathbb{R}$ verifică relaţia (1). De aici deducem că există $N\in\mathbb{N}$ astfel încât pentru orice $n\geq N$ să aibă loc inegalitatea

$$\left|\frac{1}{n}\sum_{i=1}^{n}P_{g,\varepsilon}(x_i)-\int_0^1 P_{g,\varepsilon}(x)\mathrm{d}x\right|<\frac{\varepsilon}{3}. \tag{3}$$

Din relaţiile (2) şi (3) deducem că pentru orice $n\geq N$ avem

$$\left|\frac{1}{n}\sum_{i=1}^{n} g(x_i) - \int_0^1 g(x)\mathrm{d}x\right| \le \left|\frac{1}{n}\sum_{i=1}^{n}\left(g(x_i) - P_{g,\varepsilon}(x_i)\right)\right| +$$

$$+\left|\frac{1}{n}\sum_{i=1}^{n} P_{g,\varepsilon}(x_i) - \int_0^1 P_{g,\varepsilon}(x)\mathrm{d}x\right| + \left|\int_0^1 \left(P_{g,\varepsilon}(x) - g(x)\right)\mathrm{d}x\right| <$$

$$< \frac{2\varepsilon}{3} + \int_0^1 \left|P_{g,\varepsilon}(x) - g(x)\right|\mathrm{d}x \le \varepsilon.$$

Trecând la limită în inegalitatea de mai sus, obţinem

$$\left|\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} g(x_i) - \int_0^1 g(x)\mathrm{d}x\right| \le \varepsilon.$$

Cum însă ε a fost ales arbitrar, obţinem $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n} g(x_i) = \int_0^1 g(x)\mathrm{d}x$, deci şi funcţia g are proprietatea $\mathcal{P}$.

Ca urmare, $(x_n)_n$ este şir Devin.

Problema 2. Fie m, n numere naturale nenule. Arătaţi că oricare ar fi matricele $A_1, \ldots, A_m \in \mathcal{M}_n(\mathbb{R})$ există $\varepsilon_1, \ldots, \varepsilon_m \in \{-1, 1\}$ astfel încât

$$\operatorname{Tr}\left((\varepsilon_1 A_1 + \cdots + \varepsilon_m A_m)^2\right) \ge \operatorname{Tr}\left(A_1^2\right) + \cdots + \operatorname{Tr}\left(A_m^2\right). \qquad (1)$$

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Juriul a considerat că această problemă este una de dificultate medie. Cu toate acestea în jur de 25% dintre concurenţi au obţinut maxim de puncte, fiind în cele din urmă cea mai uşoară din cele patru probleme.

Soluţia 1 (a autorului). Această soluţie se bazează pe observaţia că funcţia

$$f : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}, \quad f(A) = \operatorname{Tr}\left(A^2\right),$$

are proprietatea că

$$f(A+B) + f(A-B) = 2(f(A) + f(B)), \quad \text{oricare ar fi } A, B \in \mathcal{M}_n(\mathbb{R}). \qquad (2)$$

Într-adevăr,

$$f(A+B) + f(A-B) = \operatorname{Tr}\left((A+B)^2 + (A-B)^2\right) = \operatorname{Tr}\left(2A^2 + 2B^2\right) =$$
$$= 2\left(\operatorname{Tr}(A^2) + \operatorname{Tr}(B^2)\right) = 2(f(A) + f(B)).$$

Acum (1) se demonstrează uşor prin inducţie după $m \ge 1$.

Afirmaţia este evident adevărată pentru $m = 1$ considerând $\varepsilon_1 = 1$.

Pentru pasul de inducţie (de la m la $m+1$), fie $A_1,\dots,A_m,\ A_{m+1}\in\mathcal{M}_n(\mathbb{R})$. Din ipoteza de inducţie există $\varepsilon_1,\dots,\varepsilon_m\in\{-1,1\}$ astfel încât

$$\operatorname{Tr}\left((\varepsilon_1A_1+\cdots+\varepsilon_mA_m)^2\right)\geq\operatorname{Tr}\left(A_1^2\right)+\cdots+\operatorname{Tr}\left(A_m^2\right)\tag{3}$$

şi notăm $A=\varepsilon_1A_1+\cdots+\varepsilon_mA_m$. Folosind (2) obţinem

$$f(A+A_{m+1})+f(A-A_{m+1})=2(f(A)+f(A_{m+1})),$$

ceea ce înseamnă că cel puţin una dintre inegalităţile

$$\begin{aligned}f(A+A_{m+1})&\geq f(A)+f(A_{m+1})\\ f(A-A_{m+1})&\geq f(A)+f(A_{m+1})\end{aligned}$$

este adevărată. Aşadar există $\varepsilon_{m+1}\in\{-1,1\}$ astfel încât

$$f(A+\varepsilon_{m+1}A_{m+1})\geq f(A)+f(A_{m+1}),$$

care se rescrie astfel:

$$\begin{aligned}\operatorname{Tr}\left((\varepsilon_1A_1+\cdots+\varepsilon_mA_m+\varepsilon_{m+1}A_{m+1})^2\right)\geq\operatorname{Tr}\left((\varepsilon_1A_1+\cdots+\varepsilon_mA_m)^2\right)+\\ +\operatorname{Tr}\left(A_{m+1}^2\right).\end{aligned}\tag{4}$$

Combinând (3) şi (4) rezultă că

$$\begin{aligned}\operatorname{Tr}\left((\varepsilon_1A_1+\cdots+\varepsilon_mA_m+\varepsilon_{m+1}A_{m+1})^2\right)\geq\operatorname{Tr}\left(A_1^2\right)+\cdots+\operatorname{Tr}\left(A_m^2\right)+\\ +\operatorname{Tr}\left(A_{m+1}^2\right).\end{aligned}$$

ceea ce încheie demonstraţia.

Soluţia 2. Această soluţie a fost găsită de către membrii juriului care au corectat la această problemă.

Fie $E=\{\varepsilon=(\varepsilon_1,\dots,\varepsilon_m):\varepsilon_i\in\{-1,1\}\text{ pentru orice }i\}$. Evident avem $|E|=2^m$.

Dacă $A_1,\dots,A_m\in\mathcal{M}_n(\mathbb{R})$ şi $\varepsilon\in E$, atunci

$$(\varepsilon_1A_1+\cdots+\varepsilon_mA_m)^2=\left(A_1^2+\cdots+A_m^2\right)+\sum_{i\neq j}\varepsilon_i\varepsilon_jA_iA_j$$

şi sumând după toţi $\varepsilon\in E$ obţinem

$$\sum_{\varepsilon\in E}(\varepsilon_1A_1+\cdots+\varepsilon_mA_m)^2=2^m\left(A_1^2+\cdots+A_m^2\right)+\sum_{\varepsilon\in E}\sum_{i\neq j}\varepsilon_i\varepsilon_jA_iA_j.$$

Analizând suma dublă

$$S=\sum_{\varepsilon\in E}\sum_{i\neq j}\varepsilon_i\varepsilon_jA_iA_j=\sum_{i\neq j}\left(\sum_{\varepsilon\in E}\varepsilon_i\varepsilon_j\right)A_iA_j,$$

rezultă că $S = 0$, deoarece $\sum_{\varepsilon \in E} \varepsilon_i \varepsilon_j = 0$ pentru orice $i \neq j$ ($\varepsilon_i \varepsilon_j = 1$ pentru jumătate din elementele lui E şi $\varepsilon_i \varepsilon_j = -1$ pentru cealaltă jumătate). În concluzie

$$\sum_{\varepsilon \in E} (\varepsilon_1 A_1 + \cdots + \varepsilon_m A_m)^2 = 2^m \left(A_1^2 + \cdots + A_m^2\right)$$

şi cum Tr este o aplicaţie liniară rezultă că

$$\sum_{\varepsilon \in E} \mathrm{Tr}\,(\varepsilon_1 A_1 + \cdots + \varepsilon_m A_m)^2 = 2^m \left(\mathrm{Tr}\left(A_1^2\right) + \cdots + \mathrm{Tr}\left(A_m^2\right)\right),$$

deci media lui $\mathrm{Tr}\,(\varepsilon_1 A_1 + \cdots + \varepsilon_m A_m)^2$ peste E este $\mathrm{Tr}\left(A_1^2\right) + \cdots + \mathrm{Tr}\left(A_m^2\right)$, ceea ce este suficient pentru a obţine concluzia.

Remarcă. Urmărind argumentele din ambele soluţii, concluzia rămâne adevărată dacă Tr se înlocuieşte cu orice funcţională liniară pe $\mathcal{M}_n(\mathbb{R})$. De asemenea, putem demonstra următoarea afirmaţie generală:

Dacă $(G, +)$ este un grup şi $f : G \to \mathbb{R}$ satisface ecuaţia funcţională pătratică

$$f(a+b) + f(a-b) = 2(f(a) + f(a)) \quad \text{pentru orice } a, b \in G,$$

atunci pentru orice $m \geq 1$ şi orice $a_1, \ldots, a_m \in G$ au loc următoarele:

(1) $\sum f\left(\pm_1 a_1 \pm_2 \cdots \pm_m a_m\right) = 2^m \left(f\left(a_1\right) + \cdots + f\left(a_m\right)\right)$, unde suma se consideră după toate alegerile posibile $\pm_i$ ale semnelor $+$ şi $-$ pentru orice a_i ($i = \overline{1, m}$);

(2) există o alegere $\pm_i$ a semnelor $+$ şi $-$ pentru fiecare a_i ($i = \overline{1, m}$) astfel încât $f\left(\pm_1 a_1 \pm_2 \cdots \pm_m a_m\right) \geq f\left(a_1\right) + \cdots + f\left(a_m\right)$.

În cazul nostru, $G = \mathcal{M}_n(\mathbb{R})$ şi $f(A) = \mathrm{Tr}\left(A^2\right)$.

Problema 3. Fie $n \geq 2$ şi $A, B \in \mathcal{M}_n(\mathbb{C})$ cu proprietatea că $B^2 = B$. Arătaţi că

$$\mathrm{rank}(AB - BA) \leq \mathrm{rank}(AB + BA). \tag{1}$$

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Juriul a considerat această problemă ca fiind de dificultate medie. În jur de 8% dintre concurenţi au obţinut punctaj maxim la această problemă.

Soluţia 1 (a autorului). Deoarece $B^2 = B$, rezultă că valorile proprii ale lui B aparţin mulţimii $\{0, 1\}$. Notăm cu J_B forma Jordan a lui B. Atunci orice celulă Jordan J_λ a lui J_B satisface $J_\lambda^2 = J_\lambda$, ceea ce se întâmplă doar pentru celule de dimensiune 1. În concluzie J_B este o matrice diagonală de forma $\left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right]$, cu k eventual 0. De asemenea, fie P o matrice de asemănare între B şi J_B, i.e.,

$$J_B = P^{-1} B P, \quad B = P J_B P^{-1}.$$

Dacă $k = 0$, atunci $J_B = O_n$, deci $B = O_n$ şi inegalitatea (1) este în mod evident o egalitate.

Dacă $k \geq 1$, atunci fie $C = P^{-1}AP$ (equivalent, $A = PCP^{-1}$). Rezultă că

$$\begin{aligned} AB - BA &= P(CJ_B - J_BC)P^{-1}, \\ AB + BA &= P(CJ_B - J_BC)P^{-1}, \end{aligned}$$

deci

$$\begin{aligned} \operatorname{rank}(AB - BA) &= \operatorname{rank}(CJ_B - J_BC), \\ \operatorname{rank}(AB + BA) &= \operatorname{rank}(CJ_B + J_BC). \end{aligned}$$

Scriem $C = \left[\begin{array}{c|c} C_1 & C_2 \\ \hline C_3 & C_4 \end{array}\right]$, cu $C_1 \in \mathcal{M}_k(\mathbb{C})$, $C_4 \in \mathcal{M}_{n-k}(\mathbb{C})$. Atunci

$$\begin{aligned} CJ_B &= \left[\begin{array}{c|c} C_1 & C_2 \\ \hline C_3 & C_4 \end{array}\right] \cdot \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} C_1 & 0 \\ \hline C_3 & 0 \end{array}\right], \\ J_BC &= \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right] \cdot \left[\begin{array}{c|c} C_1 & C_2 \\ \hline C_3 & C_4 \end{array}\right] = \left[\begin{array}{c|c} C_1 & C_2 \\ \hline 0 & 0 \end{array}\right], \\ CJ_B - J_BC &= \left[\begin{array}{c|c} 0 & -C_2 \\ \hline C_3 & 0 \end{array}\right], \\ CJ_B + J_BC &= \left[\begin{array}{c|c} 2C_1 & C_2 \\ \hline C_3 & 0 \end{array}\right]. \end{aligned}$$

Obţinem

$$\begin{aligned} &\operatorname{rank}(CJ_B - J_BC) = \operatorname{rank} C_2 + \operatorname{rank} C_3, \\ &\operatorname{rank}(CJ_B + J_BC) \geq \operatorname{rank} \left[\begin{array}{c|c} 0 & C_2 \\ \hline C_3 & 0 \end{array}\right] = \operatorname{rank} C_2 + \operatorname{rank} C_3, \end{aligned}$$

ceea ce încheie demonstraţia.

Soluţia 2 (dată în concurs de Andrei Alexandru Jelea de la Universitatea Politehnica din Bucureşti).

Deoarece $B^2 = B$, avem că $B\,(B - I_n) = (B - I_n)\,B = O_n$, deci

$$\begin{aligned} (B - I_n)\,(AB + BA)\,(B - I_n) &= (B - I_n)\,AB\,(B - I_n) \\ &\quad + (B - I_n)\,BA\,(B - I_n) \\ &= O_n. \end{aligned}$$

Folosind inegalitatea Frobenius

$$\operatorname{rank} XYZ + \operatorname{rank} Y \geq \operatorname{rank} XY + \operatorname{rank} YZ,$$

obţinem că

$$\begin{aligned}\operatorname{rank}(AB+BA) &= \operatorname{rank}(B-I_n)(AB+BA)(B-I_n) + \operatorname{rank}(AB+BA)\\ &\geq \operatorname{rank}(B-I_n)(AB+BA) + \operatorname{rank}(AB+BA)(B-I_n)\\ &= \operatorname{rank}(BAB-AB) + \operatorname{rank}(BAB-BA)\\ &\geq \operatorname{rank}\big((BAB-BA)-(BAB-AB)\big)\\ &= \operatorname{rank}(AB-BA).\end{aligned}$$

Soluţia 3 (C. Băeţica şi G. Mincu). În această soluţie vom considera $\mathbb{C}$-spaţiul vectorial $V=\mathbb{C}^n$ şi $f,g:V\to V$ aplicaţiile liniare asociate matricelor A,B (în baza canonică). Din ipoteză avem că $g^2=g$.

Fie $u=fg+gf$ şi $v=fg-gf$. Inegalitatea cerută este echivalentă cu $\dim_{\mathbb{C}}\ker u\leq\dim_{\mathbb{C}}\ker v$.

Pentru a demonstra acest fapt să observăm următoarele:

1) $\ker u\cap\ker g\subseteq\ker v$;

2) $g(\ker u)\subseteq\ker v$.

1) Dacă $x\in\ker u\cap\ker g$, atunci $g(x)=0$ şi $(fg+gf)(x)=0$, ceea ce implică $(gf)(x)=0$ şi de aici obţinem $(fg-gf)(x)=0$, adică $v(x)=0$.

2) Pe de altă parte, dacă $y\in g(\ker u)$, atunci există $z\in\ker u$ astfel încât $y=g(z)$. Cum $z\in\ker u$ avem $(fg+gf)(z)=0$ şi aplicând g la stânga obţinem $(gfg)(z)+(gf)(z)=0$. Acum calculăm $v(y)=(fg-gf)(y)=$ $=(fg-gf)(g(z))=(fg)(z)-(gfg)(z)=(fg+gf)(z)=u(z)=0$.

Este însă uşor de văzut că $(\ker u\cap\ker g)\cap g(\ker u)=\{0\}$, ceea ce înseamnă că suma acestor două subspaţii ale lui $\ker v$ este directă. De aici putem conchide că

$$\dim_{\mathbb{C}}\ker u=\dim_{\mathbb{C}}(\ker u\cap\ker g)+\dim_{\mathbb{C}} g(\ker u)\leq\dim_{\mathbb{C}}\ker v,$$

ceea ce era de demonstrat.

Problema 4. (a) Fie $n\geq 1$ un număr întreg. Calculaţi $\displaystyle\int_0^1 x^{n-1}\ln x\,\mathrm{d}x$.

(b) Calculaţi

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}-\cdots\right). \tag{1}$$

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Juriul a considerat această problemă ca fiind cea mai dificilă şi aşa a fost: numai 6% *dintre concurenţi au dat o soluţie completă.*

Soluţie (a autorilor). (a) Integrând prin părţi obţinem

$$\int_0^1 x^{n-1}\ln x\,\mathrm{d}x=-\frac{1}{n^2},\quad n\in\mathbb{N}. \tag{2}$$

(b) Pentru $k \in \mathbb{N}$ notăm

$$E_k = \frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} + \cdots + \frac{(-1)^{k-1}}{(n+k)^2}.$$

Din (2) rezultă că

$$\begin{aligned} E_k &= -\int_0^1 x^n \left(1 - x + \cdots + (-x)^{k-1}\right) \ln x \,\mathrm{d}x = \\ &= -\int_0^1 x^n \frac{1-(-x)^k}{1+x} \ln x \,\mathrm{d}x = \\ &= -\int_0^1 \frac{x^n \ln x}{1+x} \,\mathrm{d}x + (-1)^k \int_0^1 \frac{x^{n+k} \ln x}{1+x} \,\mathrm{d}x. \end{aligned}$$

Făcând $k \to \infty$ se obţine

$$\frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} - \cdots = -\int_0^1 \frac{x^n \ln x}{1+x} \,\mathrm{d}x,$$

deoarece

$$0 \le -\int_0^1 \ln x \frac{x^{n+k}}{1+x} \,\mathrm{d}x \le -\int_0^1 x^k \ln x \,\mathrm{d}x = \frac{1}{(k+1)^2} \to 0 \quad (\text{când } k \to \infty).$$

Apoi evaluarea celei de-a n-a sume parţiale a seriei (1) conduce la

$$\begin{aligned} \sum_{i=0}^{n} (-1)^i \left(\frac{1}{(i+1)^2} - \frac{1}{(i+2)^2} + \frac{1}{(i+3)^2} - \cdots \right) &= -\sum_{i=0}^{n} (-1)^i \int_0^1 \frac{x^i \ln x}{1+x} \,\mathrm{d}x \\ = -\int_0^1 \frac{\ln x}{1+x} \sum_{i=0}^{n} (-x)^i \,\mathrm{d}x &= -\int_0^1 \frac{\ln x}{1+x} \cdot \frac{1-(-x)^{n+1}}{1+x} \,\mathrm{d}x = \\ = -\int_0^1 \frac{\ln x}{(1+x)^2} \,\mathrm{d}x &+ (-1)^{n+1} \int_0^1 \frac{x^{n+1} \ln x}{(1+x)^2} \,\mathrm{d}x. \end{aligned}$$

Făcând $n \to \infty$ in egalitatea precedentă rezultă că

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} - \cdots \right) = -\int_0^1 \frac{\ln x}{(1+x)^2} \,\mathrm{d}x,$$

deoarece

$$0 \le -\int_0^1 \frac{x^{n+1} \ln x}{(1+x)^2} \,\mathrm{d}x \le -\int_0^1 x^{n+1} \ln x \,\mathrm{d}x = \frac{1}{(n+2)^2} \to 0 \ (\text{când } n \to \infty).$$

Pe de altă parte

$$\int_0^1 \frac{\ln x}{(1+x)^2} \,\mathrm{d}x = -\ln 2$$

folosind integrarea prin părţi.

În concluzie, sume seriei (1) este $\ln 2$.

Soluţie alternativă pentru (b). Această soluţie se bazează pe ideea schimbării ordinii de sumare.

Termenul general al seriei (1) se poate scrie astfel: $r_n = \sum\limits_{k \geq n+1} \frac{(-1)^{k-1}}{k^2}$, care este restul de ordin n al seriei absolut convergente $\sum\limits_{k \geq 1} \frac{(-1)^{k-1}}{k^2}$.

Fixăm $m \geq 1$ şi considerăm a m-a sumă parţială a seriei (1):

$$s_m = \sum_{n=0}^{m} r_n = \sum_{n=0}^{m} \left(\sum_{k \geq n+1} \frac{(-1)^{k-1}}{k^2} \right).$$

Deoarece fiecare dintre seriile $r_0, r_1, \ldots, r_m$ este absolut convergentă, este posibil să schimbăm ordinea de sumare în s_m şi să adunăm termenii în orice ordine; în particular, este permis să interschimbăm ordinea de sumare (fixăm k, sumăm după n, apoi sumăm după k) şi asta duce la

$$\begin{aligned} s_m &= \sum_{k} \left(\sum_{\substack{n \geq 0 \\ n \leq m \\ n+1 \leq k}} \frac{(-1)^{k-1}}{k^2} \right) = \sum_{k \geq 1} \left(\sum_{n=0}^{\min\{m,k-1\}} \frac{(-1)^{k-1}}{k^2} \right) = \\ &= \sum_{k \geq 1} (1 + \min\{m, k-1\}) \cdot \frac{(-1)^{k-1}}{k^2} = \\ &= \sum_{k=1}^{m+1} k \cdot \frac{(-1)^{k-1}}{k^2} + \sum_{k \geq m+2} (m+1) \cdot \frac{(-1)^{k-1}}{k^2} = \\ &= \sum_{k=1}^{m+1} \frac{(-1)^{k-1}}{k} + (m+1) r_{m+1}. \end{aligned}$$

Acum, făcând $m \to \infty$ rezultă concluzia, folosind că $\sum\limits_{n \geq 1} \frac{(-1)^{n-1}}{n} = \ln 2$ şi $\lim\limits_{n \to \infty} n r_n = 0$ deoarece $|r_n| < \frac{1}{(n+1)^2}$.

Remarcă. Soluţia alternativă conduce la următorul rezultat general: dacă $(a_n)_{n \geq 1}$ este un şir descrescător astfel încât seriile $\sum\limits_{n \geq 1} a_n$ şi $\sum\limits_{n \geq 1} (-1)^n n a_n$ sunt convergente, atunci

$$\sum_{n \geq 0} \left(\sum_{k \geq n+1} (-1)^{k-1} a_k \right) = \sum_{n \geq 1} (-1)^{n-1} n a_n.$$

MATHEMATICAL NOTES

Multivariate weak hazard rate order according to hazard rate function

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Abstract. In this note we prove that in the particular cases of multivariate uniform distribution the properties of dilatation and translation of the weak hazard rate order are lost.

Keywords: Weak hazard rate order, hazard rate function.

MSC: 60E15.

Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}^d$ be an absolute continuous random vector, $d \geq 2$. We denote by μ its distribution $\mu(B) = P(X \in B)$, $B \in \mathcal{B}(\mathbb{R})$, by $F(x) = P(X \leq x)$ its distribution function, $F^*(x) = P(X > x)$ and by f its density. Notice that if $d \geq 2$ then one cannot find a distribution such that $F^* \equiv 1 - F$ (see [1]).

For a random vector X with F^* differentiable, we define the hazard rate function $r : L(X) \to \mathbb{R}^d$, $r(x) = \left(-\frac{\partial}{\partial x_i}\left(\ln F^*(x)\right)\right)_{i=\overline{1,d}}$, where

$$L(X) = \{x \in \mathbb{R}^d : F^*(x) > 0\}.$$

When we have random vectors X and Y, we will denote by μ and ν their distributions, by F and G their distribution functions, and by r and q their hazard rate functions, respectively.

Remark 1. It is not true that if F^* is differentiable, then the distribution μ is absolute continuous with respect to the Lebesgue measure λ^d.

Counterexample. Let $U \sim \text{Unif}\left([0,1]\right)$ and $X = \left(\cos\frac{\pi}{4}U, \sin\frac{\pi}{4}U\right)$ with $X \sim \mu$. Then F_X^* is differentiable, but μ is not absolute continuous with respect to the Lebesgue measure λ^2.

Proof. Indeed we have

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$$\begin{aligned} F_X^*(x_1,x_2) &= P((X_1,X_2)>(x_1,x_2)) = P(X_1>x_1, X_2>x_2)\\ &= P\left(\cos\left(\frac{\pi}{4}U\right)>x_1, \sin\left(\frac{\pi}{4}U\right)>x_2\right)\\ &= P\left(U<\frac{4}{\pi}\arccos x_1, U>\frac{4}{\pi}\arcsin x_2\right)\\ &= P\left(\frac{4}{\pi}\arcsin x_2 < U < \frac{4}{\pi}\arccos x_1\right) = \int_{\frac{4}{\pi}\arcsin x_2}^{\frac{4}{\pi}\arccos x_1} 1_{[0,1]}(t)\,\mathrm{d}t. \end{aligned}$$

Then F_X^* is differentiable.

Let $A=\{(x,y)\in\mathbb{R}^2 : x\in\left[0,\frac{\pi}{4}\right], y=\cos x\}$. Then it is obvious that $\lambda^2(A)=0$, but $\mu(A)>0$. $\square$

Remark 2. Let us notice that $r\geq 0$.

If $d=2$, then $r_1(x_1,x_2)=\frac{f_{X_1}(x_1,X_2>x_2)}{F^*(x_1,x_2)}$ and $r_2(x_1,x_2)=\frac{f_{X_2}(X_1>x_1,x_2)}{F^*(x_1,x_2)}$, $\forall(x_1,x_2)\in L(X)$, where we have denoted $f_{X_1}(x_1,X_2>x_2)=\int_{x_2}^{\infty} f(x_1,t)\mathrm{d}t$ and $f_{X_2}(X_1>x_1,x_2)=\int_{x_1}^{\infty} f(t,x_2)\mathrm{d}t$.

For $x,y\in\mathbb{R}^d$ we say that $x\leq y$ if $x_i\leq y_i$, $i=\overline{1,d}$.

For $x,y\in\mathbb{R}^d$ we say that $x<y$ if $x\leq y$ and $x\neq y$. For $c\in\mathbb{R}^d$ and $r\in\mathbb{R}$, $r>0$, we put

$$B[c,r]=\{x : \|x-c\|\leq r\},$$
$$B(c,r)=\{x : \|x-c\|< r\}.$$

A set $C\subset\mathbb{R}^d$ is *increasing* if $x\in C, y\in\mathbb{R}^d, y\geq x\Rightarrow y\in C$.

Let us recall the following definitions (see [3]):

Definition 1. Let X and Y be two random vectors. We say that X is *stochastic dominated* by Y and we denote $X\prec_{\mathrm{st}} Y$ iff for all increasing subsets $C\subset\mathbb{R}^d$ it holds $P(X\in C)\leq P(Y\in C)$.

Definition 2. Let X and Y be two random vectors. We say that X is *weak stochastic dominated* by Y and we denote $X\prec_{\mathrm{stw}} Y$ iff $F^*\leq G^*$.

Definition 3. Let X and Y be two random vectors. We say that X is *dual weak stochastic dominated* by Y and we denote $X\prec_{\mathrm{stdw}} Y$ iff $F\geq G$.

Definition 4. Let X and Y be two random vectors. We say that X is *smaller than* Y *in hazard rate sense* and we denote $X\prec_{\mathrm{hr}} Y$ iff $F^*(x)G^*(y)\leq F^*(x\wedge y)G^*(x\vee y)$, for all $x,y\in\mathbb{R}^d$.

Definition 5. Let X and Y be two random vectors. We say that X is *smaller than* Y *in weak hazard rate sense* and we denote $X\prec_{\mathrm{whr}} Y$ iff

$$\frac{G^*}{F^*} \text{ is non-decreasing on } L(Y).$$

The following two results are well known (see [3]).

Proposition 1. Let X and Y be two random vectors with hazard rate functions r and q, respectively. Then $X \prec_{\text{whr}} Y$ iff $r(x) \geq q(x)$, for all $x \in L(X) \cap L(Y)$.

Proposition 2. Let X and Y be two random vectors. If $X \prec_{\text{whr}} Y$, then $X \prec_{\text{stw}} Y$.

The *multivariate uniform distribution* $\text{Unif}(A)$ has the density function $f(x) = \frac{1_A(x)}{\lambda^d(A)}$, where $A \in \mathcal{B}(\mathbb{R})$ has positive finite Lebesgue measure $\lambda^d(A)$.

In [2] the author has proved that $\text{Unif}([0,a]) \prec_{\text{hr}} \text{Unif}([0,b])$ iff $a \leq b$, thus the hazard rate order is increasing in this case.

1. Main Results

It is known that if $d = 1$, $X \sim \text{Unif}(I)$, $I \subset [0, \infty)$ closed interval then $X \prec_{\text{whr}} aX$ for all $a \in \mathbb{R}, a \geq 1$, and that $X \prec_{\text{whr}} X + a$ for all $a \geq 0$.

We were surprised to notice that in the multidimensional case these properties do not hold anymore. For $d \geq 2$ it may happen that never $X \prec_{\text{whr}} aX$, for some uniform distribution.

Proposition 3. *Let* $X \sim \text{Unif}(B[(1,1);1])$. *Then there does not exist* $a \in \mathbb{R}$, $a > 1$, *such that* $X \prec_{\text{whr}} aX$.

Proof. It is obvious that $aX \sim \text{Unif}(B[(a,a);a])$.

Let us suppose that there exists $a \in \mathbb{R}$, $a > 1$, such that $X \prec_{\text{whr}} aX$ and consider r, respectively q, the hazard rate function for X, respectively aX.

Then $r(x) \geq q(x)$, for all $x \in L(X) \cap L(aX)$.

We have $r_1\left(\frac{1}{2}, \frac{2+\sqrt{3}}{2}\right) = \frac{f_{X_1}\left(\frac{1}{2}, X_2 > \frac{2+\sqrt{3}}{2}\right)}{F_X^*\left(\frac{1}{2}, \frac{2+\sqrt{3}}{2}\right)} = 0$, since $F_X^*\left(\frac{1}{2}, \frac{2+\sqrt{3}}{2}\right) > 0$ and $f_{X_1}\left(\frac{1}{2}, X_2 > \frac{2+\sqrt{3}}{2}\right) = 0$.

On the other side, $q_1\left(\frac{1}{2}, \frac{2+\sqrt{3}}{2}\right) = \frac{f_{(aX)_1}\left(\frac{1}{2}, (aX)_2 > \frac{2+\sqrt{3}}{2}\right)}{F_{aX}^*\left(\frac{1}{2}, \frac{2+\sqrt{3}}{2}\right)} > 0$, since $F_{aX}^*\left(\frac{1}{2}, \frac{2+\sqrt{3}}{2}\right) > 0$ and $f_{(aX)_1}\left(\frac{1}{2}, (aX)_2 > \frac{2+\sqrt{3}}{2}\right) > 0$.

Then $r_1\left(\frac{1}{2}, \frac{2+\sqrt{3}}{2}\right) < q_1\left(\frac{1}{2}, \frac{2+\sqrt{3}}{2}\right)$, which is a contradiction.

In conclusion it does not exist $a \in \mathbb{R}, a > 1$ such that $X \prec_{\text{whr}} aX$. □

It is obvious that if $X \sim \text{Unif}(B[(0,0);1])$ then $X \prec_{\text{whr}} X$ and $X \prec_{\text{whr}} X + b$, for all $b \geq (2,2)$.

However, for this particular case, the translation property is not true, as one can see in the following

Proposition 4. *Let* $X \sim \text{Unif}(B[(0,0);1])$. *Then there does not exist* $b \in \mathbb{R}^2$, $b > (0,0)$ *and* $b \ngeq (1,1)$, *such that* $X \prec_{\text{whr}} X + b$.

Proof. It is obvious that $X + b \sim \text{Unif}(B\,[b;1])$.

Let us suppose that there exists $b \in \mathbb{R}^2$, $b > (0,0)$ and $b \not\geq (1,1)$ such that $X \prec_{\text{whr}} X + b$ and consider r, q the hazard rate function for $X, X + b$.

Then $r\,(x) \geq q\,(x)$, for all $x \in L(X) \cap L(X + b)$.

If $b_1 < 1$ then let us take $t \in \mathbb{R}^2$ with $\|t\| = 1$,$t_1 < 0, t_2 > 0$.

We have $r_1\,(t_1, t_2) = \frac{f_{X_1}(t_1, X_2 > t_2)}{F_X^*(t_1,t_2)} = 0$, since $f_{X_1}\,(t_1, X_2 > t_2) = 0$ and $F_X^*\,(t_1, t_2) > 0$.

On the other side, $q_1\,(t_1, t_2) = \frac{f_{(X+b)_1}\left(t_1, (X+b)_2 > t_2\right)}{F_{X+b}^*(t_1,t_2)} > 0$, since we have $f_{(X+b)_1}\,(t_1, (X + b)_2 > t_2) > 0$ and $F_{X+b}^*\,(t_1, t_2) > 0$.

Then $r_1\,(t_1, t_2) < q_1\,(t_1, t_2)$, which is a contradiction.

If $b_2 < 1$ then let us take $t \in \mathbb{R}^2$ with $\|t\| = 1$, $t_1 > 0$, $t_2 < 0$.

We have $r_2\,(t_1, t_2) = \frac{f_{X_2}(X_1 > t_1, t_2)}{F_X^*(t_1,t_2)} = 0$, since $f_{X_2}\,(X_1 > t_1, t_2) = 0$ and $F_X^*\,(t_1, t_2) > 0$.

On the other side, $q_2\,(t_1, t_2) = \frac{f_{(X+b)_2}\left((X+b)_1 > t_1, t_2\right)}{F_{X+b}^*(t_1,t_2)} > 0$, since we have $f_{(X+b)_2}\,((X + b)_1 > t_1, t_2) > 0$ and $F_{X+b}^*\,(t_1, t_2) > 0$.

Then $r_2\,(t_1, t_2) < q_2\,(t_1, t_2)$, which is a contradiction.

In conclusion it does not exist $b \in \mathbb{R}^2, b > (0,0)$ and $b \not\geq (1,1)$ such that $X \prec_{\text{whr}} X + b$. □

References

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Applications of an extended theorem of Liouville

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Abstract. This note contains three applications of Liouville's extended theorem on entire functions.

Keywords: entire functions, finite order, system of equations.

MSC: 30D10; 30B10.

In the following we present three applications of Liouville's extended theorem on entire functions.

The following theorems are proved in [1].

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Theorem 1. [The extended Liouville Theorem] *If f is an entire function and if, for some integer $k \geq 0$, there exist positive constants A and B such that $|f(z)| \leq A + B|z|^k$ for all sufficiently large $|z|$, then f is a polynomial of degree at most k.*

Theorem 2. *If f is an entire function and one of the four inequalities*

$$-A|z|^n \leq \operatorname{Re} f(z) \leq A|z|^n,$$
$$-A|z|^n \leq \operatorname{Im} f(z) \leq A|z|^n$$

holds for sufficiently large $|z|$, then f is a polynomial of degree less than or equal to n.

Definition 3. *An entire function f is said to be of* finite order k *if for some k and some $R > 0$, $|f(z)| \leq \exp(|z|^k)$ for all z with $|z| > R$.*

Theorem 4. *Suppose f is an entire function of finite order k. Then either f has infinitely many zeroes or $f(z) = Q(z)e^{P(z)}$, where P and Q are polynomials.*

Remark 1. *If f is an entire function that is never zero, then $f(z) = e^{P(z)}$, where $P(z)$ is a polynomial of degree less than or equal to k (k is a finite order of f).*

Proof. We can define an entire function $P(z) = \log f(z)$ which by our hypothesis must satisfy $|\operatorname{Re} P(z)| = |\operatorname{Re} \log f(z)| = |\log|f(z)|| \leq |z|^k$, where k is a finite order of f. The statements follow from Theorem 2. □

We give three applications of these theorems.

A$_1$. Find all solutions to the infinite system of equations

$$\begin{aligned} x_1 + y_1 &= 2, \\ x_2 + 2x_1y_1 + y_2 &= 4, \\ x_3 + 3x_2y_1 + 3x_1y_2 + y_3 &= 8, \\ &\vdots \\ x_n + \binom{n}{1}x_{n-1}y_1 + \binom{n}{2}x_{n-2}y_2 + \cdots + y_n &= 2^n, \\ &\vdots \end{aligned}$$

with $x_k, y_k \geq 0$ for all k.

Solution. If the sequences $\{x_n\}, \{y_n\}$ are a solution, we consider their generating functions

$$f(z) = \sum_{n=0}^{\infty} \frac{x_n z^n}{n!}, \quad g(z) = \sum_{n=0}^{\infty} \frac{y_n z^n}{n!}, \quad \text{with } x_0 = y_0 = 1.$$

Since $x_k, y_k \geq 0$, for all k, it follows that $x_k, y_k \leq 2^k$, so that both $f(z)$ and $g(z)$ are entire functions. We have

$$f(z)g(z) = \sum_{n=0}^{\infty} C_n z^n$$

where

$$C_n = \sum_{j=0}^{n} \frac{x_{n-j}}{(n-j)!} \frac{y_j}{j!} = \sum_{j=0}^{n} \binom{n}{j} \frac{x_{n-j} y_j}{n!},$$

so (from the hypothesis)

$$f(z)g(z) = \sum_{n=0}^{\infty} \frac{2^n z^n}{n!} = e^{2z}.$$

Thus, f and g are entire functions with no zeroes.

From the above relation and Remark 1 it follows that $f(z) = e^{\alpha z + \beta}$ and $g(z) = e^{\gamma z + \delta}$. Since $f(0) = x_0 = 1$ and $g(0) = y_0 = 1$, one has $\beta = \delta = 0$ and $f(z) = e^{\alpha z}$, $g(z) = e^{\gamma z}$.

Expanding, one finds

$$f(z) = e^{\alpha z} = 1 + \alpha z + \frac{\alpha^2 z^2}{2!} + \cdots = 1 + x_1 z + \frac{x_2 z^2}{2!} + \cdots,$$

$$g(z) = e^{\gamma z} = 1 + \gamma z + \frac{\gamma^2 z^2}{2!} + \cdots = 1 + y_1 z + \frac{y_2 z^2}{2!} + \cdots.$$

Thus, there are infinitely many solutions of the form $\{x_k\}, \{y_k\}$ with $x_k, y_k \geq 0$, $x_k = \alpha^k$, $y_k = \gamma^k$, $k = 1, 2, 3, \ldots$ and $\alpha + \gamma = 2$.

Note that the system has a unique solution if x_1 and y_1 are given.

A$_2$. $e^z - z$ has infinitely many zeroes.
Solution. If $e^z - z$ had only a finite number of zeroes $a_1, a_2, \ldots, a_N$, then $e^z - z = (z - a_1)(z - a_2) \cdots (z - a_N) g(z)$, where $g(z)$ is a non-zero entire function given by

$$g(z) = \frac{e^z - z}{(z - a_1)(z - a_2) \cdots (z - a_N)}.$$

We then define the entire function $g_1(z) = \log g(z)$, with

$$\begin{aligned} \operatorname{Re} g_1(z) &= \log|g(z)| = \log\left|\frac{e^z - z}{(z-a_1)(z-a_2)\cdots(z-a_N)}\right| \\ &= \log|e^z - z| - \log|(z-a_1)(z-a_2)\cdots(z-a_N)| \\ &\le \log|e^z - z| = \log\left|1 + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right| \\ &\le \log\left(1 + \frac{|z|^2}{2!} + \frac{|z|^3}{3!} + \cdots\right) \\ &\le \log e^{|z|} \le |z| \text{ for sufficiently large } z. \end{aligned}$$

Then, according to Theorem 2, g_1 would be a linear polynomial, that is

$$\log\frac{e^z - z}{(z-a_1)(z-a_2)\cdots(z-a_N)} = az + b$$

and furthermore

$$e^z - z = (z-a_1)(z-a_2)\cdots(z-a_N)e^{az+b}.$$

Considering $z \to \infty$, it is obvious that this relation cannot hold.

A$_3$. $e^z - P(z)$ and $\sin z - P(z)$ have infinitely many zeroes for every non-zero polynomial P.

Solution. If $e^z - P(z)$ does not have infinitely many zeroes, then $e^z - P(z) = Q(z)e^{R(z)}$, where Q, R are polynomials. Considering the growth at infinty, it follows that $R(z) = z$, $Q(z) = 1$ and $P(z) = 0$.

Similarly for $\sin z - P(z)$.

References

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PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of May 2020**.

PROPOSED PROBLEMS

489. Let $m \le n$ be positive integers. For $A \in \mathcal{M}_{m,n}(\mathbb{C})$ and $B \in \mathcal{M}_{n,m}(\mathbb{C})$ define the functions

$$f_{A,B} : \mathcal{M}_n(\mathbb{C}) \longrightarrow \mathcal{M}_m(\mathbb{C}), \quad f_{A,B}(X) = AXB,$$
$$f_{B,A} : \mathcal{M}_m(\mathbb{C}) \longrightarrow \mathcal{M}_n(\mathbb{C}), \quad f_{B,A}(Y) = BYA.$$

Prove that $f_{A,B}$ is surjective (onto) if and only if $f_{A,B}$ is injective (one-to-one).

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania.

490. Let $n \in \mathbb{N}^*$. Calculate

$$\int_0^1 \left(\frac{\ln(1-x) + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n}}{x} \right)^2 \mathrm{d}x.$$

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

491. If the arithmetic mean of $a, b, c, d \ge 0$ is 1, then their quadratic mean $q = \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$ takes values in the interval $[1, 2]$.

If $q \in [1, 2]$ then we denote by $M = M_q$ the largest possible value of the geometric mean of four numbers $a, b, c, d \ge 0$ with the arithmetic mean 1 and the quadratic mean q.

Determine M in terms of q and prove that $M + q \ge 2$.

Proposed by Leonard Giugiuc, Traian National College, Drobeta Turnu Severin, Romania and Alexander Bogomolny, New Jersey, USA.

492. Let V be a vector space over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ and $f : V \to \mathbb{R} \cup \{\infty\}$ satisfying $f(x) = \infty$ iff $x = 0$ and

$$f(x + y) \ge \min\{f(x), f(y)\} \quad \forall x, y \in V.$$

For every $c \in V$ we define $g_c : V \to \mathbb{R} \cup \{\infty\}$ by $g_c(x) = f(x) + f(x + c)$.

(i) Prove that g_c satisfies the same inequality as f, viz.,

$$g_c(x+y) \geq \min\{g_c(x), g_c(y)\} \quad \forall x, y \in V.$$

Equivalently, if $x, y, z, t \in V$ with $x + y = z + t$ then

$$f(x+z) + f(x+t) \geq \min\{f(x) + f(y), f(z) + f(t)\}.$$

For any $a, b \in V$ we define $h_{a,b} : V \to \mathbb{R} \cup \{\infty\}$ by

$$h_{a,b}(x) = f(x) + f(x+a) + f(x+b).$$

(ii) If $x, y, a, b \in V$ such that $f(x) \leq f(y)$ prove that

$$h_{x,x+a+b}(y) \geq \min\{h_{a,b}(x), h_{a,b}(y)\}.$$

Let $k : V^2 \to \mathbb{R} \cup \{\infty\}$, $k(x,y) = f(x) + f(y) + f(x+y)$.

(iii) If $a, b, x, y \in V$ prove that

$$h_{a,b}(x+y) \geq \min\{h_{a,b}(x), h_{a,b}(y), k(x,y)\}$$

and

$$k(x,y) \geq \min\{h_{a,b}(x), h_{a,b}(y), h_{a,b}(x+y)\}.$$

Conclude that none of the four numbers $h_{a,b}(x)$, $h_{a,b}(y)$, $h_{a,b}(x+y)$ and $k(x,y)$ is strictly smaller than all remaining three numbers.

(iv) If $a, b, x, y, z \in V$ prove that

$$\max\{h_{y,z}(x), h_{z,x}(y), h_{x,y}(z)\} \geq \min\{h_{a,b}(x), h_{a,b}(y), h_{a,b}(z)\}.$$

Proposed by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.

493. (a) Calculate

$$\lim_{n\to\infty} n \int_0^\infty \frac{\sin x}{\mathrm{e}^{(n+1)x} - \mathrm{e}^{nx}} \mathrm{d}x.$$

(b) Let $k > -1$ be a real number. Calculate

$$\lim_{n\to\infty} n^{k+1} \int_0^\infty \frac{x^k \sin x}{\mathrm{e}^{(n+1)x} - \mathrm{e}^{nx}} \mathrm{d}x.$$

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Romania.

494. Let $n \geq 3$ and let $a_1, \ldots, a_n$ be nonnegative real numbers such that $a_1^2 + \cdots + a_n^2 = n - 1$.

(i) Prove that $a_1 + \cdots + a_n - a_1 \cdots a_n \leq n - 1$.

(ii) Prove that if $k < 1$ then the inequality $a_1 + \cdots + a_n - k a_1 \cdots a_n \leq n - 1$ is not always true.

Proposed by Leonard Giugiuc, Colegiul Naţional Traian, Drobeta Turnu Severin, România, Qing Song, Beihang University Library and Yongxi Wang, Shanxi University Affiliated High School, People's Republic of China.

495. Let $n \geq 2$ and $A, B \in \mathcal{M}_n(\mathbb{C})$ such that
$$AB - BA = c(A - B)$$
for some $c \in \mathbb{C}^*$.
a) For $n = 2$, give an example of distinct matrices A and B that satisfy the above condition.
b) Prove that A and B have the same eigenvalues.

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania, and Mihai Opincariu, Avram Iancu National College, Brad, Romania.

SOLUTIONS

472. Let $a, b, c \in [0, \frac{\pi}{2}]$ such that $a+b+c = \pi$. Prove the following inequality:
$$\sin a + \sin b + \sin c \geq 2 + 4 \left| \sin\left(\frac{a-b}{2}\right) \sin\left(\frac{b-c}{2}\right) \sin\left(\frac{c-a}{2}\right) \right|.$$

Proposed by Leonard Giugiuc, Traian National College, Drobeta Turnu Severin, Romania and Jiahao He, South China University of Technology, People's Republic of China.

Solution by the authors. Since both sides of the inequality are symmetric we may assume that $a \geq b \geq c$. Then the product from the right side is non-negative and we have to prove that
$$\sin a + \sin b + \sin c - 4 \sin\left(\frac{a-b}{2}\right) \sin\left(\frac{b-c}{2}\right) \sin\left(\frac{c-a}{2}\right) \geq 2.$$
Since $\frac{a-b}{2} + \frac{b-c}{2} + \frac{c-a}{2} = 0$ we have
$$-4 \sin\left(\frac{a-b}{2}\right) \sin\left(\frac{b-c}{2}\right) \sin\left(\frac{c-a}{2}\right) = \sin(a-b) + \sin(b-c) + \sin(c-a).$$
(In general, $4 \sin x \sin y \sin z = \sin(y+z-x) + \sin(z+x-y) + \sin(x+y-z) - \sin(x+y+z)$. When $x+y+z = 0$ this is equal to $-\sin 2x - \sin 2y - \sin 2z$.)

Also since $a+b+c = \pi$ we have $\sin a = \sin(b+c)$ and similarly for $\sin b$ and $\sin c$. Hence the inequality we want to prove also writes as
$$\sin(a+b) + \sin(a-b) + \sin(b+c) + \sin(b-c) + \sin(c+a) + \sin(c-a) \geq 2,$$
which is equivalent to $\sin a \cos b + \sin b \cos c + \sin c \cos a \geq 1$. But $c \geq b$, so $\sin c \cos a \geq \sin b \cos a$ and
$$\sin a \cos b + \sin b \cos c + \sin c \cos a \geq \sin a \cos b + \sin b(\cos a + \cos c).$$
Since $0 \leq a+c-\frac{\pi}{2} \leq a, c \leq \frac{\pi}{2}$ and $\left(a+c-\frac{\pi}{2}\right) + \frac{\pi}{2} = a+c$, by the Karamata's inequality applied to the cosine function, which is concave on $[0, \frac{\pi}{2}]$, we get
$$\cos a + \cos c \geq \cos\left(a + c - \frac{\pi}{2}\right) + \cos\frac{\pi}{2} = \cos\left(\frac{\pi}{2} - b\right) + 0 = \sin b.$$

It follows that

$$\sin a \cos b + \sin b \cos c + \sin c \cos a \geq \sin a \cos b + \sin^2 b.$$

So it suffices to prove that $\sin a \cos b + \sin^2 b \geq 1$, which is equivalent to $\sin a \cos b \geq \cos^2 b$, i.e., to $\sin a \geq \cos b = \sin\left(\frac{\pi}{2} - b\right)$. This follows from the fact that $c \leq \frac{\pi}{2}$, so $a = \pi - b - c \geq \frac{\pi}{2} - b$.

Note that we have equality when a, b, c are, in some order, $\frac{\pi}{2}, 0, \frac{\pi}{2}$. □

We also received a solution from Yury Yucra Limachi, from Puno, Peru.

473. (**Corrected**[1)]) Let $e_1, \ldots, e_n$ be the elementary symmetric polynomials in the variables $X_1, \ldots, X_n$,

$$e_k(X_1, \ldots, X_n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} X_{i_1} \cdots X_{i_k}$$

and let M be the ideal generated by $e_1, \ldots, e_n$ in $\mathbb{R}[X_1, \ldots, X_n]$.

Then every monomial $X_1^{m_1} \cdots X_n^{m_n}$ with the degree $m = m_1 + \cdots + m_n$ strictly greater than $\binom{n}{2}$ belongs to M. On the other hand, there exists a monomial of degree $\binom{n}{2}$ which does not belong to M.

`Proposed by George Stoica, New Brunswick, Canada.`

Solution by C. Băeţica. Let $R = \mathbb{R}[X_1, \ldots, X_n]$, and $I = (s_1, \ldots, s_n)$. If P is a minimal prime over I, then $s_n \in I$ and there exists $i \in \{1, \ldots, n\}$ such that $X_i \in P$. We may assume $X_n \in P$. Since $s_{n-1} \in P$ and $X_n \in P$ we get $X_1 \cdots X_{n-1} \in P$. Similarly we can suppose $X_{n-1} \in P$, and so on. Now it is easily seen that the only minimal prime ideal over I is $P = (X_1, \ldots, X_n)$. This shows that $\sqrt{I} = (X_1, \ldots, X_n)$, and therefore the height of I is n. Since R is Cohen-Macaulay the grade of I is also n, and thus $s_1, \ldots, s_n$ form a regular sequence. By induction on n one can show that $H_{R/I}(t) = \dfrac{(1-t)\cdots(1-t^n)}{(1-t)^n}$, a polynomial of degree $n(n-1)/2$. (Here $H_{R/I}(t)$ stands for the Hilbert series of R/I.) In particular, the homogeneous parts of R/I of degree $d > n(n-1)/2$ are zero, and thus every homogeneous polynomial of degree $d > n(n-1)/2$ belongs to I.

For the second part of the question we define the homogeneous polynomials $h_i(X_1, \ldots, X_n)$, $1 \leq i \leq n$, as the sum of all monomials of total degree i in $X_1, \ldots, X_n$. By Proposition 5, page 350 from Cox D., Little J., O'Shea D., *Ideals, Varieties, and Algorithms*, Springer, 2015, we learn that $h_j(X_j, \ldots, X_n)$, $1 \leq j \leq n$, form a Gröbner basis for I with respect to the lexicographic order on R with $X_1 > \cdots > X_n$. In particular, the initial ideal of I is generated by the monomials $X_1, X_2^2, \ldots, X_n^n$. Now it is immediate that the monomial $X_2 X_3^2 \cdots X_n^{n-1}$ does not belong to I.

[1)]In the 1-2/2018 issue the problem appeared with $\binom{n}{k}$ instead of $\binom{n}{2}$, which doesn't make sense.

Solution by C.N. Beli. We denote by X the multivariable $(X_1,\dots,X_n)$ so that $\mathbb{R}[X_1,\dots,X_n]=\mathbb{R}[X]$. We have $\mathbb{R}[X]=\bigoplus_{k\geq 0}\mathbb{R}[X]_k$, where $\mathbb{R}[X]_k$ is the set of all homogeneous polynomials of degree k. Since M is generated by homogeneous ideals, it is homogeneous as well, so $M=\bigoplus_{k\geq 0}M_k$, where $M_k=M\cap\mathbb{R}[X]_k$. We must prove that $\mathbb{R}[X]_k\subseteq M$, i.e., $M_k=\mathbb{R}[X]_k$, holds for $k>\binom{n}{2}$, but not for $k=\binom{n}{2}$.

We put $\mathbb{N}=\mathbb{Z}_{\geq 0}$ and $\mathbb{N}^*=\mathbb{Z}_{\geq 1}$. If $i=(i_1,\dots,i_n)\in\mathbb{N}^n$ then we denote $X^i=X_1^{i_1}\cdots X_n^{i_n}$. Every polynomial $P\in\mathbb{R}[X]$ writes as $P=\sum_{i\in I(P)}a_iX^i$, where $I(P)\subseteq\mathbb{N}^n$ is a finite set and $a_i\in\mathbb{R}\setminus\{0\}$, $\forall i\in I(P)$. The set $I(P)$ is called the support of P.

On $\mathbb{N}^n$ we define the lexicographic order by $(i_1,\dots,i_n)<(j_1,\dots,j_n)$ if there is $1\leq h\leq n$ such that $i_l=j_l$ for $l<h$ and $i_h<j_h$. If $i,j\in\mathbb{N}^n$ we say that $i\leq j$ if $i<j$ or $i=j$.

If $i=(i_1,\dots,i_n)\in\mathbb{N}^n$ we put $o(i)=(j_1,\dots,j_n)\in\mathbb{N}^n$, where $j_1,\dots,j_n$ is the sequence $i_1,\dots,i_n$ in decreasing order.

For example, $o(0,2,3,0,2)=(3,2,2,0,0)$. If $P\in\mathbb{R}[X]\setminus\{0\}$ then we define $f(P)\in(\mathbb{N}^n,\leq)$ by $f(P)=\max_{i\in I(P)}o(i)$.

Lemma 1. Let $i\in\mathbb{N}^n$ with $\deg X^i=m$.

(i) If $i\in\mathbb{N}^{*n}$ then $X^i\in M_m$.

(ii) If $i\in\mathbb{N}^n$ with $o(i)=(j_1,\dots,j_n)$ such that $j_h-j_{h+1}\geq 2$ for some $1\leq h\leq n-1$ then $X^i\equiv P \pmod{M_m}$ for some $P\in\mathbb{R}[X]_m$ such that either $f(P)<f(X^i)=o(i)$ or $P=0$.

Proof. (i) If $i=(i_1,\dots,i_n)\in\mathbb{N}^{*n}$ and $j=(i_1-1,\dots,i_n-1)$ then $j\in\mathbb{N}^n$ and we have $X^i=X^j(X_1\cdots X_n)=X^je_n\in M$. Since also $X^i\in\mathbb{R}[X]_m$, we have $X^i\in M_m$.

(ii) Let $i=(i_1,\dots,i_n)$. By permuting the variables $X_1,\dots,X_n$, we may assume that $X^i=X_1^{i_1}\cdots X_n^{i_n}$ satisfies $i_1\geq\cdots\geq i_n$, i.e., that $o(i)=i$ and $j_s=i_s$ $\forall s$.

Let $1\leq h\leq n-1$ such that $i_h-i_{h+1}\geq 2$. Let $j=(i_1-1,\dots,i_h-1,i_{h+1},\dots,i_n)$. If $j=(j_1,\dots,j_n)$ then $i=(j_1+1,\dots,j_h+1,j_{h+1},\dots,j_n)$. Since the sequence $i_1,\dots,i_n$ is decreasing and $i_h-i_{h+1}\geq 2$, we have $j_1\geq\cdots\geq j_h>j_{h+1}\geq\cdots\geq j_n$.

We have $\deg X^j=\sum_s j_s=\sum_s i_s-h=m-h$. Also $e_h\in\mathbb{R}[X]_h$. Hence $X^je_h\in\mathbb{R}[X]_m$. Since also $X^je_h\in M$, we have $X^je_h\in M_m$. It follows that $X^i\equiv P \pmod{M_m}$, where $P=X^i-X^je_h$. Since both $X^i,X^je_h\in\mathbb{R}[X]_m$, we have $P\in\mathbb{R}[X]_m$. We must prove that $f(P)<i$.

Note that $X_1\cdots X_h$ is a monomial in e_h so $X^jX_1\cdots X_h=X^i$ is a monomial of X^je_h. It is canceled in $P=X^i-X^je_h$. We must prove that all the other monomials X^k that appear in X^je_h satisfy $o(k)<i$.

We write $e_h = \sum_{c\in T} X^c$, where $T = \{(c_1,\dots,c_n) \in \{0,1\}^n \mid \sum_s c_s = h\}$. We have $e_h = X_1 \dots X_h + \sum_{c\in T'} X^c$, where $T' = T \setminus \{(1,\dots,1,0,\dots,0)\}$. (In $(1,\dots,1,0,\dots,0)$ we have 1 on the first h positions.) Then

$$P = X^i - X^j(X_1 \dots X_h + \sum_{c\in T'} X^c) = -X^j \sum_{c\in T'} X^c = -\sum_{c\in T'} X^{j+c}$$

and we must prove that $o(j + c) < i$ for every $c \in T'$.

Let $c = (c_1,\dots,c_n) \in T'$. For $1 \le s \le h$ we have $j_s + c_s \ge j_s \ge j_h$ and for every $h + 1 \le s \le n$ we have $j_s + c_s \le j_s + 1 \le j_{h+1} + 1 \le j_h$. Hence the largest h entries of $j + c$ are, in some order, $j_1 + c_1,\dots,j_h + c_h$. So, if $o(j + c) = k = (k_1,\dots,k_n)$, then $k_1,\dots,k_h$ are $j_1 + c_1,\dots,j_h + c_h$ written in decreasing order. For short, $(k_1,\dots,k_h) = o(j_1 + c_1,\dots,j_h + c_h)$. Note that $c_1,\dots,c_h \in \{0,1\}$, but they cannot be all 1, since this would mean $(c_1,\dots,c_n) = (1,\dots,1,0,\dots,0) \notin T'$. Therefore for every $1 \le s \le h$ we have $j_s + c_s \le j_s + 1 = i_s$ and at least one of these inequalities is strict. It follows that in $(\mathbb{N}^h, \le)$ we have $o(j_1 + c_1,\dots,j_h + c_h) < o(i_1,\dots,i_h)$, i.e., $(k_1,\dots,k_h) < (i_1,\dots,i_h)$. But this implies that in $(\mathbb{N}^h, \le)$ we have $(k_1,\dots,k_n) < (i_1,\dots,i_n)$, i.e., $o(j + c) < i$, as claimed. □

Lemma 2. Let $i \in \mathbb{N}^n$ such that $\deg X^i \ge \binom{n}{2}$ and let $o(i) = j = (j_1,\dots,j_n)$. Then one of the following statements holds:

(1) $i \in \mathbb{N}^{*n}$.

(2) There is some $1 \le h \le n-1$ such that $j_h - j_{h+1} \ge 2$.

(3) $j = \alpha := (n-1, n-2,\dots,1,0)$. In particular, $\deg X^i = \binom{n}{2}$.

Proof. We have $j_1 \ge \cdots \ge j_n \ge 0$. If $j_n \ge 1$ then $\{j_1,\dots,j_n\} = \{i_1,\dots,i_n\} \subseteq \mathbb{N}^*$ so (1) holds. Therefore we will assume that $j_n = 0$.

Suppose now that (2) does not hold, so $j_h \le j_{h+1} + 1$ for $1 \le h \le n-1$. Since $j_n = 0$, this implies inductively that $j_{n-1} \le 1$, $j_{n-2} \le 2$,..., $j_1 \le n-1$. Then we get

$$\binom{n}{2} \le \deg X^i = \sum_h i_h = \sum_h j_h \le (n-1) + (n-2) + \cdots + 1 + 0 = \binom{n}{2}.$$

So all inequalities must be equalities, i.e., $(j_1,\dots,j_n) = (n-1, n-2,\dots,1,0)$, hence we have (3). □

Now back to the proof, suppose that $P = \sum_{i\in I(P)} a_i X^i \in \mathbb{R}[X]_m$, with $\deg P = m > \binom{n}{2}$. Let $j = (j_1,\dots,j_n) = f(P)$. We have $P = P' + P''$, with $P' = \sum_{i\in I'} a_i X^i$ and $P'' = \sum_{i\in I''} a_i X^i$, where I' and I'' are the sets of all indices $i \in I(P)$ such that $o(i) = j$ and $o(i) < j$, respectively. Obviously, $f(P'') < j$ (or $P'' = 0$, if $I'' = \emptyset$). For every $i \in I'$, since $\deg X^i = m > \binom{n}{2}$, we are in one of the cases (1) and (2) of Lemma 2. Then, by Lemma 1, we have that $X^i \equiv Q_i \pmod{M_n}$ for some $Q_i \in \mathbb{R}[X]_m$ with $f(Q_i) < o(i) = j$.

Then $P' \equiv \sum_{i\in I'} a_iQ_i \pmod{M_n}$, so $P = P' + P'' \equiv Q \pmod{M_n}$, where $Q = \sum_{i\in I'} a_iQ_i + P''$. Since $f(Q_i) < j\ \forall i \in I'$ and $f(P'') < j$ or they are 0, same will happen with Q.

If $Q = 0$ then $P \in M_n$ and we are done. Otherwise we apply the same procedure to Q and the invariant $f(P)$ will decrease further. Since it cannot decrease indefinitely, eventually we get $P \equiv 0 \pmod{M_n}$, so $P \in M_n$, as claimed.

For the second statement, we will prove that $X^i \notin M_{\binom{n}{2}}$ if $o(i) = \alpha = (n-1, n-2, \dots, 1, 0)$.

Let S_n be the symmetric group on n letters. For any $\sigma \in S_n$ and $i = (i_1, \dots, i_n) \in \mathbb{N}^n$ we denote $i\sigma := (i_{\sigma(1)}, \dots, i_{\sigma(n)})$. (If we regard i and σ as functions $i : \{1, \dots, n\} \to \mathbb{N}$ and $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ then $i\sigma$ is the composition $i \circ \sigma$.) This is a right action of S_n on $\mathbb{N}^n$, i.e., $(i\sigma)\tau = i(\sigma\tau)$ $\forall \sigma, \tau \in S_n$.

Then for any $i \in \mathbb{N}^n$ we have $o(i) = \alpha$ iff $i = \alpha\sigma$ for some $\sigma \in S_n$. Moreover, since the entries of α are mutually distinct, σ is uniquely determined. So there is a bijection $S_n \to o^{-1}(\alpha)$, given by $\sigma \mapsto \alpha\sigma$.

We define a linear map $\psi : \mathbb{R}[X]_{\binom{n}{2}} \to \mathbb{R}$ by $\sum_i a_iX^i = \sum_{\sigma\in S_n} \varepsilon(\sigma)a_{\alpha\sigma}$. On monomials, ψ is given by $X^{\alpha\sigma} \mapsto \varepsilon(\sigma)\ \forall \sigma \in S_n$ and $X^i \mapsto 0$ if $i \notin o^{-1}(\alpha)$.

Lemma 3. We have $M_{\binom{n}{2}} \subseteq \ker\psi$.

Proof. As a vector space, $M_{\binom{n}{2}}$ is generated by the products X^je_h, with $1 \le h \le n$ and $j = (j_1, \dots, j_n) \in \mathbb{N}^n$ such that $\deg X^j = \binom{n}{2} - h$. So it suffices to prove that such products belong to $\ker\psi$.

As seen in the proof of Lemma 1, we have $e_h = \sum_{c\in T} X^c$, where $T = \{(c_1, \dots, c_n) \in \{0,1\}^n \mid \sum_s c_s = h\}$, so $X^je_h = \sum_{c\in T} X^{j+c}$. Hence, if we put $A = \{\sigma \in S_n \mid \alpha\sigma = j + c \text{ for some } c \in T\}$, then $f(X^je_h) = \sum_{\sigma\in A} \varepsilon(\sigma)$.

Note that $j_1, \dots, j_n$ cannot be mutually distinct since this would imply that $\binom{n}{2} - h = \deg X^j = i_1 + \dots + i_n \ge 0 + 1 + \dots + n - 1 = \binom{n}{2}$. So there are $1 \le s_1 < s_2 \le n$ such that $j_{s_1} = j_{s_2}$. We denote by τ the transposition $(s_1, s_2) \in S_n$. Then $H = \{1, \tau\}$ is a subgroup of S_n and S_n writes as a disjoint union of left cosets from S_n/H, $S_n = \bigcup_{t=1}^{n!/2} \sigma_tH = \bigcup_{t=1}^{n!/2} \{\sigma_t, \sigma_t\tau\}$.

Let $\sigma \in A$ and let $c = (c_1, \dots, c_n) \in T$ such that $j + c = \alpha\sigma$. We define $c' = (c'_1, \dots, c'_n)$ by $c'_{s_1} = c_{s_2}$, $c'_{s_2} = c_{s_1}$ and $c'_s = c_s$ for $s \ne s_1, s_2$. Obviously $c' \in \{0,1\}^n$ and $\sum_{s=1}^n c'_s = \sum_{s=1}^n c_s = h$, so $c' \in T$. We denote $i = (i_1, \dots, i_n) = j + c$ and $i' = (i'_1, \dots, i'_n) = j + c'$. Since $j_{s_1} = j_{s_2}$, we have $i'_{s_1} = j_{s_1} + c'_{s_1} = j_{s_2} + c_{s_2} = i_{s_2}$ and similarly $i'_{s_2} = i_{s_1}$. If $s \ne s_1, s_2$ then

$i'_s = j_s + c'_s = j_s + c_s = i_s$. In conclusion, $i'_s = i_{\tau(s)}$ $\forall s$, i.e., $i' = i\tau$. This means $j + c' = (j + c)\tau = \alpha\sigma\tau$, so $\sigma\tau \in A$.

Hence if $\sigma \in A$ then the whole left coset $\sigma H = \{\sigma, \sigma\tau\}$ is contained in A. It follows that A is a union of left cosets $A = \bigcup_{t\in B} \{\sigma_t, \sigma_t\tau\}$ for some $B \subseteq \{1, \ldots, n!/2\}$. Hence

$$f(X^j e_h) = \sum_{\sigma\in A} \varepsilon(\sigma) = \sum_{t\in B} (\varepsilon(\sigma_t) + \varepsilon(\sigma_t\tau)) = \sum_{t\in B} 0 = 0.$$

(Since τ is odd, σ_t and $\sigma_t\tau$ have opposite parities, so $\varepsilon(\sigma_t) = -\varepsilon(\sigma_t\tau)$.) □

As a consequence of Lemma 3, if $i \in \mathbb{N}^n$ with $o(i) = \alpha$ then $i = \alpha\sigma$ for some $\sigma \in S_n$, so $\psi(X^i) = \psi(X^{\alpha\sigma}) = \varepsilon(\sigma) \neq 0$, whence $X^i \notin \ker\psi$, so $X^i \notin M_{\binom{n}{2}}$.

In particular $X^\alpha = X_1^{n-1} X_2^{n-2} \cdots X_{n-1} \notin M_{\binom{n}{2}}$. □

We can actually prove that $M_{\binom{n}{2}} = \ker\psi$. For this we need some preliminary results.

Lemma 4. Let $P \in \mathbb{R}[X]_{\binom{n}{2}}$.

(i) If $f(P) < \alpha$ then $P \in M_{\binom{n}{2}}$.

(ii) If $f(P)$ is arbitrary then there are $b_\sigma \in \mathbb{R}$, with $\sigma \in S_n$, such that $P \equiv \sum_{\sigma\in S_n} b_\sigma X^{\alpha\sigma} \pmod{M_{\binom{n}{2}}}$ and $\psi(P) = \sum_{\sigma\in S_n} \varepsilon(\sigma) b_\sigma$.

Proof. Note that if $f(P) \neq \alpha$ then, by the same proof from the case when $\deg P = m > \binom{n}{2}$, there is a polynomial $Q \in \mathbb{R}[X]_{\binom{n}{2}}$ with $P \equiv Q \pmod{M_{\binom{n}{2}}}$, such that $f(Q) < f(P)$ or $Q = 0$. When $f(P) = \alpha$ this no longer applies because of the obstruction posed by the special case (3) of Lemma 2.

If $\deg P < \alpha$ then $\deg P \neq \alpha$, so there is Q with $P \equiv Q \pmod{M_{\binom{n}{2}}}$, such that $f(Q) < f(P)$ or $Q = 0$. If $Q = 0$ then we are done. Otherwise $f(Q) < f(P) < \alpha$, so the procedure can be repeated. Since $f(P)$ cannot decrease indefinitely, eventually we get $f(P) \equiv 0 \pmod{M_{\binom{n}{2}}}$, i.e., $P \in M_{\binom{n}{2}}$ and we have (i).

For the proof of (ii) we show first that for any $P \in \mathbb{R}[X]_{\binom{n}{2}}$ we have $P \equiv Q \pmod{M_{\binom{n}{2}}}$ for some $Q \in \mathbb{R}[X]_{\binom{n}{2}}$ with $f(Q) \leq \alpha$ or $Q = 0$. If $f(P) \leq \alpha$ or $P = 0$ then we just take $Q = P$. So we may assume that $f(P) > \alpha$. Since $f(P) \neq \alpha$, we have $P \equiv Q \pmod{M_{\binom{n}{2}}}$ for some $Q \in \mathbb{R}[X]_{\binom{n}{2}}$ such that $f(Q) < f(P)$ or $Q = 0$. If $f(Q) \leq \alpha$ or $Q = 0$ then we are done. Otherwise $f(Q) > \alpha$, so we can apply the same procedure to Q. At each step $f(P)$ decreases. Eventually we get some Q with $f(Q) \leq \alpha$ or $Q = 0$.

Let a_i be the coefficient of X^i in Q. Since $f(Q) \leq \alpha$ or $Q = 0$, every X^i that appears with non-zero coefficient in Q satisfies $o(i) \leq \alpha$. Then

$Q = \sum_{o(i)=\alpha} a_i X^i + \sum_{o(i)<\alpha} a_i X^i$. But if $o(i) < \alpha$ then $f(X^i) = o(i) < \alpha$ so, by (i), $X^i \in M_{\binom{n}{2}}$. Hence $P \equiv Q = \sum_{o(i)=\alpha} a_i X^i + \sum_{o(i)<\alpha} a_i X^i \equiv \sum_{o(i)=\alpha} a_i X^i$ (mod $M_{\binom{n}{2}}$).

Since $\{i \in \mathbb{N}^n \mid o(i) = \alpha\} = \{\alpha\sigma \mid \sigma \in S_n\}$, we get $P \equiv \sum_{o(i)=\alpha} a_i X^i = \sum_{\sigma\in S_n} b_\sigma X^{\alpha\sigma}$ (mod $M_{\binom{n}{2}}$), where $b_\sigma = a_{\alpha\sigma}$. By Lemma 3, we have $P - \sum_{\sigma\in S_n} b_\sigma X^{\alpha\sigma} \in M_{\binom{n}{2}} \subseteq \ker\psi$. Therefore $\psi(P) = \psi(\sum_{\sigma\in S_n} b_\sigma X^{\alpha\sigma}) = \sum_{\sigma\in S_n} \varepsilon(\sigma) b_\sigma$. □

Lemma 5. If $1 \leq h \leq n-1$ and $\tau \in S_n$, $\tau = (h, h+1)$, then $X^\alpha + X^{\alpha\tau} \in M_{\binom{n}{2}}$.

Proof. Note that $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\alpha_s = n-s$. Let $j = (j_1, \dots, j_n)$,

$$\begin{aligned} j &= (\alpha_1 - 1, \dots, \alpha_h - 1, \alpha_{h+1}, \dots, \alpha_n) \\ &= (n-2, n-3, \dots, n-h-1, n-h-1, n-h-2, \dots, 1, 0). \end{aligned}$$

We have $\deg X^j = \deg X^\alpha - h = \binom{n}{2} - h$. Then $\deg X^j e_h$ is homogeneous of degree $\deg X^j + h = \binom{n}{2}$ and $X^j e_h \in M$, so $X^j e_h \in M_{\binom{n}{2}}$.

We write as usual $e_h = \sum_{c\in T} X^c$, with $T = \{c = (c_1, \dots, c_n) \in \{0,1\}^n \mid \sum_s c_s = h\}$, so that $X^j e_h = \sum_{c\in T} X^{j+c}$.

Let $c', c'' \in T$, $c' = (1, \dots, 1, 0, \dots, 0)$ and $c'' = (1, \dots, 1, 0, 1, 0, \dots, 0)$. If $c = c'$ then for $s \leq u$ we have $j_s + c'_s = \alpha_s - 1 + 1 = \alpha_s$, and for $s \geq u+1$ we have $j_s + c'_s = \alpha_s + 0 = \alpha_s$. Hence $j + c' = \alpha$. If $c = c''$ note that $c''_s = c'_s$, so $j_s + c''_s = j_s + c'_s = \alpha_s$ for $s \neq h, h+1$. Also $j_h + c''_h = \alpha_h - 1 + 0 = \alpha_{h+1}$ and $j_{h+1} + c''_{h+1} = \alpha_{h+1} + 1 = \alpha_h$. Hence $j_s + c''_s = \alpha_{\tau(s)}$ $\forall s$, i.e., $j + c'' = \alpha\tau$.

Since $X^j e_h = \sum_{c\in T} X^{j+c}$, we have $X^\alpha + X^{\alpha\tau} = X^{j+c'} + X^{j+c''} = X^j e_h - \sum_{c\in T\setminus\{c',c''\}} X^{j+c}$. We have $X^j e_h \in M_{\binom{n}{2}}$, so if we prove that $\sum_{c\in T\setminus\{c',c''\}} X^{j+c} \in M_{\binom{n}{2}}$ then $X^\alpha + X^{\alpha\tau} \in M_{\binom{n}{2}}$ and we are done. By Lemma 4(i) it is enough to prove that

$$f\Big(\sum_{c\in T\setminus\{c',c''\}} X^{j+c}\Big) = \max_{c\in T\setminus\{c',c''\}} o(j+c) < \alpha.$$

So we must prove that $o(j+c) < \alpha$ $\forall c \in T \setminus \{c', c''\}$.

Let $c \in T \setminus \{c', c''\}$ and let $o(j+c) = k = (k_1, \dots, k_n)$. We must prove that $k < \alpha$. Assume first that $c_1 = \cdots = c_{u-1} = 1$. Since $c_1 + \cdots + c_n = u$ there is precisely one index $u \leq s \leq n$ such that $c_s = 1$. This index cannot be u or $u+1$ since this would imply that $c = c'$ or c''. So $c_u = c_{u+1} = 0$. For $1 \leq s \leq u-1$ we have $j_s + c_s = \alpha_s - 1 + 1 = \alpha_s$. Also

$j_u + c_u = \alpha_u - 1 + 0 = \alpha_u - 1$, $j_{u+1} + c_{u+1} = \alpha_{u+1} + 0 = \alpha_u - 1$ and if $s \geq u + 2$ then $j_s + c_s = \alpha_s + c_s \leq \alpha_{u+2} + 1 = \alpha_u - 2 + 1 = \alpha_u - 1$. Since $\alpha_1 > \alpha_2 > \cdots > \alpha_{u-1} > \alpha_u - 1$, we have that $k_1, \dots, k_u$, the largest u entries of $j + c$ in decreasing order, are $\alpha_1, \alpha_2, \dots, \alpha_{u-1}, \alpha_u - 1$. This implies that $(k_1, \dots, k_n) < (\alpha_1, \dots, \alpha_n)$, i.e., $k < \alpha$.

If not all $c_1, \dots, c_{u-1}$ are 1 then let $1 \leq v \leq u - 1$ be minimal with the property that $c_v = 0$. For $1 \leq s \leq v-1$ we have $j_s + c_s = \alpha_s - 1 + 1 = \alpha_s$. Also $j_v + c_v = \alpha_v - 1 + 0 = \alpha_v - 1$. For $v+1 \leq s \leq n$ we use the fact that the sequence j is decreasing and $v + 1 \leq u$ so $j_s + c_s \leq j_{v+1} + 1 = \alpha_{v+1} - 1 + 1 = \alpha_v - 1$. Since $\alpha_1 > \alpha_2 > \cdots > \alpha_{v-1} > \alpha_v - 1$, we have that $k_1, \dots, k_v$, the largest v entries of $j + c$ in decreasing order, are $\alpha_1, \alpha_2, \dots, \alpha_{v-1}, \alpha_v - 1$. This implies that $(k_1, \dots, k_n) < (\alpha_1, \dots, \alpha_n)$, i.e., $k < \alpha$. □

Lemma 6. For every $\sigma \in S_n$ we have $X^\alpha \equiv \varepsilon(\sigma) X^{\alpha\sigma} \pmod{M_{\binom{n}{2}}}$.

Proof. For $\phi \in S_n$ put $X_\phi = (X_{\phi(1)}, \dots, X_{\phi(n)})$. If $i = (i_1, \dots, i_n) \in \mathbb{N}^n$ then $X_\phi^{i\phi} = \prod_{s=1}^{n} X_{\phi(s)}^{i_{\phi(s)}} = \prod_{t=1}^{n} X_t^{i_t} = X^i$.

Then, as a consequence of Lemma 4, if $\phi, \tau \in S_n$ such that τ is a transposition of the form $(u, u+1)$, with $1 \leq u \leq n - 1$, then $X_\phi^{\alpha\phi} + X_\phi^{\alpha\tau\phi} = X^\alpha + X^{\alpha\tau} \in M_{\binom{n}{2}}$. But $M_{\binom{n}{2}}$ is invariant to permutations of the variables $X_1, \dots, X_n$, so $X_\phi^{\alpha\phi} + X_\phi^{\alpha\tau\phi} \in M_{\binom{n}{2}}$ remains true if we replace X by $X_{\phi^{-1}}$, which sends X_ϕ to X. Hence we have $X^{\alpha\phi} + X^{\alpha\tau\phi} \in M_{\binom{n}{2}}$, i.e.,

$$X^{\alpha\phi} \equiv -X^{\alpha\tau\phi} \pmod{M_{\binom{n}{2}}}.$$

Let now $\sigma \in S_n$. Then σ writes as $\sigma = \tau_k \cdots \tau_1$, where each $\tau_l \in S_n$ is a transposition of the form $(u, u + 1)$. Since each τ_l is odd, we have $\varepsilon(\sigma) = (-1)^k$.

For $1 \leq l \leq k$ we take $\tau = \tau_l$ and $\phi = \tau_{l-1} \cdots \tau_1$ in the congruence above. We get $X^{\alpha\tau_{l-1}\cdots\tau_1} \equiv -X^{\alpha\tau_l\cdots\tau_1} \pmod{M_{\binom{n}{2}}}$. Hence we have

$$X^\alpha \equiv -X^{\alpha\tau_1} \equiv X^{\alpha\tau_2\tau_1} \equiv \cdots \equiv (-1)^k X^{\alpha\tau_k\cdots\tau_1} \pmod{M_{\binom{n}{2}}}.$$

But $\sigma = \tau_k \cdots \tau_1$ and $\varepsilon(\sigma) = (-1)^k$, so we have $X^\alpha \equiv \varepsilon(\sigma) X^{\alpha\sigma} \pmod{M_{\binom{n}{2}}}$. □

We now prove our main result.

Proposition. $M_{\binom{n}{2}} = \ker \psi$. Equivalently, a homogeneous polynomial of degree $\binom{n}{2}$ belongs to M if and only if $\psi(P) = 0$.

Proof. The inclusion $M_{\binom{n}{2}} \subseteq \ker \psi$ is just Lemma 3. Conversely, assume that $P \in \ker \psi$. By Lemma 4(ii) there are $b_\sigma \in \mathbb{R}$ for $\sigma \in S_n$ such that $P \equiv \sum_{\sigma \in S_n} b_\sigma X^{\alpha\sigma} \pmod{M_{\binom{n}{2}}}$ and $0 = \psi(P) = \sum_{\sigma \in S_n} \varepsilon(\sigma) b_\sigma$.

Then we have $b_1 = -\sum_{\sigma\in S_n\setminus\{1\}} \varepsilon(\sigma)b_\sigma$, so

$$P \equiv b_1X^\alpha + \sum_{\sigma\in S_n\setminus\{1\}} b_\sigma X^{\alpha\sigma} = \Bigg(-\sum_{\sigma\in S_n\setminus\{1\}} \varepsilon(\sigma)b_\sigma\Bigg)X^\alpha + \sum_{\sigma\in S_n\setminus\{1\}} b_\sigma X^{\alpha\sigma}$$
$$= \sum_{\sigma\in S_n\setminus\{1\}} b_\sigma(X^{\alpha\sigma} - \varepsilon(\sigma)X^\alpha) \pmod{M_{\binom{n}{2}}}$$

But for every $\sigma \in S_n \setminus \{1\}$, by Lemma 6, we have $X^\alpha - \varepsilon(\sigma)X^{\alpha\sigma} \in M_{\binom{n}{2}}$, so $b_\sigma(X^{\alpha\sigma} - \varepsilon(\sigma)X^\alpha) = -\varepsilon(\sigma)b_\sigma(X^\alpha - \varepsilon(\sigma)X^{\alpha\sigma}) \equiv 0 \pmod{M_{\binom{n}{2}}}$, whence $P \equiv 0 \pmod{M_{\binom{n}{2}}}$, i.e., $P \in M_{\binom{n}{2}}$. □

Note. The first part of this problem was posted on math.stackexchange at the address https://math.stackexchange.com/questions/84780.

In the original posting from 2011 the condition was that the degree is $> n(n-1)$ but, in comments, it was improved to $> \frac{n(n-1)}{2}$.

474. Calculate $\sum_{n=1}^{\infty}(2n-1)\left[\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots\right)^2-\frac{1}{n^2}\right]$.

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

Solution by the authors. The series equals 3. We need Abel's summation by parts formula which states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of real or complex numbers and $A_n = \sum_{k=1}^{n} a_k$, then $\sum_{k=1}^{n} a_kb_k = A_nb_{n+1} + \sum_{k=1}^{n} A_k(b_k - b_{k+1})$, or $\sum_{k=1}^{\infty} a_kb_k = \lim_{n\to\infty} A_nb_{n+1} + \sum_{k=1}^{\infty} A_k(b_k - b_{k+1})$.

We apply this formula for

$$a_n = 2n-1 \quad\text{and}\quad b_n = \left(\frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots\right)^2-\frac{1}{n^2}.$$

We have $A_n = n^2$ and

$$b_n - b_{n+1} = \frac{1}{n^2}\left(\frac{1}{n^2}+\frac{2}{(n+1)^2}+\frac{2}{(n+2)^2}+\cdots\right)+\frac{1}{(n+1)^2}-\frac{1}{n^2}.$$

It follows that

$$\sum_{n=1}^{\infty}(2n-1)\left[\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots\right)^2-\frac{1}{n^2}\right]$$
$$= \lim_{n\to\infty} n^2\left[\left(\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots\right)^2-\frac{1}{(n+1)^2}\right]$$

$$
\begin{aligned}
&\quad + \sum_{n=1}^{\infty} n^2 \left[\frac{1}{n^2} \left(\frac{1}{n^2} + \frac{2}{(n+1)^2} + \cdots \right) + \frac{1}{(n+1)^2} - \frac{1}{n^2} \right] \\
&= \sum_{n=1}^{\infty} \left[\left(\frac{2}{n^2} + \frac{2}{(n+1)^2} + \cdots \right) + \frac{n^2}{(n+1)^2} - \frac{1}{n^2} - 1 \right] \\
&= 2\sum_{n=1}^{\infty} \left[\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots \right) - \frac{1}{n} \right] + \sum_{n=1}^{\infty} \left(\frac{2}{n} - \frac{1}{n^2} - \frac{2n+1}{(n+1)^2} \right) \\
&= 2\sum_{n=1}^{\infty} \left[\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots \right) - \frac{1}{n} \right] + \sum_{n=1}^{\infty} \left(\frac{2}{n} - \frac{2}{n+1} + \frac{1}{(n+1)^2} - \frac{1}{n^2} \right) \\
&= 2\sum_{n=1}^{\infty} \left[\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots \right) - \frac{1}{n} \right] + 1. \qquad (1)
\end{aligned}
$$

We used in our calculations the limit

$$
\begin{aligned}
&\lim_{n\to\infty} n^2 \left[\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \right)^2 - \frac{1}{(n+1)^2} \right] \\
&\qquad = \lim_{n\to\infty} n^2 \left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \right)^2 - 1 \\
&\qquad = 0,
\end{aligned}
$$

where the limit $\lim_{n\to\infty} n \left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} \cdots \right) = 1$ can be proved by an application of Cesàro–Stolz lemma, the 0/0 case.

Now we calculate the sum $\sum_{n=1}^{\infty} \left[\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots \right) - \frac{1}{n} \right]$.

We apply yet again Abel's summation formula, this time with $a_n = 1$ and $b_n = \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots \right) - \frac{1}{n}$, and we get

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \left[\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots \right) - \frac{1}{n} \right] \\
&\qquad = \lim_{n\to\infty} n \left[\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \right) - \frac{1}{n+1} \right] \\
&\qquad\quad + \sum_{n=1}^{\infty} n \left(\frac{1}{n^2} - \frac{1}{n} + \frac{1}{n+1} \right) \\
&\qquad = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1. \qquad (2)
\end{aligned}
$$

Combining (1) and (2) the problem is solved. □

475. We say that a function $f:\mathbb{R}\to\mathbb{R}$ has the property (P) if it is continuous and

$$2f(f(x)) = 3f(x) - x \quad \text{for all } x\in\mathbb{R}.$$

a) Prove that if f has property (P) then $M = \{x\in\mathbb{R} : f(x) = x\}$ is a nonempty interval.

b) Find all functions with property (P).

Proposed by Dan Moldovan and Bogdan Moldovan, Cluj-Napoca, Romania.

Solution by the authors. a) We have that

$$f(x) = f(y) \Rightarrow f(f(x)) = f(f(y)) \Rightarrow x = y.$$

It follows that the function is injective and, being continuous, it is strictly monotone. From $f(x) = \dfrac{2f(f(x))+x}{3}$ it follows that f is increasing (as a sum of increasing functions).

Let $x_0\in\mathbb{R}$. We define the sequence of iterates $(x_n)_{n\geq 0}$: $x_{n+1} = f(x_n)$, for all $n\in\mathbb{N}$. After replacing in the hypothesis, we get $2x_{n+2} = 3x_{n+1} - x_n$, meaning that

$$x_n = c_1\cdot\frac{1}{2^n} + c_2\cdot 1^n,$$

where $c_1 = 2x_0 - 2f(x_0)$ and $c_2 = 2f(x_0) - x_0$.

We have $\lim\limits_{n\to\infty} x_n = c_2$ and, because f is continuous, from $x_{n+1} = f(x_n)$ it follows that $c_2 = f(c_2)$. So the function f has $c_2 = 2f(x_0) - x_0$ as a fixed point. Moreover, since x_0 has been chosen arbitrarily, it follows that $2f(x) - x$ is a fixed point of f, for all $x\in\mathbb{R}$, hence

$$2f(x) - x\in M \quad \text{for all } x\in\mathbb{R}. \tag{1}$$

Now, let $a, b\in M$ and $c\in(a,b)$. We define the function

$$h:\mathbb{R}\to\mathbb{R}, \quad h(x) = 2f(x) - x - c.$$

Obviously, h is continuous, and $h(a)h(b) = (a-c)(b-c) < 0$. So there exists $x_1\in(a,b)$ such that $h(x_1) = 0$, meaning that $c = 2f(x_1) - x_1$, which by (1) ensures that c is a fixed point of f.

Concluding, M is a nonempty interval.

b) Consider $\alpha = \inf M$ and $\beta = \sup M$. If $\alpha = -\infty$ and $\beta = +\infty$, then $f(x) = x$ for all $x\in\mathbb{R}$, which clearly has property (P).

Assume next that β is finite and let $x_0 > \beta$. Then

$$2f(x_0) - x_0 \leq \beta < x_0 \tag{2}$$

since $2f(x_0) - x_0\in M$ by (1). It follows that the sequence $(x_n)_{n\geq 0}$ of iterates defined above is strictly decreasing because $c_1 > 0$. Moreover, the sequence decreases to $c_2 = 2f(x_0) - x_0\in M$ as $n\to\infty$. If $c_2\neq\beta$, then there exists $k\in\mathbb{N}$ such that $x_k\in[c_2,\beta)\subseteq M$, hence $x_k\in M$ and then $(x_n)_{n\geq k}$ is constant, in contradiction with its strict monotonicity. Therefore, $c_2 = \beta$ which gives us $f(x_0) = \frac{x_0+\beta}{2}$ (for all $x_0 > \beta$).

Similarly, if α is finite we consider $x_0 < \alpha$ and following the same reasoning we get $f(x_0) = \frac{x_0+\alpha}{2}$ (for all $x_0 < \alpha$). In conclusion, the functions verifying property (P) are:

$$f(x) = \begin{cases} \frac{x+\alpha}{2}, & \text{if } x \le \alpha \\ x, & \text{if } x \in (\alpha, \beta) \\ \frac{x+\beta}{2}, & \text{if } x \ge \beta \end{cases} \qquad (x \in \mathbb{R})$$

(easy to check that property (P) is satisfied), with the observation that α, β may be infinite (cases when the corresponding branches are missing). $\square$

Solution by Leonard Giugiuc. If $n \ge 1$ denote by $f_n = f \circ \cdots \circ f$, where the number of f's is n. We also put $f_0(x) = x$.

By replacing in the given relation x by $f_n(x)$ one gets $2f_{n+2} = 3f_{n+1} - f_n$ $\forall n \ge 0$, so we have a second degree linear recurrence, from which we get

$$f_n = \left(\frac{2^n - 1}{2^{n-1}}\right) f_1 - \left(\frac{2^{n-1} - 1}{2^{n-1}}\right) f_0 = \left(\frac{2^n - 1}{2^{n-1}}\right) f - \left(\frac{2^{n-1} - 1}{2^{n-1}}\right) x.$$

As a consequence, for every $x \in \mathbb{R}$ we have $\lim_{n\to\infty} f_n(x) = g(x)$, where $g(x) = 2f(x) - x$. Since f is continuous, g is continuous too.

We have $g(f(x)) = 2f(f(x)) - f(x) = 3f(x) - x - f(x) = 2f(x) - x = g(x)$, i.e., $g \circ f = g$. By composing to the right with f_k for some $k \ge 0$ we get $g \circ f \circ f_k = g \circ f_k$, i.e., $g \circ f_{k+1} = g \circ f_k$. It follows that for $n \ge 0$ we have $g = g \circ f = g \circ f_2 = \cdots = g \circ f_n$.

Then for every $n \ge 0$ we have $g(x) = g(f_n(x))$. Since g is continuous and $\lim_{n\to\infty} f_n(x) = g(x)$, we have $g(g(x)) = \lim_{n\to\infty} g(f_n(x)) = \lim_{n\to\infty} g(x) = g(x)$, so $g \circ g = g$.

If g is constant, say $g(x) = a$ $\forall x \in \mathbb{R}$, then $2f(x) - x = a$, so $f(x) = \frac{x+a}{2}$.

We assume now that g is not a constant function. Since g is continuous, $g(\mathbb{R})$ is a proper interval $I \subset \mathbb{R}$. For every $x \in I$ we have $x = g(y)$ for some $y \in \mathbb{R}$ and we get $g(x) = g(g(y)) = g(y) = x$, i.e., $2f(x) - x = x$. So $f(x) = x$ $\forall x \in I$.

If $I = \mathbb{R}$, i.e., if g is surjective, then $g(x) = x$ $\forall x \in \mathbb{R}$.

Suppose now that $I \subset \mathbb{R}$. Assume first that I is bounded from above and let $b = \sup I$. Then there is a sequence x_n in I with $x_n \to b$. By taking limits in the relation $g(x_n) = x_n$, we get $g(b) = b$, so $b \in I$. As a consequence, $f(b) = b$

We claim that $f(y) \le b$ for every $y \ge b$. When $y = b$ we have $f(b) = b$. Suppose that there is $y > b$ such that $f(y) < b$. Then $(f(y), b) \cap I \ne \emptyset$. Take $t \in (f(y), b) \cap I$. Since $f(b) = b > t > f(y)$ and f is continuous, there is $x \in (b, y)$ with $f(x) = t$. Since $t \in I$ we Have $f(t) = t$. Then

$$2f(f(x)) = 3f(x) - x \Rightarrow 2f(t) = 3t - x \Rightarrow 2t = 3t - x \Rightarrow x = t.$$

But $t < b < x$. Contradiction. Thus $f(x) \geq b$ $\forall x \geq b$. Recursively $f_n(x) \geq b$ $\forall x \geq b$ $\forall n \geq 0$. (If $f_n(x) \geq b$ then $f_{n+1}(x) = f(f_n(x)) \geq b$.)

It follows that for every $x \geq b$ we have $g(x) = \lim_{n\to\infty} f_n(x) \geq b$. On the other hand, $g(x) \in g(\mathbb{R}) = I$, so that $g(x) \leq \sup I = b$. Hence for every $x \geq b$ we have $2f(x) - x = g(x) = b$, whence $f(x) = \frac{x+b}{2}$.

If I is bounded from bellow and $a = \inf I$ then, by a similar reasoning, we get $a \in I$ and for every $x \leq a$ we gave $2f(x) - x = g(x) = a$, so $f(x) = \frac{x+a}{2}$.

In conclusion, we have three cases: $I = (-\infty, b]$, $I = [a, \infty)$ and $I = [a, b]$, where $a, b \in \mathbb{R}$ are parameters, $a < b$. In each of these cases, we have a different formula for f, namely

$$f(x) = \begin{cases} x & \text{if } x \leq b, \\ \frac{x+b}{2} & \text{if } x \geq b, \end{cases} \quad f(x) = \begin{cases} \frac{x+a}{2} & \text{if } x \leq a, \\ x & \text{if } x \geq b, \end{cases}$$

respectively

$$f(x) = \begin{cases} \frac{x+a}{2} & \text{if } x \leq a, \\ x & \text{if } a \leq x \leq b, \\ \frac{x+b}{2} & \text{if } x \geq b. \end{cases}$$

Together with these, we have the functions $f(x) = x$ and $f(x) = \frac{x+a}{2}$, previously obtained. Obviously all these functions satisfy the required condition. □

476. Calculate the integral

$$\int_0^\infty \frac{\arctan x}{\sqrt{x^4+1}} \mathrm{d}x.$$

`Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.`

Solution by the author. We denote

$$A = \int_0^\infty \frac{\arctan x}{\sqrt{x^4+1}} \mathrm{d}x \quad \text{and } B = \int_0^\infty \frac{\operatorname{arcctan} x}{\sqrt{x^4+1}} \mathrm{d}x.$$

Then

$$A + B = \int_0^\infty \frac{\arctan x + \operatorname{arcctan} x}{\sqrt{x^4+1}} \mathrm{d}x = \frac{\pi}{2} \int_0^\infty \frac{1}{\sqrt{x^4+1}} \mathrm{d}x.$$

To compute the integral $C = \int_0^\infty \frac{1}{\sqrt{x^4+1}} \mathrm{d}x$ we use the following formula for Euler's beta function

$$B(p, q) = \int_0^\infty y^{p-1}(1+y)^{p+q} \mathrm{d}y.$$

We make the change of variables $x^4 = y$ in the integral C and we get

$$C = \frac{1}{4} \int_0^\infty y^{-3/4}(1+y)^{1/2} \mathrm{d}y = \frac{1}{4} B(1/4, 1/4).$$

We use the well known formula $B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ and we get

$$C=\frac{1}{4}\cdot\frac{\Gamma(1/4)^2}{\Gamma(1/2)}=\frac{1}{4\sqrt{\pi}}\Gamma(1/4)^2.$$

It follows that $A+B=\frac{\pi}{2}C=\frac{\sqrt{\pi}}{8}\Gamma(1/4)^2$. But $\arctan x=\operatorname{arcctan}\frac{1}{x}$ so, after making the change of variables $t=\frac{1}{x}$, we get $A=B$. Thus our integral equals $A=\frac{\sqrt{\pi}}{16}\Gamma(1/4)^2$. □

Solution by Moubinool Omarjee, Lycée Henri IV, Paris, France. We denote by A our integral and we make the substitution $y=\frac{1}{x}$. We get

$$A=\int_0^\infty\frac{\arctan x}{\sqrt{x^4+1}}\mathrm{d}x=\int_0^\infty\frac{\arctan\frac{1}{y}}{\sqrt{y^4+1}}\mathrm{d}y=\int_0^\infty\frac{\frac{\pi}{2}-\arctan y}{\sqrt{y^4+1}}\mathrm{d}y$$
$$=\frac{\pi}{2}\int_0^\infty\frac{1}{\sqrt{y^4+1}}\mathrm{d}y-A.$$

Hence

$$A=\frac{\pi}{2}\int_0^\infty\frac{1}{\sqrt{x^4+1}}\mathrm{d}x.$$

But it is a well-known result that $\int_0^\infty\frac{1}{\sqrt{x^4+1}}\mathrm{d}x=\frac{4\Gamma(5/4)^2}{\sqrt{\pi}}$ (see wolfram alpha.) We therefore get $\int_0^\infty\frac{\operatorname{arctg}x}{\sqrt{x^4+1}}\mathrm{d}x=\sqrt{\pi}\Gamma^2(5/4)=1.4561\ldots$. □

We also received a solution from Daniel Văcaru, from Piteşti, Romania. He proved that the integral is equal to $\frac{\pi}{2}\int_0^1\frac{1}{\sqrt{x^4+1}}\mathrm{d}x$, but left the result in this form.

477. For every complex matrix A we denote by A^* its adjoint, i.e., the transposed of its conjugate, $A^*=\bar{A}^T$. If A is square and $A=A^*$ we say that A is self-adjoint (or Hermitian). In this case for every complex vector x we have $x^*Ax\in\mathbb{R}$. If A,B are self adjoint we say that $A\geq B$ if $x^*Ax\geq x^*Bx$ for every complex vector x.

For a complex matrix A we denote $|A|^2=AA^*$. Note that $|A|^2$ is self-adjoint and ≥ 0. (If $A^*x=y=(y_1,\ldots,y_n)^T$ then $x^*|A|^2x=y^*y=(\bar{y}_1,\ldots,\bar{y}_n)(y_1,\ldots,y_n)^T=|y_1|^2+\cdots+|y_n|^2\geq 0$.)

Let A be a square matrix with complex coefficients and I the identity matrix of the same order. Then the following statements are equivalent:

(i) $|I+zA|^2=|I-zA|^2$ for all $z\in\mathbb{C}$;

(ii) $|I+zA|^2\geq I$ for all $z\in\mathbb{C}$;

(iii) $A=0$.

Are these statements still equivalent if we replace “complex” by “real” throughout?

`Proposed by George Stoica, New Brunswick, Canada.`

Solution by the author. Obviously (iii) $\Rightarrow$ (ii), (i).

Let us prove that (ii) $\Rightarrow$ (iii). Since $|I+zA|^2 = I+zA+\bar{z}A^*+|z|^2|A|^2$, the inequality $|I+zA|^2 \geq I$ writes as

$$zA+\bar{z}A^*+|z|^2|A|^2 \geq 0 \text{ for all } z \in \mathbb{C}.$$

For natural $m \geq 1$ and $z = 1/m, -1/m, i/m, -i/m$, the above inequality becomes

$$A+A^*+\frac{1}{m}|A|^2 \geq 0, \qquad A+A^*-\frac{1}{m}|A|^2 \leq 0$$

and

$$iA-iA^*+\frac{1}{m}|A|^2 \geq 0, \qquad iA-iA^*-\frac{1}{m}|A|^2 \leq 0.$$

Letting $m \to \infty$ in the first two inequalities, and then in the last two inequalities, gives

$$A+A^*=0 \text{ and } iA-iA^*=0,$$

which imply that $A=0$, i.e., (iii) is proved.

Finally, let us prove that (i) $\Rightarrow$ (iii). The relation $|I+zA|^2 = |I-zA|^2$ for all $z \in \mathbb{C}$, writes as $zA+\bar{z}A^*=0$ for all $z \in \mathbb{C}$. For $z=1, i$, we conclude that

$$A+A^*=0 \text{ and } iA-iA^*=0,$$

which yield that $A=0$, i.e., (iii) is true.

The statements are no longer equivalent if one replaces "complex" by "real" in the problem. If we try to reproduce the proof from the complex case then we can no longer give z the values $z = \pm i/m$ or i. So, by a similar reasoning, we obtain that (i) is equivalent to $A+A^T=0$ and that (ii) implies $A+A^T=0$. But in fact (ii) is equivalent to $A+A^T=0$. Indeed, if $A+A^T=0$ then $|I+zA|^2 = I+z^2|A|^2 \geq I$. So both (i) and (ii) are equivalent to $A+A^T=0$, but they do not imply (iii).

E.g., for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ we have $|I+zA|^2 = (1+z^2)I \geq I$, but $A \neq 0$. $\square$

478. Determine the largest positive constant k such that for every $a, b, c \geq 0$ with $a^2+b^2+c^2=3$ we have

$$(a+b+c)^2+k|(a-b)(b-c)(c-a)| \leq 9.$$

`Proposed by Leonard Giugiuc, Traian National College, Drobeta Turnu Severin, Romania.`

Solution by the author. We denote by $k_{\max}$ the required largest constant.

The inequality can also be written as

$$(a+b+c)^2\sqrt{\frac{a^2+b^2+c^2}{3}}+k|(a-b)(b-c)(c-a)| \leq 9\left(\frac{a^2+b^2+c^2}{3}\right)^{3/2}.$$

Since this new form of the inequality is homogeneous it should hold for every $a, b, c \geq 0$ with $a^2 + b^2 + c^2 \neq 0$.

By homogeneity, we may assume that $a+b+c = 3$. Then $0 \leq ab+bc+ca \leq \frac{(a+b+c)^2}{3} = 3$, so $ab + bc + ca = 3(1-q)$ for some $q \in [0, 1]$. Together with $a + b + c = 3$, this implies $\frac{a^2+b^2+c^2}{3} = 1 + 2q$.

Then the required equality becomes

$$k|(a-b)(b-c)(c-a)| \leq 18q\sqrt{1+2q}.$$

We denote $x = a-1$, $y = b-1$, $z = c-1$. Then $0 \leq a, b, c \leq 3$, $a + b + c = 3$ and $ab + bc + ca = 3(1-q)$ translate to $-1 \leq x, y, z \leq 2$, $x + y + z = 0$ and $xy + yz + zx = -3q$. (We have $xy + yz + zx = ab + bc + ca - 2(a+b+c) + 3 = 3(1-q) - 2 \cdot 3 + 3 = -3q$.)

We also have $(a-b)(b-c)(c-a) = (x-y)(y-z)(z-x)$.

If we put $P_j = x^j + y^j + z^j$ ($j \in \mathbb{N}$) then $P_1 = 0$, $P_2 = 6q$, $P_3 = 3xyz$ and $P_4 = 18q^2$. We denote $p = xyz$.

Lemma. $(x-y)^2(y-z)^2(z-x)^2 = 3P_2P4+2P_1P_2P_3-(P_2^3+3P_3^2+P_1^2P_4)$.

This result is well known so we won't prove it.

By the above considerations and notations, we get $(x-y)^2(y-z)^2(z-x)^2 = 27(4q^3 - p^2)$. Therefore, our inequality writes as

$$k\sqrt{4q^3 - p^2} \leq 2\sqrt{3}q\sqrt{1+2q}.$$

Note that if $q = 0$ then $P_2 = x^2 + y^2 + z^2 = 6q = 0$, so $x = y = z = 0$. If $q = 1$ then $ab+bc+ca = 3(1-q) = 0$. Since $a, b, c \geq 0$ and $a+b+c = 3$, this implies that two of the numbers a, b, c are 0 and the third is 3. Hence x, y, z are, in some order, $-1, -1, 2$. In both cases $4q^3 - p^2 = 0$, so the inequality $k\sqrt{4q^3 - p^2} \leq 2\sqrt{3}q\sqrt{1+2q}$ holds for $k > 0$ arbitrary. So we may assume that $q \in (0, 1)$.

As we will see, if $q \in (0, 1)$ then x, y, z can be chosen such that $4q^3 - p^2 \neq 0$. In this case our inequality writes as

$$k \leq \frac{2\sqrt{3}q\sqrt{1+2q}}{\sqrt{4q^3 - p^2}}.$$

Note that for q fixed the above upper bound is big when p^2 is large. So if $p_q^2 = \min\{p^2 \mid \exists -1 \leq x, y, z \leq 2,\ x+y+z = 0,\ xy+yz+zx = -3q,\ xyz = p\}$ then k should satisfy the inequalities $k \leq f(q)$ for every $q \in (0, 1)$, where $f : (0, 1) \to \mathbb{R}$,

$$f(q) = \frac{2\sqrt{3}q\sqrt{1+2q}}{\sqrt{4q^3 - p_q^2}}.$$

In conclusion, $k_{\max} = \min_{0<q<1} f(q)$.

Apparently, the smallest possible value of p_q^2 is 0. But, as we will see, this happens only when $q \leq \frac{1}{3}$. So we discuss two cases.

Case 1. $0 < q \le \frac{1}{3}$. We take $x = -\sqrt{3q}$, $y = 0$, $z = \sqrt{3q}$. Then the conditions $-1 \le x, y, z \le 2$, $x + y + z = 0$, and $xy + yz + zx = -3q$ are fulfilled. We also have $p = xyz = 0$. Hence $p_q = 0$ and we have $f(q) = \frac{2\sqrt{3}q\sqrt{1+2q}}{\sqrt{4q^3}} = \sqrt{3}\sqrt{q^{-1}+2}$. Then $\min_{0<q\le 1/3} f(q) = f(\frac{1}{3}) = \sqrt{15}$.

Case 2. $\frac{1}{3} \le q < 1$. We have $0 \le (x+1)(y+1)(z+1) = xyz + (xy + yz + zx) + (x + y + z) + 1 = p - 3q + 1$, so $p \ge 3q - 1 \ge 0$ and $p^2 \ge (3q-1)^2$. We prove that this lower bound for p^2 can be attained for every $\frac{1}{3} \le q < 1$.

We must have $(x+1)(y+1)(z+1) = 0$ so, say, $x = -1$. Then $x+y+z = 0$ and $xy + yz + zx = -3q$ write as $-1 + y + z = 0$ and $-1(y + z) + yz = -3q$, which are equivalent to $y + z = 1$ and $yz = 1 - 3q$. Thus y, z are the roots of $X^2 - X + 1 - 3q$ so, say, $y = \frac{1-\sqrt{3(4q-1)}}{2}$ and $z = \frac{1+\sqrt{3(4q-1)}}{2}$. Since $\frac{1}{3} \le q < 1$, we have $1 \le 3(4q-1) < 9$, so $-1 < y \le 0$ and $1 \le z < 2$. In conclusion, $-1 \le x, y, z, \le 2$, so x, y, z satisfy all the required conditions.

Hence $p_q^2 = (3q-1)^2$, and we get

$$f(q) = \frac{2\sqrt{3}q\sqrt{1+2q}}{\sqrt{4q^3 - (3q-1)^2}}.$$

Note that

$$\sqrt{4q^3 - (3q-1)^2} = \sqrt{(q-1)^2(4q-1)} = (1-q)\sqrt{4q-1},$$

therefore we have

$$f(q) = 2\sqrt{3}\frac{q\sqrt{2q+1}}{(1-q)\sqrt{4q-1}}.$$

We make the substitution $u = \sqrt{\frac{2q+1}{4q-1}}$. Then $u \in (1, \sqrt{5}]$ and we have $q = \frac{u^2+1}{2(2u^2-1)}$. It follows that $f(q) = h(u)$, with $h : (1, \sqrt{5}] \to \mathbb{R}$ defined by

$$h(u) = \frac{2}{\sqrt{3}} \cdot \frac{u^3 + u}{u^2 - 1}.$$

We have $h'(u) = \frac{2}{\sqrt{3}}\frac{u^4-4u^2-1}{(u^2-1)^2}$. The only zero of h' in the interval $(1, \sqrt{5}\,]$ is $\sqrt{2+\sqrt{5}}$. We have that h' is ≤ 0 on the interval $\big(1, \sqrt{2+\sqrt{5}}\,\big]$ and ≥ 0 on $\big[\sqrt{2+\sqrt{5}}, \sqrt{5}\,\big]$. In conclusion,

$$\min_{1/3\le q<1} f(q) = \min_{1<u\le\sqrt{5}} h(u) = h\big(\sqrt{2+\sqrt{5}}\big) = \frac{1+\sqrt{5}}{\sqrt{3}}\sqrt{2+\sqrt{5}}.$$

If ϕ is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$, then note that $2 + \sqrt{5} = \phi^3$, so we have $\min_{1/3\le q<1} f(q) = \frac{2\phi^2\sqrt{\phi}}{\sqrt{3}}$.

From Case 1 we have $\min_{0<q\le 1/3} f(q) = \sqrt{15} > \min_{1/3\le q<1} f(q)$. Hence $k_{\max} = \min_{0<q<1} f(q) = \min_{1/3\le q<1} f(q) = \frac{2\phi^2\sqrt{\phi}}{\sqrt{3}}$. □

Note from the editor. The proof of the formula for $(x-y)^2(y-z)^2(z-x)^2$ given by the author is unnecessarily complicated. The product $\Delta=(x-y)^2(y-z)^2(z-x)^2$ is the discriminant of the polynomial $(X-x)(X-y)(X-z)=X^3-3qX-p$. (We have $x+y+z=0$, $xy+yz+zx=-3q$ and $xyz=p$.)

It is known that the discriminant of the polynomial X^3+cX+d is $-4c^3-27d^2$. In our case, $c=-3q$ and $d=-p$, so $\Delta=-4(-3q)^3-27(-p)^2=27(4q^3-p^2)$.

Solution by Mihai Cipu. We may assume without loss of generality that $b=a+x$, $c=b+y$ for some x, $y\geq 0$. Since

$$a+b+c=3a+2x+y,\quad a^2+b^2+c^2=3a^2+2(2x+y)a+2x^2+2xy+y^2,$$

$$3(a^2+b^2+c^2)-(a+b+c)^2=2(x^2+xy+y^2),$$

the desired inequality is equivalent to

$$\sqrt{3}\,xy(x+y)k\leq 2(x^2+xy+y^2)\sqrt{3a^2+2(2x+y)a+2x^2+2xy+y^2}.$$

As the right hand side of the previous inequality decreases with a, we may assume that $a=0$. When we denote $t=y/x$, we deduce that the maximal value of k is

$$k_{\max}=\inf_{t>0}\frac{2}{\sqrt{3}}f(t),$$

where $f(t)=\dfrac{(t^2+t+1)\sqrt{t^2+2t+2}}{t(t+1)}$.

Routine computations give

$$f'(t)=\frac{(t+2)(t^4+t^3-2t-1)}{t^2(t+1)^2\sqrt{t^2+2t+2}}.$$

The polynomial $P(t)=t^4+t^3-2t-1$ has unique positive zero u, takes negative values for $t<u$ and positive values for $t>u$. Therefore, one has $k_{\max}=2f(u)/\sqrt{3}$.

The wanted value $v=f(u)$ can be computed by noting that u is a common zero for the polynomials $P(t)$ and $Q(t)$, where

$$Q(t)=(t^2+t+1)^2(t^2+2t+2)-v^2t^2(t+1)^2.$$

Therefore, v is a positive zero for their resultant. Using a package like PARI/GP[1)], one easily finds $\operatorname{Res}(P,Q)=(v^4-11v^2-1)^2$, whence

$$v=\sqrt{\frac{11+5\sqrt{5}}{2}}.$$

Then the value $k_{\max}$ is determined as in the first solution.

[1)] The package is a widely used computer algebra system freely downloadable from the address https://pari.math.u-bordeaux.fr/download.html.

479. Let p be an odd prime number and $A \in \mathcal{M}_p(\mathbb{Q})$ a matrix such that $\det(A^p + I_p) = 0$ and $\det(A + I_p) \neq 0$. Prove that:

a) $\mathrm{Tr}(A)$ is an eigenvalue of $A + I_p$.

b) $\det(A + I_p) - \det(A - I_p) = (p-1)\,\mathrm{Tr}(A) + 2$.

Proposed by Vlad Mihaly, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

Solution by the author. a) Let $-1, \xi_1, \xi_2, \ldots, \xi_{p-1}$ be the complex roots of -1 of order p. Because $\det(A^p + I_p) = 0$ and $\det(A + I_p) \neq 0$, we obtain that there exists $\xi \in \{\xi_1, \xi_2, \ldots, \xi_{p-1}\}$ an eigenvalue of A. Let p_A be the characteristic polynomial of A and let $P(X) = X^{p-1} - X^{p-2} + \cdots - X + 1$. Since $p_A, P \in \mathbb{Q}[X]$, ξ is a root for both p_A and P, while P is an irreducible polynomial over $\mathbb{Q}[X]$, it follows that $P \mid p_A$. So, all the roots of P are eigenvalues of A, hence

$$\sigma(A) = \{\xi, \xi^2, \ldots, \xi^{p-1}, r\},$$

where r is an eigenvalue of A.

We have $\mathrm{Tr}(A) = r + \xi + \xi^2 + \cdots + \xi^{p-1} = r + 1 \in \mathbb{Q}$. Since r is an eigenvalue of A, we have that $\mathrm{Tr}(A) = r + 1$ is an eigenvalue of $A + I_p$.

b) From

$$\det(A - I_p) = (\xi - 1)(\xi^2 - 1) \cdot \ldots \cdot (\xi^{p-1} - 1)(r - 1) = P_2(1)(r-1) = r - 1,$$

$$\det(A + I_p) = (\xi + 1)(\xi^2 + 1) \cdot \ldots \cdot (\xi^{p-1} + 1)(r + 1) = P_2(-1)(r+1) = (r+1)p,$$

and $\mathrm{Tr}(A) = r + 1$, we find

$$\det(A + I_p) - \det(A - I_p) = (p-1)\,\mathrm{Tr}(A) + 2.$$

□

480. Let k, n be natural numbers, $x_1, x_2, \ldots, x_k$ be distinct complex numbers and a matrix $A \in \mathcal{M}_n(\mathbb{C})$ such that $(A - x_1 I_n)(A - x_2 I_n) \cdots (A - x_k I_n) = O_n$. Prove that $\mathrm{rank}(A - x_1 I_n) + \mathrm{rank}(A - x_2 I_n) + \cdots + \mathrm{rank}(A - x_k I_n) = n(k-1)$.

Proposed by Dan Moldovan and Vasile Pop, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

Solution by the authors. We proceed step by step.

1) If λ is an eigenvalue of matrix A and $P \in \mathbb{C}[X]$ such that $P(A) = O_n$, then $P(\lambda) = 0$ ($AX = \lambda X,\ X \neq O \Rightarrow P(A)X = P(\lambda)X \Rightarrow P(\lambda)X = O \Rightarrow P(\lambda) = 0$).

Therefore if $\lambda_1, \lambda_2, \ldots, \lambda_p$ are all the distinct eigenvalues of A, then $\{\lambda_1, \lambda_2, \ldots, \lambda_p\} \subseteq \{x_1, x_2, \ldots, x_k\}$ and we may assume that one has $\lambda_1 = x_1, \lambda_2 = x_2, \ldots, \lambda_p = x_p$ and $p \leq k$.

2) If $p < k$, then for all $i \in \{p+1, \ldots, k\}$, x_i is not an eigenvalue of A so the matrix $A - \lambda_i I_n$ is invertible and $\mathrm{rank}(A - \lambda_i I_n) = n$, hence the

problem can be reduced to the problem with the first p values x_i:
prove that if holds $(A - x_1 I_n)(A - x_2 I_n) \cdots (A - x_p I_n) = O_n$ then
$\mathrm{rank}(A - x_1 I_n) + \mathrm{rank}(A - x_2 I_n) + \cdots + \mathrm{rank}(A - x_p I_n) = n(p - 1)$.

3) It is sufficient to consider instead of A its Jordan canonical form $J_A = P^{-1}AP$ and again the problem can be further reduced to the following one: show that if $(J_A - x_1 I_n)(J_A - x_2 I_n) \cdots (J_A - x_p I_n) = O_n$, then $\mathrm{rank}(J_A - x_1 I_n) + \mathrm{rank}(J_A - x_2 I_n) + \cdots + \mathrm{rank}(J_A - x_p I_n) = n(p - 1)$.

4) We will show that J_A is a diagonal matrix: if J_{λ_1} is the direct sum of all Jordan blocks from J_A corresponding to the eigenvalue λ_1 and its dimension is $m \times m$, then $(J_{\lambda_1} - x_1 I_m)(J_{\lambda_1} - x_2 I_m) \cdots (J_{\lambda_1} - x_p I_m) = O_n$ and the matrices $J_{\lambda_1} - x_2 I_m, \ldots, J_{\lambda_1} - x_p I_m$ are invertible, hence $J_{\lambda_1} = \lambda_1 I_m$. Similarly for $\lambda_2, \ldots, \lambda_p$. Thus, J_A is a diagonal matrix.

5) When $J_A = \mathrm{diag}[\lambda_1 I_{k_1}|, \ldots, |\lambda_p I_{k_p}]$, then $\mathrm{rank}(J_A - \lambda_1 I_n) = n - k_1$, $\ldots, \mathrm{rank}(J_A - \lambda_p I_n) = n - k_p$ and hence $\mathrm{rank}(J_A - \lambda_1 I_n) + \mathrm{rank}(J_A - \lambda_2 I_n) + \cdots + \mathrm{rank}(J_A - \lambda_p I_n) = pn - (k_1 + k_2 + \cdots + k_p) = pn - n = (p - 1)n$. □

Solution by Moubinool Omarjee, Lycée Henri IV, Paris, France. Since in $\mathbb{C}[T]$ we have $\gcd(T - x_i, T - x_j) = 1 \ \forall i \neq j$, the relation $(A - x_1 I_n) \cdots (A - x_k I_n) = 0$ implies that

$$\mathbb{C}^n = \ker(A - x_1 I_n) \oplus \cdots \oplus \ker(A - x_k I_n).$$

We take dimensions and we get

$$n = \sum_{j=1}^{k} \dim \ker(A - x_j I_n).$$

By the rank theorem, this implies

$$n = \sum_{j=1}^{k} (n - \mathrm{rank}\,(A - x_j I_n)).$$

This gives $\mathrm{rank}\,(A - x_1 I_n) + \cdots + \mathrm{rank}\,(A - x_k I_n) = n(k - 1)$. □

We also received a solution from Leonard Giugiuc.

481. Let K be a field and let $n \geq 1$. Let $A, B \in M_n(K)$ such that $[A, B]$ commutes with A or B.

If $\mathrm{char}\, K = 0$ or $\mathrm{char}\, K > n$ then it is known that $[A, B]$ is nilpotent, i.e., $[A, B]^n = 0$.

Prove that this result no longer holds if $0 < \mathrm{char}\, K \leq n$.

(Here by $[\cdot, \cdot]$ we mean the commutator $[A, B] = AB - BA$.)

`Proposed by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.`

Solution by the author. First we give a proof of the already known result so that we can see what doesn't work when $0 < \mathrm{char}\, K \leq n$.

Let $C = [A, B]$. We prove that every positive power of C writes as the commutator of two matrices, so its trace is 0. (In general, $\operatorname{tr} XY = \operatorname{tr} YX$, so $\operatorname{tr}(XY - YX) = 0$.)

Let $i \geq 1$. We have $C^i = C^{i-1}(AB - BA) = C^{i-1}AB - C^{i-1}BA$. If C commutes with A then so does C^{i-1} and we have $C^i = AC^{i-1}B - C^{i-1}BA = [A, C^{i-1}B]$. If C commutes with B then so does C^{i-1} and we have $C^i = C^{i-1}AB - BC^{i-1}A = [C^{i-1}A, B]$. So, in both cases, C^i is the commutator of two matrices and we have $\operatorname{tr}(C^i) = 0$.

Let $\alpha_1, \ldots, \alpha_n$ be the roots of the characteristic polynomial $P_C(X)$ in a certain extension L of K. Then the roots of $P_{C^i}(X)$ are $\alpha_1^i, \ldots, \alpha_n^i$, so $\operatorname{tr}(C^i) = 0$ writes as $\alpha_1^i + \cdots + \alpha_n^i = 0$.

Let $S_1, \ldots, S_n$ be the elementary symmetric polynomials in n variables and let $\Pi_i(X_1, \ldots, X_n) = X_1^i + \cdots + X_n^i$. We have the Newton-Girard formulae $kS_k = \sum_{i=1}^{k} (-1)^{i-1} S_{k-i} \Pi_i$. Let $s_i = S_i(\alpha_1, \ldots, \alpha_n)$. We have $\Pi_i(\alpha_1, \ldots, \alpha_n) = 0$ $\forall i \geq 1$, so if we take $(X_1, \ldots, X_n) = (\alpha_1, \ldots, \alpha_n)$ in the Newton-Girard formulae then we get $ks_k = 0$ $\forall 1 \leq k \leq n$, so $s_k = 0$. Here we used the fact that either $\operatorname{char} K = 0$ or $\operatorname{char} K > n \geq k$, so that $k \in K^\times$. Then $P_C(X) = X^n - s_1 X^{n-1} + \cdots + (-1)^n s_n = X^n$, so $C^n = 0$, as claimed.

The proof doesn't work when $\operatorname{char} K = p \leq n$ because in this case $ps_p = 0$ does not imply $s_p = 0$. Note that it is enough to consider the case $n = p$. Indeed, if $A, B \in M_p(K)$ are a counter-example in the case $n = p$ then for $n \geq p$ we have a counter-example given by the matrices

$$\overline{A} = \left(\begin{array}{c|c} A & 0_{p,n-p} \\ \hline 0_{n-p,p} & 0_{n-p,n-p} \end{array}\right) \quad \text{and} \quad \overline{B} = \left(\begin{array}{c|c} B & 0_{p,n-p} \\ \hline 0_{n-p,p} & 0_{n-p,n-p} \end{array}\right).$$

Suppose now that $n = p = \operatorname{char} K$. With the notations above, $ks_k = 0$ implies $s_k = 0$ for $0 \leq k \leq p-1$, so $P_C(X) = X^p + (-1)^p s_p = X^p - s_p$. (Even when $p = 2$, because in characteristic 2 we have $1 = -1$.) If $\alpha = \sqrt[p]{s_p}$ then $P_C = X^p - \alpha^p = (X - \alpha)^p$, so $\alpha_1 = \cdots = \alpha_p = \alpha$ and the minimal polynomial of C has the form $(X - \alpha)^i$ for some $1 \leq i \leq p$. We will provide a counter example with $i = 1$ and $\alpha = 1$, i.e., when the minimal polynomial is $X - 1$ and we have $[A, B] = C = I_p$. Then obviously C commutes with both A and B.

For convenience, we index the rows and columns of the $p \times p$ matrices not by $1, \ldots, p$, but by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$. Let $e_{i,j}$ with $i, j \in \mathbb{Z}_p$ be the canonical basis of $M_p(K)$, where $e_{i,j}$ has 1 on the (i, j) position and 0 everywhere else. We have $e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l}$. Then we take $A = \sum_{i \in \mathbb{Z}_p} e_{i,i+1}$ and $B = \sum_{i \in \mathbb{Z}_p} ie_{i+1,i}$. (Note that, since $\operatorname{char} K = p$, we have $\mathbb{Z}_p \subseteq K$, so $B \in M_p(\mathbb{Z}_p) \subseteq M_P(K)$.) Then $AB = \sum_{i \in \mathbb{Z}_p} ie_{i,i}$ and $BA = \sum_{i \in \mathbb{Z}_p} ie_{i+1,i+1} = \sum_{i \in \mathbb{Z}_p} (i-1)e_{i,i}$.

Hence $[A, B] = AB - BA = \sum_i e_{i,i} = I_p$, as claimed. □