GAZETA MATEMATICĂ

SERIA A

ANUL XXXVI (CXV) Nr. 1 – 2/ 2018

ARTICOLE

Computing exponential and trigonometric functions of matrices in $\mathcal{M}_2(\mathbb{C})$

Ovidiu Furdui[1)]

Abstract. In this paper we give a new technique for the calculation of the matrix exponential function e^A as well as the matrix trigonometric functions $\sin A$ and $\cos A$, where $A \in \mathcal{M}_2(\mathbb{C})$. We also determine the real logarithm of the matrix xI_2, when $x \in \mathbb{R}^*$, as well as the real logarithm of scalar multiple of rotation and reflection matrices. The real logarithm of a real circulant matrix and a symmetric matrix are also determined.

Keywords: Square matrices of order two, matrix exponential function, matrix trigonometric functions, the real logarithm.

MSC: 15A16.

1. Introduction and the main results

Let $A \in \mathcal{M}_2(\mathbb{C})$. In this paper we give a new technique for the calculation of the matrix exponential function e^A as well as the matrix trigonometric functions $\sin A$ and $\cos A$. Our method, which we believe is new in the literature?!, is based on an application of the Cayley–Hamilton Theorem and the power series expansion of the exponential and the trigonometric functions. The organization of the paper is as follows: in the first section we give formulas for e^A, $\sin A$ and $\cos A$ in terms of the eigenvalues of A and in the second section we solve several exponential equations in $\mathcal{M}_2(\mathbb{R})$ and we compute the real logarithm of special real matrices. The next theorems are the main results of this section.

Theorem 1. ***The exponential function*** e^A**.**

[1)]Department of Mathematics, Technical University of Cluj-Napoca, Romania, `Ovidiu.Furdui@math.utcluj.ro`, `ofurdui@yahoo.com`

Let $A \in \mathcal{M}_2(\mathbb{C})$ and let $\alpha, \beta \in \mathbb{C}$ be the eigenvalues of A. The following formula holds

$$\mathrm{e}^A = \begin{cases} \dfrac{\mathrm{e}^\alpha - \mathrm{e}^\beta}{\alpha - \beta} A + \dfrac{\alpha \mathrm{e}^\beta - \beta \mathrm{e}^\alpha}{\alpha - \beta} I_2, & \textit{if} \quad \alpha \neq \beta, \\ \mathrm{e}^\alpha A + (\mathrm{e}^\alpha - \alpha \mathrm{e}^\alpha) I_2, & \textit{if} \quad \alpha = \beta. \end{cases}$$

Theorem 2. ***The trigonometric function*** $\sin A$***.***
Let $A \in \mathcal{M}_2(\mathbb{C})$ and let $\alpha, \beta \in \mathbb{C}$ be the eigenvalues of A. The following formula holds

$$\sin A = \begin{cases} \dfrac{\sin\alpha - \sin\beta}{\alpha - \beta} A + \dfrac{\alpha \sin\beta - \beta \sin\alpha}{\alpha - \beta} I_2, & \textit{if} \quad \alpha \neq \beta, \\ (\cos\alpha) A + (\sin\alpha - \alpha\cos\alpha) I_2, & \textit{if} \quad \alpha = \beta. \end{cases}$$

Theorem 3. ***The trigonometric function*** $\cos A$***.***
Let $A \in \mathcal{M}_2(\mathbb{C})$ and let $\alpha, \beta \in \mathbb{C}$ be the eigenvalues of A. The following formula holds

$$\cos A = \begin{cases} \dfrac{\cos\alpha - \cos\beta}{\alpha - \beta} A + \dfrac{\alpha \cos\beta - \beta \cos\alpha}{\alpha - \beta} I_2, & \textit{if} \quad \alpha \neq \beta, \\ (\sin\alpha) A + (\cos\alpha + \alpha\sin\alpha) I_2, & \textit{if} \quad \alpha = \beta. \end{cases}$$

Before we prove these theorems we collect a lemma we need in our analysis.

Lemma 4. *Let $a \in \mathbb{C}$ and let $A \in \mathcal{M}_2(\mathbb{C})$. The following statements hold*:

(a) $\mathrm{e}^{aI_2} = \mathrm{e}^a I_2$;

(b) $\sin(aI_2) = (\sin a) I_2$;

(c) $\cos(aI_2) = (\cos a) I_2$;

(d) if $A, B \in \mathcal{M}_2(\mathbb{C})$ commute, then $\mathrm{e}^{A+B} = \mathrm{e}^A \mathrm{e}^B$;

(e) if $A, B \in \mathcal{M}_2(\mathbb{C})$ commute, then $\sin(A+B) = \sin A \cos B + \cos A \sin B$;

(f) if $A, B \in \mathcal{M}_2(\mathbb{C})$ commute, then $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

Proof. The proof of this lemma can be found in [4, pp. 189–192]. □

Now we are ready to prove these theorems.
Proof. First we consider the case $\alpha \neq \beta$. We have, based on the Cayley–Hamilton Theorem, that $(A - \alpha I_2)(A - \beta I_2) = O_2$. Let $X = A - \alpha I_2$. It follows that $X(X + (\alpha - \beta) I_2) = O_2$ which implies that $X^2 = (\beta - \alpha) X$. This

in turn implies that $X^n = (\beta-\alpha)^{n-1}X$, for all $n \geq 1$. We have,

$$\begin{aligned}
e^A &= e^{\alpha I_2 + X} \\
&= e^{\alpha I_2} e^X \\
&= e^{\alpha} \sum_{n=0}^{\infty} \frac{X^n}{n!} \\
&= e^{\alpha} \left(I_2 + \sum_{n=1}^{\infty} \frac{(\beta-\alpha)^{n-1} X}{n!} \right) \\
&= e^{\alpha} \left(I_2 + \frac{e^{\beta-\alpha}-1}{\beta-\alpha} X \right) \\
&= e^{\alpha} \left(I_2 + \frac{e^{\beta-\alpha}-1}{\beta-\alpha} (A-\alpha I_2) \right) \\
&= \frac{e^{\alpha}-e^{\beta}}{\alpha-\beta} A + \frac{\alpha e^{\beta} - \beta e^{\alpha}}{\alpha-\beta} I_2.
\end{aligned}$$

Now we consider the case $\alpha = \beta$. The Cayley–Hamilton Theorem implies that $(A-\alpha I_2)^2 = O_2$. Let $X = A - \alpha I_2$. It follows that $A = X + \alpha I_2$ and $X^2 = O_2$. We have,

$$\begin{aligned}
e^A &= e^{\alpha I_2 + X} \\
&= e^{\alpha I_2} e^X \\
&= e^{\alpha} I_2 \left(I_2 + \frac{X}{1!} + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots \right) \\
&= e^{\alpha}(I_2 + X) \\
&= e^{\alpha}(I_2 + A - \alpha I_2) \\
&= e^{\alpha} A + (e^{\alpha} - \alpha e^{\alpha}) I_2,
\end{aligned}$$

and Theorem 1 is proved. □

Now we give the proof of Theorem 2.

Proof. One method of proving the theorem is based on the use of *Euler matrix formula*

$$\sin A = \frac{e^{iA} - e^{-iA}}{2i}$$

combined to Theorem 1. We leave these details to the interested reader. Instead we prove the theorem by using a power series technique. First we consider the case $\alpha \neq \beta$. As in the proof of Theorem 1 we have, based on the Cayley–Hamilton Theorem, that $(A-\alpha I_2)(A-\beta I_2) = O_2$. Let $X = A - \alpha I_2$. It follows that $X(X + (\alpha-\beta)I_2) = O_2$ which implies that $X^2 = (\beta-\alpha)X$.

This in turn implies that $X^n = (\beta-\alpha)^{n-1}X$, for all $n \geq 1$. We have,

$$\begin{aligned}\sin A = \sin(X+\alpha I_2) &= \sin X\cos(\alpha I_2) + \cos X\sin(\alpha I_2)\\ &= \sin X(\cos\alpha) + \cos X(\sin\alpha).\end{aligned}$$

On the other hand,

$$\begin{aligned}\sin X &= \sum_{n=0}^{\infty}(-1)^n\frac{X^{2n+1}}{(2n+1)!}\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(\beta-\alpha)^{2n}}{(2n+1)!}X\\ &= \frac{1}{\beta-\alpha}\sum_{n=0}^{\infty}(-1)^n\frac{(\beta-\alpha)^{2n+1}}{(2n+1)!}X\\ &= \frac{\sin(\beta-\alpha)}{\beta-\alpha}X\end{aligned}$$

and

$$\begin{aligned}\cos X &= \sum_{n=0}^{\infty}(-1)^n\frac{X^{2n}}{(2n)!}\\ &= I_2 + \sum_{n=1}^{\infty}(-1)^n\frac{(\beta-\alpha)^{2n-1}}{(2n)!}X\\ &= I_2 + \frac{1}{\beta-\alpha}\sum_{n=1}^{\infty}(-1)^n\frac{(\beta-\alpha)^{2n}}{(2n)!}X\\ &= I_2 + \frac{\cos(\beta-\alpha)-1}{\beta-\alpha}X.\end{aligned}$$

Putting all these together we have, after simple calculations, that

$$\begin{aligned}\sin A &= \frac{\sin(\beta-\alpha)}{\beta-\alpha}(\cos\alpha)(A-\alpha I_2) + \left[I_2 + \frac{\cos(\beta-\alpha)-1}{\beta-\alpha}(A-\alpha I_2)\right]\sin\alpha\\ &= \frac{\sin\alpha-\sin\beta}{\alpha-\beta}A + \frac{\alpha\sin\beta-\beta\sin\alpha}{\alpha-\beta}I_2.\end{aligned}$$

Now we consider the case $\alpha = \beta$. The Cayley–Hamilton Theorem implies that $(A-\alpha I_2)^2 = O_2$. Let $X = A-\alpha I_2$. It follows that $A = X+\alpha I_2$ and $X^2 = O_2$. We have,

$$\begin{aligned}\sin A &= \sin(\alpha I + X)\\ &= \sin X\cos(\alpha I_2) + \cos X\sin(\alpha I_2)\\ &= \sin X\cos\alpha + \cos X\sin\alpha\\ &= X\cos\alpha + I_2\sin\alpha\\ &= (\cos\alpha)A + (\sin\alpha - \alpha\cos\alpha)I_2.\end{aligned}$$

We used that $\sin X = X$ and $\cos X = I_2$. $\square$

The proof of Theorem 3 which is similar to the proof of Theorem 2 is left as an exercise to the interested reader.

Corollary 5. ***Functions of reflection matrices.***

Let $a, b \in \mathbb{R}$. The following equalities hold:

(1) $$e^{\begin{pmatrix} a & b \\ b & -a \end{pmatrix}} = \left(\cosh\sqrt{a^2+b^2}\right) I_2 + \frac{\sinh\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} \begin{pmatrix} a & b \\ b & -a \end{pmatrix};$$

(2) $$\sin\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \frac{\sin\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} \begin{pmatrix} a & b \\ b & -a \end{pmatrix};$$

(3) $$\cos\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \left(\cos\sqrt{a^2+b^2}\right) I_2.$$

Remark 6. Observe that

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \sqrt{a^2+b^2} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

where $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$. Thus, any matrix of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ is a scalar multiple of a reflection matrix. The reason why the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ is called a reflection matrix is given in [4, p. 14].

Corollary 7. *Let $a, b, c \in \mathbb{R}$. The following statements hold*

$$e^{\begin{pmatrix} a & b \\ c & a \end{pmatrix}} = \begin{cases} e^a \begin{pmatrix} \cosh\sqrt{bc} & \frac{b}{\sqrt{bc}}\sinh\sqrt{bc} \\ \frac{c}{\sqrt{bc}}\sinh\sqrt{bc} & \cosh\sqrt{bc} \end{pmatrix} & \textit{if} \quad bc > 0, \\ e^a \begin{pmatrix} \cos\sqrt{-bc} & \frac{b}{\sqrt{-bc}}\sin\sqrt{-bc} \\ \frac{c}{\sqrt{-bc}}\sin\sqrt{-bc} & \cos\sqrt{-bc} \end{pmatrix} & \textit{if} \quad bc < 0. \end{cases}$$

Corollary 8. *Let $a, b, c \in \mathbb{R}$. The following statements hold*

$$\sin\begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{cases} \begin{pmatrix} \sin a\cos\sqrt{bc} & \frac{b}{\sqrt{bc}}\cos a\sin\sqrt{bc} \\ \frac{c}{\sqrt{bc}}\cos a\sin\sqrt{bc} & \sin a\cos\sqrt{bc} \end{pmatrix} & \textit{if } bc > 0, \\ \begin{pmatrix} \sin a\cosh\sqrt{-bc} & \frac{b}{\sqrt{-bc}}\cos a\sinh\sqrt{-bc} \\ \frac{c}{\sqrt{-bc}}\cos a\sinh\sqrt{-bc} & \sin a\cosh\sqrt{-bc} \end{pmatrix} & \textit{if } bc < 0. \end{cases}$$

Corollary 9. *Let $a, b, c \in \mathbb{R}$. The following statements hold*

$$\cos\begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{cases} \begin{pmatrix} \cos a \cos\sqrt{bc} & -\dfrac{b}{\sqrt{bc}}\sin a \sin\sqrt{bc} \\ -\dfrac{c}{\sqrt{bc}}\sin a \sin\sqrt{bc} & \cos a\cos\sqrt{bc} \end{pmatrix} & \text{if } bc > 0, \\ \begin{pmatrix} \cos a \cosh\sqrt{-bc} & -\dfrac{b}{\sqrt{-bc}}\sin a \sinh\sqrt{-bc} \\ -\dfrac{c}{\sqrt{-bc}}\sin a \sinh\sqrt{-bc} & \cos a\cosh\sqrt{-bc} \end{pmatrix} & \text{if } bc < 0. \end{cases}$$

Corollary 10. ***Functions of rotation matrices.***

Let $a, b \in \mathbb{R}$. The following equalities hold:

(1) $\mathrm{e}^{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}} = \mathrm{e}^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}$;

(2) $\sin\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \sin a \cosh b & -\cos a \sinh b \\ \cos a \sinh b & \sin a \cosh b \end{pmatrix}$;

(3) $\cos\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \cos a \cosh b & \sin a \sinh b \\ -\sin a \sinh b & \cos a \cosh b \end{pmatrix}$.

Remark 11. Observe that

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \sqrt{a^2+b^2}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

where $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \frac{a}{\sqrt{a^2+b^2}}$. Thus, any matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is a scalar multiple of a rotation matrix. The reason why the matrix $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ is called a rotation matrix is given in [4, Problem 1.61, p. 31].

Remark 12. We mention that if $x \in \mathbb{R}$ and $A \in \mathcal{M}_2(\mathbb{R})$ the calculation of the matrix functions e^{Ax}, $\sin(Ax)$ and $\cos(Ax)$ has been done, by a completely different method, in [4, Appendix A, pp. 354–358].

If f is a function which has the Maclaurin series $f(z) = \sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$, $|z| < R$, where $R \in (0, \infty]$ and $A \in \mathcal{M}_2(\mathbb{C})$ is such that its eigenvalues α and β verify the conditions $|\alpha| < R$ and $|\beta| < R$, then

$$f(A) = \begin{cases} \dfrac{f(\alpha) - f(\beta)}{\alpha - \beta} A + \dfrac{\alpha f(\beta) - \beta f(\alpha)}{\alpha - \beta} I_2, & \text{if } \alpha \neq \beta, \\ f(\alpha)A + (f(\alpha) - \alpha f'(\alpha)) I_2, & \text{if } \alpha = \beta. \end{cases} \tag{1}$$

For a proof of this remarkable formula, which is based on the calculation of the nth power of a square matrix of order two the reader is referred to [4, Theorem 4.7, p. 194].

We mention that formula (1) holds in a more general case: when $\alpha \neq \beta$ the formula holds for complex value functions f defined on $\operatorname{spec}(A) = \{\alpha, \beta\}$, while if $\alpha = \beta$ the formula is valid if f is differentiable on $\operatorname{spec}(A) = \{\alpha\}$ (see [2, Notes p. 221] and [3, P.11, Sect. 6.1]).

2. The real logarithm of special matrices

Let $A \in \mathcal{M}_2(\mathbb{R})$. We say that $B \in \mathcal{M}_2(\mathbb{R})$ is a real logarithm of A if $\mathrm{e}^B = A$ (see [1, p. 718]). In this section we solve in $\mathcal{M}_2(\mathbb{R})$ various exponential matrix equations and hence we determine the real logarithm of special real matrices.

Theorem 13. ***Two exponential equations.***

(a) *Let $A \in \mathcal{M}_2(\mathbb{R})$. The solution of the equation $\mathrm{e}^A = xI_2$, where $x \in \mathbb{R}^*$, is given by*

$$A = P \begin{pmatrix} \ln(x(-1)^k) & k\pi \\ -k\pi & \ln(x(-1)^k) \end{pmatrix} P^{-1},$$

where $P \in \mathrm{GL}_2(\mathbb{R})$ and k is an even integer if $x > 0$ and an odd integer if $x < 0$.

(b) *The equation $\mathrm{e}^A = \begin{pmatrix} x & y \\ y & 0 \end{pmatrix}$, where $x, y \in \mathbb{R}$, does not have solutions in $\mathcal{M}_2(\mathbb{R})$.*

Proof. (a) If α, β are the real eigenvalues of A, then there exists $P \in \mathrm{GL}_2(\mathbb{R})$ such that

$$A = P \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} P^{-1} \quad \text{or} \quad A = P \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} P^{-1},$$

according to whether the eigenvalues of A are distinct or not (see [4, Theorem 2.10, p. 79]). It follows that

$$\mathrm{e}^A = P \mathrm{e}^{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}} P^{-1} = xI_2$$

and this implies that

$$\begin{pmatrix} \mathrm{e}^\alpha & 0 \\ 0 & \mathrm{e}^\beta \end{pmatrix} = xI_2.$$

Thus, $\mathrm{e}^\alpha = \mathrm{e}^\beta = x$ which implies that $x > 0$ and $\alpha = \beta = \ln x$. This shows that $A = (\ln x) I_2$.

If the eigenvalues of A are $\alpha + i\beta$ and $\alpha - i\beta$, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^*$, then there exists $P \in \mathrm{GL}_2(\mathbb{R})$ such that (see [4, Theorem 2.10, p. 79])

$$A = P \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} P^{-1}.$$

This implies that

$$e^{\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}} = xI_2$$

and it follows, based on part (1) of Corollary 10, that

$$e^{\alpha} \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} = xI_2.$$

Thus, $e^{\alpha}\cos\beta = x$ and $e^{\alpha}\sin\beta = 0$. It follows that $\sin\beta = 0 \Rightarrow \beta = k\pi$, $k \in \mathbb{Z}$. The first equation implies that $e^{\alpha} = x(-1)^k \Rightarrow \alpha = \ln(x(-1)^k)$, where k is an even integer when $x > 0$ and k is an odd integer when $x < 0$.

(b) Passing to determinants we get that

$$\det e^{A} = \det \begin{pmatrix} x & y \\ y & 0 \end{pmatrix} \quad \Rightarrow \quad e^{\operatorname{Tr}(A)} = -y^2,$$

which is impossible over $\mathbb{R}$. □

Remark 14. Part (a) of Theorem 13 states that the real logarithm of the matrix xI_2 is given by

$$\ln(xI_2) = P \begin{pmatrix} \ln(x(-1)^k) & k\pi \\ -k\pi & \ln(x(-1)^k) \end{pmatrix} P^{-1},$$

where $P \in \mathrm{GL}_2(\mathbb{R})$ and k is an even integer if $x > 0$ and an odd integer if $x < 0$. Part (b) of the theorem shows that the matrix $\begin{pmatrix} x & y \\ y & 0 \end{pmatrix}$, $x, y \in \mathbb{R}$, does not have a real logarithm.

We mention that if $A \in \mathcal{M}_2(\mathbb{C})$ the equation $e^{A} = zI_2$, where $z \in \mathbb{C}^*$, has been studied in [4, Problem 4.36, p. 219]. Also, the trigonometric equations $\sin A = I_2$ and $\cos A = I_2$ over the real square matrices have been studied in Appendix B of [4]. The equations $\sin A = xI_2$ and $\cos A = xI_2$, with $x \in \mathbb{R}$, can be solved similarly to the technique given in Appendix B of [4].

Theorem 15. ***The real logarithm of scalar multiple of rotation matrices.***

(a) *Let $A \in \mathcal{M}_2(\mathbb{R})$. The solution of the equation*

$$e^{A} = \begin{pmatrix} 0 & -y \\ y & 0 \end{pmatrix}, \quad y \in \mathbb{R}^*,$$

is given by

$$A = \begin{pmatrix} \ln((-1)^k y) & -(2k+1)\dfrac{\pi}{2} \\ (2k+1)\dfrac{\pi}{2} & \ln((-1)^k y) \end{pmatrix},$$

where $k \in \mathbb{Z}$ is an even integer when $y > 0$ and an odd integer when $y < 0$.

(b) *Let $A \in \mathcal{M}_2(\mathbb{R})$ and let $x, y \in \mathbb{R}^*$. The solution of the equation*

$$e^A = \begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

is given by

$$A = \begin{pmatrix} \ln\sqrt{x^2+y^2} & -(\theta+2k\pi) \\ \theta+2k\pi & \ln\sqrt{x^2+y^2} \end{pmatrix},$$

where $k \in \mathbb{Z}$ and $\theta \in (0, 2\pi)$ is such that

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \text{ and } \sin\theta = \frac{y}{\sqrt{x^2+y^2}}.$$

Proof. (a) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since A commutes with $\begin{pmatrix} 0 & -y \\ y & 0 \end{pmatrix}$ we get that $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. We have, based on part (1) of Corollary 10, that

$$e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix} = \begin{pmatrix} 0 & -y \\ y & 0 \end{pmatrix}$$

and it follows that $e^a \cos b = 0$ and $e^a \sin b = y$. The first equation implies that $b = (2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$, and the second equation shows that $e^a = (-1)^k y$. Thus, $a = \ln((-1)^k y)$, where k is an even integer when $y > 0$ and an odd integer when $y < 0$.

(b) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since A commutes with $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ we get that $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. We have, based on part (1) of Corollary 10, that

$$e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

and it follows that $e^a \cos b = x$ and $e^a \sin b = y$. This implies that $e^{2a} = x^2+y^2 \Rightarrow a = \ln\sqrt{x^2+y^2}$ and $\cos b = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin b = \frac{y}{\sqrt{x^2+y^2}}$. Let $\theta \in (0, 2\pi)$ be such that $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$. It follows that

$$\begin{cases} \sin b = \sin\theta \\ \cos b = \cos\theta \end{cases} \Leftrightarrow \begin{cases} b = \theta + 2k\pi \quad \text{or} \quad b = \pi - \theta + 2k\pi, \\ b = \theta + 2n\pi \quad \text{or} \quad b = -\theta + 2n\pi, \end{cases}$$

where $k, n \in \mathbb{Z}$. Since $x, y \neq 0$ these cases imply that $b = \theta + 2k\pi$, $k \in \mathbb{Z}$. $\square$

Remark 16. Part (a) of Theorem 15 shows that the real logarithm of the real matrix $\begin{pmatrix} 0 & -y \\ y & 0 \end{pmatrix}$ is given by

$$\ln\begin{pmatrix} 0 & -y \\ y & 0 \end{pmatrix} = \begin{pmatrix} \ln((-1)^k y) & -(2k+1)\frac{\pi}{2} \\ (2k+1)\frac{\pi}{2} & \ln((-1)^k y) \end{pmatrix},$$

where $k \in \mathbb{Z}$ is an even integer when $y > 0$ and an odd integer when $y < 0$.

Part (b) of the theorem shows that the real logarithm of the matrix $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ is

$$\ln \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} \ln \sqrt{x^2+y^2} & -(\theta + 2k\pi) \\ \theta + 2k\pi & \ln \sqrt{x^2+y^2} \end{pmatrix},$$

where $k \in \mathbb{Z}$ and $\theta \in (0, 2\pi)$ is such that $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$.

Corollary 17. ***The real logarithm of a rotation matrix.***

Let $\alpha \in (0, 2\pi) \setminus \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$. The solution, in $\mathcal{M}_2(\mathbb{R})$, of the equation

$$\mathrm{e}^A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

is given by

$$A = (\alpha + 2k\pi) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad k \in \mathbb{Z}.$$

The previous corollary shows that the real logarithm of the rotation matrix $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ is

$$\ln \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = (\alpha + 2k\pi) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Theorem 18. ***The real logarithm of a circulant matrix.***

Let $A \in \mathcal{M}_2(\mathbb{R})$ and let $x, y \in \mathbb{R}$, $y \neq 0$. The equation

$$\mathrm{e}^A = \begin{pmatrix} x & y \\ y & x \end{pmatrix}$$

has solutions in $\mathcal{M}_2(\mathbb{R})$ if and only if $x > 0$ and $-x < y < x$ and in this case the solution is given by

$$A = \begin{pmatrix} \ln \sqrt{x^2-y^2} & \ln \sqrt{\frac{x+y}{x-y}} \\ \ln \sqrt{\frac{x+y}{x-y}} & \ln \sqrt{x^2-y^2} \end{pmatrix}.$$

Proof. Since A commutes with $\begin{pmatrix} x & y \\ y & x \end{pmatrix}$ we get that $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. We have, based on Corollary 7, that

$$\mathrm{e}^A = \mathrm{e}^a \begin{pmatrix} \cosh b & \sinh b \\ \sinh b & \cosh b \end{pmatrix}.$$

It follows that

$$\begin{cases} \mathrm{e}^a \cdot \dfrac{\mathrm{e}^b + \mathrm{e}^{-b}}{2} = x > 0 \\ \mathrm{e}^a \cdot \dfrac{\mathrm{e}^b - \mathrm{e}^{-b}}{2} = y \end{cases} \quad \Rightarrow \quad \begin{cases} \mathrm{e}^{a+b} = x + y > 0, \\ \mathrm{e}^{a-b} = x - y > 0. \end{cases}$$

Thus, $-x < y < x$ and a calculation shows that $a = \ln\sqrt{x^2 - y^2}$ and $b = \ln\sqrt{\frac{x+y}{x-y}}$. □

Corollary 19. *Let $A \in \mathcal{M}_2(\mathbb{R})$ and let $y \in \mathbb{R}$, $y > -\frac{1}{2}$. The solution of the equation*

$$e^A = \begin{pmatrix} y+1 & y \\ y & y+1 \end{pmatrix}$$

is given by

$$A = \ln\sqrt{2y+1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Proof. In Theorem 18 we let $x = y + 1 > 0$. □

Theorem 20. *Let $x, y \in \mathbb{R}$ be such that $xy < 0$. The solution of the equation*

$$e^A = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$$

is given by

$$A = \begin{pmatrix} \ln\sqrt{-xy} & \mp\dfrac{\pi}{2}\sqrt{-\dfrac{x}{y}}(2k+1) \\ \pm\dfrac{\pi}{2}\sqrt{-\dfrac{y}{x}}(2k+1) & \ln\sqrt{-xy} \end{pmatrix},$$

where $k \geq 0$ is an integer.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$. Since A commutes with $\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$ we have that $A = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$. Observe that Corollary 7 implies that $bc < 0$ and we get that

$$e^a \begin{pmatrix} \cos\sqrt{-bc} & \dfrac{b}{\sqrt{-bc}}\sin\sqrt{-bc} \\ \dfrac{c}{\sqrt{-bc}}\sin\sqrt{-bc} & \cos\sqrt{-bc} \end{pmatrix} = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix},$$

which implies that

$$\begin{cases} \cos\sqrt{-bc} = 0, \\ e^a \dfrac{b}{\sqrt{-bc}}\sin\sqrt{-bc} = x, \\ e^a \dfrac{c}{\sqrt{-bc}}\sin\sqrt{-bc} = y. \end{cases} \tag{2}$$

The first equation implies that $\sqrt{-bc} = \frac{\pi}{2}(2k+1)$, where $k \geq 0$ is an integer. It follows that $e^a \frac{b}{\sqrt{-bc}}(-1)^k = x$ and $e^a \frac{c}{\sqrt{-bc}}(-1)^k = y$. These two equations imply that $e^{2a} = -xy \Rightarrow a = \ln\sqrt{-xy}$. Dividing the last two equations in (2) we get that $\frac{b}{c}=\frac{x}{y}\Rightarrow b=\frac{x}{y}c$. On the other hand, $bc=-\left(\frac{\pi}{2}(2k+1)\right)^2$ and it follows that $c^2=-\frac{y}{x}\left(\frac{\pi}{2}(2k+1)\right)^2$. This implies that $c=\pm\frac{\pi}{2}\sqrt{-\frac{y}{x}}(2k+1)$ and $b=\frac{x}{y}c=-\left(\sqrt{-\frac{x}{y}}\right)^2\left(\pm\frac{\pi}{2}\sqrt{-\frac{y}{x}}(2k+1)\right)=\mp\frac{\pi}{2}\sqrt{-\frac{x}{y}}(2k+1)$. □

Theorem 21. ***The real logarithm of triangular matrices.***
Let $x,y\in\mathbb{R}^*$. The equation

$$e^A=\begin{pmatrix}x & y\\ 0 & x\end{pmatrix}$$

has solutions in $\mathcal{M}_2(\mathbb{R})$ if and only if $x>0$ and in this case the solution is given by

$$A=\begin{pmatrix}\ln x & \frac{y}{x}\\ 0 & \ln x\end{pmatrix}.$$

Proof. Let $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. Since A commutes with $\begin{pmatrix}x & y\\ 0 & x\end{pmatrix}$ we get that $A=\begin{pmatrix}a & b\\ 0 & a\end{pmatrix}$. In this case both eigenvalues of A are equal to a. It follows, based on Theorem 1, that

$$e^A=\begin{pmatrix}e^a & be^a\\ 0 & e^a\end{pmatrix}=\begin{pmatrix}x & y\\ 0 & x\end{pmatrix}.$$

Thus, $e^a=x>0$ and $be^a=y$. This implies that $a=\ln x$ and $b=\frac{y}{x}$. □

Remark 22. The preceding theorem shows that the matrix $\begin{pmatrix}x & y\\ 0 & x\end{pmatrix}$ has a real logarithm if and only if $x>0$.

Theorem 23. ***The real logarithm of symmetric matrices.***
Let $X\in\mathcal{M}_2(\mathbb{R})$ be a symmetric matrix such that $X\neq aI_2$, $a\in\mathbb{R}^*$. The equation $e^A=X$ has solutions in $\mathcal{M}_2(\mathbb{R})$ if and only if $\operatorname{Tr}(A)>0$ *and* $\det A>0$ *and in this case the solution is given by*

$$A=\frac{\ln\lambda_1-\ln\lambda_2}{\lambda_1-\lambda_2}X+\frac{\lambda_1\ln\lambda_2-\lambda_2\ln\lambda_1}{\lambda_1-\lambda_2}I_2, \tag{3}$$

where λ_1,λ_2 are the eigenvalues of X.

Proof. Since $X\neq aI_2$ and A commutes with X we get, based on Theorem 1.1 in [4, p. 15], that $A=\alpha X+\beta I_2$, for some $\alpha,\beta\in\mathbb{R}$. The equation $e^A=X$ implies that $e^{\alpha X+\beta I_2}=X\Rightarrow e^{\alpha X}=e^{-\beta}X$. Since X is symmetric and $X\neq aI_2$, $a\in\mathbb{R}^*$, we know that X has real distinct eigenvalues (see [4, Theorem 2.5, p. 73]). Let λ_1,λ_2 be the eigenvalues of X. Observe that, since A commutes with X, A is also symmetric and it has real eigenvalues. If μ_1 and μ_2 are the eigenvalues of A, then $\{e^{\mu_1},e^{\mu_2}\}=\{\lambda_1,\lambda_2\}$. Thus, both eigenvalues of X should be positive real numbers, i.e., $\operatorname{Tr}(A) > 0$ and $\det A > 0$. The equation $e^{\alpha X} = e^{-\beta} X$ combined to Theorem 1 show that

$$\frac{e^{\alpha\lambda_1} - e^{\alpha\lambda_2}}{\alpha\lambda_1 - \alpha\lambda_2}(\alpha X) + \frac{\alpha\lambda_1 e^{\alpha\lambda_2} - \alpha\lambda_2 e^{\alpha\lambda_1}}{\alpha\lambda_1 - \alpha\lambda_2} I_2 = e^{-\beta} X.$$

This implies that

$$\frac{e^{\alpha\lambda_1} - e^{\alpha\lambda_2}}{\lambda_1 - \lambda_2} X + \frac{\lambda_1 e^{\alpha\lambda_2} - \lambda_2 e^{\alpha\lambda_1}}{\lambda_1 - \lambda_2} I_2 = e^{-\beta} X$$

and, since $X \neq aI_2$, $a \neq 0$, we have that

$$\frac{e^{\alpha\lambda_1} - e^{\alpha\lambda_2}}{\lambda_1 - \lambda_2} = e^{-\beta} \quad \text{and} \quad \frac{\lambda_1 e^{\alpha\lambda_2} - \lambda_2 e^{\alpha\lambda_1}}{\lambda_1 - \lambda_2} = 0.$$

It follows that $\lambda_1 e^{\alpha\lambda_2} - \lambda_2 e^{\alpha\lambda_1} = 0 \Rightarrow e^{\alpha(\lambda_1 - \lambda_2)} = \frac{\lambda_1}{\lambda_2} \Rightarrow \alpha = \frac{\ln\lambda_1 - \ln\lambda_2}{\lambda_1 - \lambda_2}$. On the other hand,

$$\begin{aligned}
\beta &= \ln \frac{\lambda_1 - \lambda_2}{e^{\alpha\lambda_1} - e^{\alpha\lambda_2}} \\
&= \ln \frac{\lambda_1 - \lambda_2}{\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_1}{\lambda_1 - \lambda_2}} - \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_2}{\lambda_1 - \lambda_2}}} \\
&= \ln \frac{\lambda_1}{\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_1}{\lambda_1 - \lambda_2}}} \\
&= \frac{\lambda_1 \ln\lambda_2 - \lambda_2 \ln\lambda_1}{\lambda_1 - \lambda_2}.
\end{aligned}$$

Thus,

$$A = \alpha X + \beta I_2 = \frac{\ln\lambda_1 - \ln\lambda_2}{\lambda_1 - \lambda_2} X + \frac{\lambda_1 \ln\lambda_2 - \lambda_2 \ln\lambda_1}{\lambda_1 - \lambda_2} I_2,$$

and the theorem is proved. □

Remark 24. Theorem 23 shows that a symmetric matrix which is not a scalar multiple of the identity matrix has a real logarithm if and only if its eigenvalues are both positive real numbers and in this case the real logarithm is given by formula (3).

References

[1] D. N. Bernstein, *Matrix Mathematics. Theory, Facts and Formulas*, Princeton University Press, 2009.

[2] S. R. Garcia and R. A. Horn, *A Second Course in Linear Algebra*, Cambridge University Press, 2017.

[3] R. A. Horn and C. H. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1994.

[4] V. Pop and O. Furdui, *Square Matrices of Order Two. Theory, Applications, and Problems*, Springer, 2017.

Stirling type formulas

ION-DENYS CIOROGARU[1)]

Abstract. In this paper we prove two Stirling type formulas:

$$\prod_{i+j\le n;i,j\in\mathbb{N}}(i+j)\sim\frac{g_k}{\sqrt{2\pi}}\cdot\frac{n^{\frac{n^2}{2}-\frac{n}{2}-\frac{5}{12}}}{e^{\frac{n^2}{4}-n}}$$

and

$$\prod_{i,j=1}^{n}(i+j)\sim\frac{g_k}{\sqrt[12]{2e\pi^6}}\cdot\frac{n^{n^2-\frac{5}{12}}\cdot 4^{n^2+n}}{e^{\frac{3n^2}{2}}},$$

where g_k is the Glaisher-Kinkelin constant.

Keywords: Stirling formula, Glaisher-Kinkelin constant.
MSC: Primary 40A25; Secondary 41A60.

1. INTRODUCTION

In this paper the notation and notion used are standard, in particular $\mathbb{N}=\{1,2,...\}$ is the set of all natural numbers.

Definition 1. *Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with the property that $\exists n_0\in\mathbb{N}$ such that $b_n\neq 0,\forall n\geq n_0$. We will say that the sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ is* equivalent *with $(b_n)_{n\in\mathbb{N}}$ and we write $a_n\sim b_n$ if and only if $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=1$.*

From the well-known result that if a sequence of real numbers is convergent then every subsequence is convergent and has the same limit, we get:

Proposition 2. *If $a_n\sim b_n$ then $a_{2n+1}\sim b_{2n+1}$.*

We need the following two results.

Proposition 3. *Let $f:(0,\infty)\to\mathbb{R}$ be a function. The following formula holds*

$$\sum_{i+j\le n;i,j\in\mathbb{N}}f(i+j)=\sum_{k=1}^{n}(k-1)f(k).$$

Proof. We have:

$$\sum_{i+j\le n;i,j\in\mathbb{N}}f(i+j)=\sum_{i=1}^{n-1}\left(\sum_{j=1}^{n-i}f(i+j)\right)=\sum_{j=1}^{n-1}f(1+j)+\sum_{j=1}^{n-2}f(2+j)$$

[1)] Mihai Viteazul Secondary School Nr. 29 Constanţa, Str. Cişmelei, Nr. 13, Constanţa and Department of Mathematics, Ovidius University of Constanţa, Bd. Mamaia 124, 900527 Constanţa, Romania, cidenys@gmail.com

$$+\cdots+\sum_{j=1}^{n-(n-1)} f\left((n-1)+j\right) = f(2)+2f(3)+\cdots+(n-1)\,f(n)$$

$$=\sum_{k=1}^{n}(k-1)\,f(k)\,.$$

□

Proposition 4. *Let $f:(0,\infty)\to\mathbb{R}$ be a function. The following formula holds*

$$\sum_{i,j=1}^{n} f(i+j)=\sum_{k=1}^{n}(k-1)\,f(k)+\sum_{k=n+1}^{2n+1}(2n+1-k)\,f(k)\,.$$

Proof. We have

$$\sum_{i,j=1}^{n} f(i+j)=\sum_{i=1}^{n}\left(\sum_{j=1}^{n} f(i+j)\right)=\sum_{j=1}^{n} f(1+j)+\sum_{j=1}^{n} f(2+j)+\cdots$$

$$+\sum_{j=1}^{n} f(n+j)=f(2)+2f(3)+\cdots+(n-1)\,f(n)+nf(n+1)$$

$$+(n-1)f(n+2)+\cdots+f(n+n)=\sum_{k=1}^{n}(k-1)f(k)+\sum_{k=0}^{n}(n-k)\,f(n+k+1)\,.$$

Changing $n+k+1=i$, we get

$$\sum_{k=0}^{n}(n-k)\,f(n+k+1)=\sum_{i=n+1}^{2n+1}(2n+1-i)\,f(i)$$

and so

$$\sum_{i,j=1}^{n} f(i+j)=\sum_{k=1}^{n}(k-1)\,f(k)+\sum_{k=n+1}^{2n+1}(2n+1-k)\,f(k)\,.$$

□

We recall two well-known results. Their proofs can be found, for example, in [7].

The Glaisher-Kinkelin theorem. *The sequence $(x_n)_{n\in\mathbb{N}}$ where*

$$x_n=\frac{1^1\cdot 2^2\cdots n^n}{n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}\cdot e^{-\frac{n^2}{4}}}$$

is convergent and its limit $\lim\limits_{n\to\infty} x_n = g_k$ is called the Glaisher-Kinkelin constant.

The Stirling formula. *The following formula holds*

$$\prod_{i=1}^{n} i \sim \sqrt{2\pi n} \cdot \frac{n^n}{e^n}.$$

We will use these results to obtain the two Stirling type formulas stated in the Abstract.

2. The main results

Proposition 5. *The following formula holds*

$$\prod_{\substack{i+j\le n;\\ i,j\in\mathbb{N}}} (i+j) \sim \frac{g_k}{\sqrt{2\pi}} \cdot \frac{n^{\frac{n^2}{2}-\frac{n}{2}-\frac{5}{12}}}{e^{\frac{n^2}{4}-n}}.$$

Proof. Using Proposition 3, we get

$$\ln \prod_{\substack{i+j\le n;\\ i,j\in\mathbb{N}}} (i+j) = \sum_{\substack{i+j\le n;\\ i,j\in\mathbb{N}}} \ln(i+j) = \sum_{k=1}^{n} (k-1)\ln k$$

$$= \sum_{k=1}^{n} k\ln k - \sum_{k=1}^{n} \ln k = \ln \prod_{k=1}^{n} k^k - \ln \prod_{k=1}^{n} k = \ln \frac{\prod_{k=1}^{n} k^k}{\prod_{k=1}^{n} k}.$$

Thus, $\displaystyle\prod_{\substack{i+j\le n;\\ i,j\in\mathbb{N}}} (i+j) = \frac{\prod_{k=1}^{n} k^k}{\prod_{k=1}^{n} k}.$

From Glaisher-Kinkelin's theorem and Stirling's formula we obtain

$$\prod_{i+j\le n; i,j\in\mathbb{N}} (i+j) \sim \frac{g_k \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \cdot e^{-\frac{n^2}{4}}}{\sqrt{2\pi} \cdot \frac{n^n}{e^n} \cdot \sqrt{n}} = \frac{g_k}{\sqrt{2\pi}} \cdot \frac{n^{\frac{n^2}{2}-\frac{n}{2}-\frac{5}{12}}}{e^{\frac{n^2}{4}-n}}.$$

□

Proposition 6. *The following formula holds*

$$\prod_{k=n+1}^{2n+1} k \sim 2\sqrt{2} \cdot \frac{4^n \cdot n^{n+1}}{e^n}.$$

Proof. Using Proposition 2 in Stirling's formula, we will have

$$\prod_{k=1}^{2n+1} k \sim \sqrt{2\pi} \cdot \frac{(2n+1)^{2n+1}}{e^{2n+1}} \cdot \sqrt{2n+1}.$$

From the equality $\prod\limits_{k=n+1}^{2n+1} k = \dfrac{\prod\limits_{k=1}^{2n+1} k}{\prod\limits_{k=1}^{n} k}$ and Stirling's formula we obtain

$$\prod_{k=n+1}^{2n+1} k \sim \frac{\sqrt{2\pi} \cdot \dfrac{(2n+1)^{2n+1}}{e^{2n+1}} \cdot \sqrt{2n+1}}{\sqrt{2\pi} \cdot \dfrac{n^n}{e^n} \cdot \sqrt{n}} = \frac{(2n+1)^{2n+1} \cdot \sqrt{2 + \dfrac{1}{n}}}{n^n \cdot e^{n+1}}$$

$$= \frac{(2n)^{2n+1} \cdot \left(1 + \dfrac{1}{2n}\right)^{2n+1} \cdot \sqrt{2 + \dfrac{1}{n}}}{n^n \cdot e^{n+1}}.$$

Since $\left(1 + \frac{1}{2n}\right)^{2n+1} \sim e$ and $\sqrt{2 + \frac{1}{n}} \sim \sqrt{2}$, we get

$$\prod_{k=n+1}^{2n+1} k \sim 2\sqrt{2} \cdot \frac{4^n \cdot n^{n+1}}{e^n}.$$

□

Proposition 7. *The following formula holds* $\left(1 + \frac{1}{2n}\right)^{\frac{(2n+1)^2}{2}} \sim \sqrt[4]{e^3} \cdot e^n$.

Proof. Indeed, we have

$$\lim_{n\to\infty} \frac{\left(1 + \dfrac{1}{2n}\right)^{\frac{(2n+1)^2}{2}}}{e^{\frac{(2n+1)^2}{4n}}} = \lim_{n\to\infty} \left[\frac{\left(1 + \dfrac{1}{2n}\right)^{2n}}{e} \right]^{\dfrac{(2n+1)^2}{4n}}$$

$$= \lim_{n\to\infty} \left[1 + \left(\frac{\left(1 + \dfrac{1}{2n}\right)^{2n}}{e} - 1 \right) \right]^{\frac{(2n+1)^2}{4n}} = e^{\lim\limits_{n\to\infty} \left(\frac{\left(1+\frac{1}{2n}\right)^{2n}}{e} - 1 \right) \dfrac{(2n+1)^2}{4n}}.$$

Also,

$$\lim_{n\to\infty} \left(\frac{\left(1 + \frac{1}{2n}\right)^{2n}}{e} - 1 \right) \frac{(2n+1)^2}{4n} = \lim_{n\to\infty} n \left(\frac{\left(1 + \frac{1}{2n}\right)^{2n}}{e} - 1 \right) \frac{(2n+1)^2}{4n^2}$$

$$= \lim_{n\to\infty} n\left(\frac{\left(1+\frac{1}{2n}\right)^{2n}}{e} - 1\right).$$

From the L'Hospital rule we have

$$\lim_{x\to 0} \frac{1}{2x}\left(\frac{(1+x)^{\frac{1}{x}}}{e} - 1\right) = \frac{1}{2}\lim_{x\to 0}\frac{e^{\frac{\ln(1+x)}{x}-1}-1}{x}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{e^{\frac{\ln(1+x)}{x}-1}-1}{\frac{\ln(1+x)}{x}-1}\cdot\frac{\frac{\ln(1+x)}{x}-1}{x} = \frac{1}{2}\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{\dfrac{1}{1+x}-1}{2x} = -\frac{1}{4}.$$

From the characterization of the limit of a function at a point with sequences, we deduce that $\lim\limits_{n\to\infty}\frac{\left(1+\frac{1}{2n}\right)^{\frac{(2n+1)^2}{2}}}{e^{\frac{(2n+1)^2}{4n}}} = e^{-\frac{1}{4}}$, so that

$$\left(1+\frac{1}{2n}\right)^{\frac{(2n+1)^2}{2}} \sim e^{\frac{(2n+1)^2}{4n}-\frac{1}{4}} = e^{n+\frac{3}{4}+\frac{1}{4n}}.$$

Since $e^{\frac{1}{4n}} \sim 1$, we obtain $\left(1+\frac{1}{2n}\right)^{\frac{(2n+1)^2}{2}} \sim \sqrt[4]{e^3}\cdot e^n$. □

Proposition 8. *The following formula holds*

$$\prod_{k=n+1}^{2n+1} k^k \sim 2e\sqrt[12]{2}\cdot\frac{2^{2n^2+3n}\cdot n^{\frac{3n^2}{2}+\frac{5n}{2}+1}}{e^{\frac{3n^2}{4}}}.$$

Proof. Using Proposition 2 in Glaisher-Kinkelin formula, we have

$$\prod_{k=1}^{2n+1} k^k \sim g_k\cdot(2n+1)^{\frac{(2n+1)^2}{2}+\frac{2n+1}{2}+\frac{1}{12}}\cdot e^{-\frac{(2n+1)^2}{4}}.$$

Then from the equality $\prod\limits_{k=n+1}^{2n+1} k^k = \frac{\prod_{k=1}^{2n+1} k^k}{\prod_{k=1}^{n} k^k}$ we deduce that

$$\prod_{k=n+1}^{2n+1} k^k \sim \frac{g_k\cdot(2n+1)^{\frac{(2n+1)^2}{2}+\frac{2n+1}{2}+\frac{1}{12}}\cdot e^{-\frac{(2n+1)^2}{4}}}{g_k\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}\cdot e^{-\frac{n^2}{4}}} = \frac{(2n+1)^{\frac{(2n+1)^2}{2}+\frac{2n+1}{2}+\frac{1}{12}}}{n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}\cdot e^{\frac{(n+1)(3n+1)}{4}}}$$

$$= \frac{(2n)^{\frac{(2n+1)^2}{2}+\frac{2n+1}{2}+\frac{1}{12}}\cdot\left(1+\frac{1}{2n}\right)^{\frac{(2n+1)^2}{2}}\cdot\left(1+\frac{1}{2n}\right)^{n+\frac{7}{12}}}{n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}\cdot e^{\frac{(n+1)(3n+1)}{4}}}$$

$$= \frac{2^{2n^2+3n+\frac{13}{12}} \cdot n^{\frac{3n^2}{2}+\frac{5n}{2}+1} \cdot \left(1+\frac{1}{2n}\right)^{\frac{(2n+1)^2}{2}} \cdot \left(1+\frac{1}{2n}\right)^{n+\frac{7}{12}}}{e^{\frac{(n+1)(3n+1)}{4}}}.$$

From Proposition 7 and $\left(1+\frac{1}{2n}\right)^{n+\frac{7}{12}} \sim \sqrt{e}$ we obtain

$$\prod_{k=n+1}^{2n+1} k^k \sim \frac{2^{2n^2+3n+\frac{13}{12}} \cdot n^{\frac{3n^2}{2}+\frac{5n}{2}+1} \cdot \sqrt[4]{e^3} \cdot e^n \cdot \sqrt{e}}{e^{\frac{(n+1)(3n+1)}{4}}}$$

$$= 2e\sqrt[12]{2} \cdot \frac{2^{2n^2+3n} \cdot n^{\frac{3n^2}{2}+\frac{5n}{2}+1}}{e^{\frac{3n^2}{4}}}.$$

□

Proposition 9. *The following formula holds* $\left(1+\frac{1}{2n}\right)^{4n^2+5n+\frac{3}{2}} \sim e^2 \cdot e^{2n}$.

Proof. We have

$$\lim_{n\to\infty} \frac{\left(1+\frac{1}{2n}\right)^{4n^2+5n+\frac{3}{2}}}{e^{2n+2}} = \lim_{n\to\infty} \left[\left(\frac{\left(1+\frac{1}{2n}\right)^{\frac{(2n+1)^2}{2}}}{\sqrt[4]{e^3} \cdot e^n}\right)^2 \cdot \frac{\left(1+\frac{1}{2n}\right)^{n+1}}{\sqrt{e}}\right].$$

By Proposition 7,

$$\left(1+\frac{1}{2n}\right)^{\frac{(2n+1)^2}{2}} \sim \sqrt[4]{e^3} \cdot e^n \text{ and so: } \lim_{n\to\infty} \left[\frac{\left(1+\frac{1}{2n}\right)^{\frac{(2n+1)^2}{2}}}{\sqrt[4]{e^3} \cdot e^n}\right]^2 = 1,$$

and since $\lim_{n\to\infty} \frac{\left(1+\frac{1}{2n}\right)^{n+1}}{\sqrt{e}} = 1$ we will obtain that $\lim_{n\to\infty} \frac{\left(1+\frac{1}{2n}\right)^{4n^2+5n+\frac{3}{2}}}{e^{2n+2}} = 1$.

□

Proposition 10. *The following formula holds*

$$\left(\prod_{k=n+1}^{2n+1} k\right)^{2n+1} \sim \frac{2e\sqrt{2}}{\sqrt[12]{e}} \cdot \frac{2^{4n^2+5n} \cdot n^{2n^2+3n+1}}{e^{2n^2+n}}.$$

Proof. From Stirling's Theorem, see for example [7, Theorem 5], we have

$$\prod_{k=1}^{n} k = \sqrt{2\pi n} \cdot \frac{n^n}{e^n} \cdot e^{\frac{1}{12n}} \cdot e^{R_n},$$

where $|R_n| \leq \frac{\sqrt{3}}{216n^2}, \forall n \in \mathbb{N}$. Changing n to $2n+1$, we get

$$\prod_{k=1}^{2n+1} k = \sqrt{2\pi\,(2n+1)} \cdot \frac{(2n+1)^{2n+1}}{e^{2n+1}} \cdot e^{\frac{1}{24n+12}} \cdot e^{R_{2n+1}},$$

where $|R_{2n+1}| \leq \frac{\sqrt{3}}{216(2n+1)^2}, \forall n \in \mathbb{N}$. Using these relations, we will obtain

$$\prod_{k=n+1}^{2n+1} k = \frac{\prod\limits_{k=1}^{2n+1} k}{\prod\limits_{k=1}^{n} k} = \frac{\sqrt{2\pi\,(2n+1)} \cdot \frac{(2n+1)^{2n+1}}{e^{2n+1}} \cdot e^{\frac{1}{12(2n+1)}} \cdot e^{R_{2n+1}}}{\sqrt{2\pi n} \cdot \frac{n^n}{e^n} \cdot e^{\frac{1}{12n}} \cdot e^{R_n}}$$

$$= \frac{(2n+1)^{2n+\frac{3}{2}} \cdot n^{-\left(n+\frac{1}{2}\right)}}{e^{n+1} \cdot e^{\frac{1}{12n}-\frac{1}{12(2n+1)}}} \cdot \alpha_n = \frac{(2n)^{2n+\frac{3}{2}} \left(1+\frac{1}{2n}\right)^{2n+\frac{3}{2}} \cdot n^{-\left(n+\frac{1}{2}\right)}}{e^{n+1} \cdot e^{\frac{1}{12n}-\frac{1}{12(2n+1)}}} \cdot \alpha_n$$

$$= \frac{2^{2n+\frac{3}{2}} \cdot \left(1+\frac{1}{2n}\right)^{2n+\frac{3}{2}} \cdot n^{n+1}}{e^{n+1} \cdot e^{\frac{n+1}{12n(2n+1)}}} \cdot \alpha_n,$$

where $\alpha_n = e^{R_{2n+1}-R_n}$. Then we deduce that

$$\left(\prod_{k=n+1}^{2n+1} k\right)^{2n+1} = \frac{2^{\left(2n+\frac{3}{2}\right)(2n+1)} \cdot \left(1+\frac{1}{2n}\right)^{4n^2+5n+\frac{3}{2}} \cdot n^{(n+1)(2n+1)}}{e^{(n+1)(2n+1)} \cdot e^{\frac{n+1}{12n}}} \cdot \beta_n,$$

where $\beta_n = \alpha_n^{2n+1} = e^{(2n+1)(R_{2n+1}-R_n)}$. Since $|R_n| \leq \frac{\sqrt{3}}{216n^2}$, we will have $|R_n| \cdot (2n+1) \leq \frac{\sqrt{3}(2n+1)}{216n^2}$, and when $n \to \infty$, we have that $R_n \cdot (2n+1) \to 0$. In the same way, $R_{2n+1} \cdot (2n+1) \to 0$, so $\beta_n \sim 1$. Also, $e^{\frac{n+1}{12n}} \sim e^{\frac{1}{12}}$. By using all that and Proposition 9, we get

$$\left(\prod_{k=n+1}^{2n+1} k\right)^{2n+1} \sim \frac{2^{\left(2n+\frac{3}{2}\right)(2n+1)} \cdot e^{2n+2} \cdot n^{(n+1)(2n+1)}}{e^{(n+1)(2n+1)} \cdot e^{\frac{1}{12}}}$$

$$= \frac{2e\sqrt{2}}{\sqrt[12]{e}} \cdot \frac{2^{4n^2+5n} \cdot n^{2n^2+3n+1}}{e^{2n^2+n}}.$$

□

Proposition 11. *The following formula holds*

$$\prod_{i,j=1}^{n} (i+j) \sim \frac{g_k}{\sqrt[12]{2e\pi^6}} \cdot \frac{n^{n^2-\frac{5}{12}} \cdot 4^{n^2+n}}{e^{\frac{3n^2}{2}}}.$$

Proof. Using Proposition 4 we obtain $\ln \prod_{i,j=1}^{n} (i+j) = \sum_{i,j=1}^{n} \ln(i+j) =$

$$= \sum_{k=1}^{n} (k-1)\ln k + \sum_{k=n+1}^{2n+1} (2n+1-k)\ln k = \ln \frac{\left(\prod_{k=1}^{n} k^k\right) \cdot \left(\prod_{k=n+1}^{2n+1} k\right)^{2n+1}}{\left(\prod_{k=1}^{n} k\right) \cdot \left(\prod_{k=n+1}^{2n+1} k^k\right)}$$

and hence $\prod_{i,j=1}^{n} (i+j) = \frac{\left(\prod_{k=1}^{n} k^k\right) \cdot \left(\prod_{k=n+1}^{2n+1} k\right)^{2n+1}}{\left(\prod_{k=1}^{n} k\right) \cdot \left(\prod_{k=n+1}^{2n+1} k^k\right)}$.

From Glaisher-Kinkelin, Stirling, Propositions 8 and 10, we obtain

$$\prod_{i,j=1}^{n} (i+j) \sim \frac{\left(g_k \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \cdot e^{-\frac{n^2}{4}}\right) \cdot \left(\frac{2e\sqrt{2}}{\sqrt[12]{e}} \cdot \frac{2^{4n^2+5n} \cdot n^{2n^2+3n+1}}{e^{2n^2+n}}\right)}{\left(\sqrt{2\pi n} \cdot \frac{n^n}{e^n}\right) \cdot \left(2e\,\sqrt[12]{2} \cdot \frac{2^{2n^2+3n} \cdot n^{\frac{3n^2}{2}+\frac{5n}{2}+1}}{e^{\frac{3n^2}{4}}}\right)},$$

so

$$\prod_{i,j=1}^{n} (i+j) \sim \frac{g_k}{\sqrt[12]{2e\pi^6}} \cdot \frac{n^{n^2-\frac{5}{12}} \cdot 4^{n^2+n}}{e^{\frac{3n^2}{2}}}.$$

□

Acknowledgments. The author thanks Prof. Univ. Dr. Dumitru Popa for his helpful suggestions that make possible to obtain these formulas.

References

[1] Gh. Gussi, O. Stănăşilă, T. Stoica, *Matematică, Manual pentru clasa a XI-a*, Editura Didactică şi Pedagogică, Bucureşti, 1996.
[2] D. Popa, *Exerciţii de analiză matematică*, Biblioteca Societăţii de Ştiinţe Matematice din România, Editura Mira, Bucureşti 2007.
[3] D. Popa, The Euler-Maclaurin summation formula for functions of class C^3, *Gazeta Matematică Seria A*, Nr. 3–4, 2016, 1–12.

Some generalizations and refinements of the RHS of Gerretsen inequality

MARIUS DRĂGAN[1)], NECULAI STANCIU[2)]

Abstract. This paper presents some generalizations of the RHS of the Gerretsen inequality.

Keywords: Blundon inequality, Gerretsen inequality, geometric inequalities, the best constant.

MSC: Primary 51M16; Secondary 26D05.

In [5] appears the inequality $a^2 + b^2 + c^2 \leq 8R^2 + 4r^2$ due to J.C. Gerretsen.

In [6] L. Panaitopol proves that the inequality of Gerretsen is the best if we suppose that

$$a^2 + b^2 + c^2 \leq \alpha R^2 + \beta Rr + \gamma r^2,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta = 0$.

In [7] the same result as in [6] is shown for $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta \geq 0$.

In [8] R.A. Satnoianu gave the following generalization of Gerretsen inequality

$$a^n + b^n + c^n \leq 2^{n+1}R^n + 2^n\left(3^{1+\frac{n}{2}} - 2^{n+1}\right)r^n, \text{ for any } n \geq 0.$$

The purpose of this article is to give a proof of Satnoianu's inequality in the case $n = 6$, i.e.,

$$a^6 + b^6 + c^6 \leq 128 \cdot R^6 - 3008 \cdot r^6,$$

then we prove that this inequality is the best if we suppose that

$$a^6 + b^6 + c^6 \leq \alpha_1 R^6 + \alpha_2 R^5 r + \cdots + a_6 Rr^5 + \alpha_7 r^6,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_7 \in \mathbb{R}$ and $\alpha_2, \alpha_3, \ldots, \alpha_7 \geq 0$.

Also we give some refinements for this inequality and we shall prove these refinements are the best of their type.

Theorem 1 (fundamental inequality of Blundon's inequality). *For any triangle ABC the inequalities $s_1 \leq s \leq s_2$ hold, where s_1, s_2 represent the semiperimeter of two isosceles triangles $A_1B_1C_1$ and $A_2B_2C_2$, which have the same circumradius R and inradius r as the triangle ABC and the sides*

$$a_1 = 2\sqrt{(R + r - d)(R - r + d)},\ b_1 = c_1 = \sqrt{2R(R + r - d)},$$

$$a_2 = 2\sqrt{(R + r + d)(R - r - d)},\ b_2 = c_2 = \sqrt{2R(R + r + d)},$$

where $d = \sqrt{R^2 - 2Rr}$.

A proof of this theorem is given in [3].

1) Mircea cel Bătrân Highschool, Bucureşti, Romania,

2) G.E. Palade Secondary School, Buzău, Romania, marius.dragan2005@yahoo.com

Theorem 2 (some minimal and maximal bounds for the sum $a^6+b^6+c^6$)**.** *In any triangle ABC is true the double inequality*

$$2^4(R+r-d)^3\left[4(R-r+d)^3+R^3\right] \le a^6+b^6+c^6 \le 2^4(R+r+d)^3\left[4(R-r-d)^3+R^3\right]. \tag{1}$$

Proof. If we replace $x=a^2$, $y=b^2$, $z=c^2$ in the identity

$$\sum_{\text{cyc}} x^3 = 3xyz + \sum_{\text{cyc}} x\left(\sum_{\text{cyc}} x^2 - \sum_{\text{cyc}} yz\right),$$

then we obtain

$$\sum_{\text{cyc}} a^6 = 3a^2b^2c^2 + \sum_{\text{cyc}} a^2\left(\sum_{\text{cyc}} a^4 - \sum_{\text{cyc}} b^2c^2\right).$$

We consider the functions

$f,g,h,F:[s_1,s_2]\to\mathbb{R},\ f(s)=3a^2b^2c^2=3(4Rrs)^2,\ g(s)=2(s^2-r^2-4Rr),$

$$\begin{aligned} h(s) &= \sum_{\text{cyc}} a^4 - \sum_{\text{cyc}} (bc)^2 \\ &= \left(\sum_{\text{cyc}} a\right)^4 - 4\sum_{\text{cyc}} ab\left(\sum_{\text{cyc}} a\right)^2 + 6abc\sum_{\text{cyc}} a + \left(\sum_{\text{cyc}} ab\right)^2 \\ &= s^4-2r(4R+7r)s^2+(4R+r)^2r^2, \\ F(s) &= f(s)+g(s)h(s). \end{aligned}$$

Since $h'(s)=4s(s^2-4Rr-7r^2)$ and $s^2\ge s_1^2\ge 16Rr-5r^2>4Rr+7r^2$, it results that h is increasing on $[s_1,s_2]$.

Also f and g are increasing on $[s_1,s_2]$. Hence, F is increasing on $[s_1,s_2]$, and then $F(s_1)\le F(s)\le F(s_2)$ or

$$a_1^6+b_1^6+c_1^6 \le a^6+b^6+c^6 \le a_2^6+b_2^6+c_2^6. \tag{2}$$

If we replace a_1,b_1,c_1,a_2,b_2,c_2 from Theorem 1 in (2) we obtain (1). □

Theorem 3 (some maximal bound for the sum $a^6+b^6+c^6$)**.** *In any triangle ABC holds the inequality*

$$a^6+b^6+c^6 \le 128R^6-3008r^6. \tag{3}$$

Proof. From (2) it follows that to prove (3) it will be sufficient to prove that

$$16(R+r+d)^3\left[4(R-r-d)^3+R^3\right] \le 128R^6-3008r^6. \tag{4}$$

If we put $x=\dfrac{R}{r}$, then the inequality (4) can be written as

$$16\left(x+1+\sqrt{x^2-2x}\right)^3\left[4\left(x-1-\sqrt{x^2-2x}\right)^3+x^3\right] \le 128x^6-3008, \forall\, x\ge 2,$$

which is successively equivalent to

$$(64x^5+64x^4+48x^3-2048x^2+2560x-384)\sqrt{x(x-2)}$$
$$\le 64x^6+48x^5-2064x^3+4068x^2-1920x-2944,$$

$$16(x-2)(4x^4+12x^3+27x^2-74x+12)\sqrt{x(x-2)}$$
$$\le 16(x-2)(4x^5+8x^4+19x^3-91x^2+106x+92),$$

$$(4x^4+12x^3+27x^2-74x+12)\sqrt{x(x-2)}$$
$$\le 4x^5+8x^4+19x^3-91x^2+106x+92,$$

$$48x^8+240x^7+816x^6+780x^5+241x^4-1772x^3-9204x^2+1972x+8464\ge 0,$$

$\forall\, x\ge 2$, which is true since

$$816x^6-9204x^2=816x^2\left(x^4-\frac{9204}{816}\right)\ge 816x^2(x^4-16)\ge 0$$

and

$$780x^5-1772x^3=780x^3\left(x^2-\frac{1772}{780}\right)\ge 780x^3(x^2-4)\ge 0,\ \forall\, x\ge 2.$$

□

Theorem 4 (the best maximal bound for the sum $a^6+b^6+c^6$ is $128R^6-3008r^6$)**.** *If $\alpha_1,\alpha_2,\ldots,\alpha_7\in\mathbb{R}$ and $\alpha_2,\alpha_3,\ldots,\alpha_7\ge 0$ with the property that the inequality* (5) *is true in every triangle ABC*

$$a^6+b^6+c^6\le \alpha_1R^6+\alpha_2R^5r+\alpha_3R^4r^2+\alpha_4R^3r^3+\alpha_5R^2r^4+\alpha_6Rr^5+\alpha_7r^6, \tag{5}$$

then we have the inequality

$$\alpha_1R^6+\alpha_2R^5r+\alpha_3R^4r^2+\alpha_4R^3r^3+\alpha_5R^2r^4+\alpha_6Rr^5+\alpha_7r^6\ge 128R^6-3008r^6$$

in any triangle ABC.

Proof. In the case of equilateral triangle, from (5) we have that

$$2^6\alpha_1+2^5\alpha_2+\cdots+2\alpha_6+\alpha_7\ge 72^2. \tag{6}$$

If we consider the case of isosceles triangle ABC, $b=c=1$, $a=0$, $R=\dfrac{1}{2}$, $r=0$, then by (5) we deduce

$$\alpha_1\ge 128. \tag{7}$$

Taking into account $R\ge 2r$, (6) and (7), we successively obtain that

$$(\alpha_1-128)R^6+\alpha_2R^5r+\alpha_3R^4r^2+\alpha_4R^3r^3+\alpha_5R^2r^4+\alpha_6Rr^5+(\alpha_7+3008)r^6$$
$$\ge [(\alpha_1-128)2^6+\alpha_2\cdot 2^5+\alpha_3\cdot 2^4+\alpha_4\cdot 2^3+\alpha_5\cdot 2^2+\alpha_6\cdot 2+\alpha_7+3008]r^6\ge 0,$$

or

$$\alpha_1 R^6 + \alpha_2 R^5 r + \alpha_3 R^4 r^2 + \alpha_4 R^3 r^3 + \alpha_5 R^2 r^4 + \alpha_6 R r^5 + \alpha_7 r^6 \geq 128R^6 - 3008r^6.$$

□

Theorem 5 (the best constant for certain Gerretsen type inequality)**.** *The best real constant k such that the inequality*

$$a^6 + b^6 + c^6 \leq 128R^6 + kRr^5 - (2k + 3008)r^6 \tag{8}$$

is true in every triangle ABC is $k_1 \cong -5269.275$.

Proof. Inequality (1) yields that if (8) is true in any triangle ABC, then

$$16(R + r + d)^3[4(R - r - d)^3 + R^3] \leq 128R^6 + kRr^5 - (2k + 3008)r^6 \tag{9}$$

is true in any triangle ABC. The inequality (9) is successively equivalent to

$$16(R + r + d)^3[4(R - r - d)^3 + R^3] - 128R^6 + 3008r^6 \leq kr^5(R - 2r),$$

$$\frac{1}{x-2}\left[16(x + 1 + \sqrt{x^2 - 2x})^3[4(x - 1 - \sqrt{x^2 - 2x})^3 + x^3] - 128x^6 + 3008\right] \leq k,$$

$\forall\, x \geq 2$,

$$16(-4x^5 - 8x^4 - 19x^3 + 91x^2 - 106x - 92)$$
$$+16\sqrt{x^2 - 2x}\,(4x^4 + 12x^3 + 27x^2 - 74x + 12) \leq k,$$

so the best constant is the maximum of the function $f_1 : [2, \infty) \to \mathbb{R}$,

$$f_1(x) = 16(-4x^5 - 8x^4 - 19x^3 + 19x^2 - 106x - 92)$$
$$+ 16\sqrt{x^2 - 2x}\,(4x^4 + 12x^3 + 27x^2 - 74x + 12),$$

i.e., $k_1 = \max_{x \geq 2} f_1(x) \cong -5269.275$. □

Remark 1. The best integer constant for which the inequality (8) is true in every triangle ABC is $k_1' = -5269$. Hence, in any triangle ABC holds the following inequality

$$a^6 + b^6 + c^6 \leq 128R^6 - 5269Rr^5 + 7530r^6.$$

Theorem 6. *In any triangle ABC holds the following inequality*

$$a^6 + b^6 + c^6 \leq 128R^6 + kRr^5 - (2k + 3008)r^6, \textit{ for each } k \in [k_1, \infty),$$

where k_1 represents the best constant for the inequality (8).

Proof. If we consider the increasing function
$F : [k_1, \infty) \to \mathbb{R}$, $F(k) = 128R^6 - 3008r^6 + kr^5(R - 2r)$, then by (8) we infer that

$$a^6 + b^6 + c^6 \leq F(k_1) \leq F(k) \tag{10}$$

for any $k \geq k_1$. □

Remark 2. If we take $k = 0$, then by (10) we obtain $a^6 + b^6 + c^6 \le F(k_1)$, i.e., a refinement of inequality (3).

A generalization of Theorems 5 and 6. *For $n \in \{1, 2, 3, 4, 5\}$, find the best real constant k such that the inequality*

$$a^6 + b^6 + c^6 \le 128R^6 + kR^n r^{6-n} - (2^n k + 3008)r^6 \tag{11}$$

is true in any triangle ABC.

Proof. By (1) we have that if the inequality (11) is true in any triangle ABC, then

$$16(R+r+d)^3[4(R-r-d)^3+R^3] \le 128R^6+kR^n r^{6-n}-(2^n k+3008)r^6 \tag{12}$$

is true in any triangle ABC. If we denote $x = \dfrac{R}{r}$, then the inequality (12) becomes successively

$$\frac{16(x+1+\sqrt{x^2-2x})^3[4(x-1-\sqrt{x^2-2x})^3+x^3]-128x^6+3008}{x^n-2^n} \le k, \forall\, x \ge 2,$$

or

$$\frac{16(-4x^5-8x^4-19x^3+91x^2-1-106x-92)+(4x^4+12x^3+27x^2-74x+12)16\sqrt{x^2-2x}}{x^{n-1}+x^{n-2}\cdot 2+\cdots+x\cdot 2^{n-2}+2^{n-1}} \le k.$$

So, the best constant is the maximum of the functions $f_n : [2,\infty) \to \mathbb{R}$,

$$f_n(x)=\frac{16(-4x^5-8x^4-19x^3+91x^2-106x-92)+16(4x^4+12x^3+27x^2-74x+12)\sqrt{x^2-2x}}{x^{n-1}+x^{n-2}\cdot 2+\cdots+x\cdot 2^{n-2}+2^{n-1}}.$$

If $n = 1$, then we find the best constant from Theorem 5.
If $n = 2$, then (using Wolfram Alpha) we find $k_2 \cong -1297.57$.
If $n = 3$, then (using Wolfram Alpha) we find $k_3 \cong -419.402$.
If $n = 4$, then (using Wolfram Alpha) we find $k_4 \cong -96$.
If $n = 5$, then (using Wolfram Alpha) we find $k_5 \cong 0$. □

Remarks.

1) The best integer constant for (11) in the case $n = 2$ is $k_2 = -1297$, so the inequality

$$a^6 + b^6 + c^6 \le 128R^6 - 1297R^2r^4 + 2180r^6 \tag{13}$$

is true in any triangle ABC;

2) The best integer constant for (11) in the case $n = 3$ is $k_3 = -419$, therefore the inequality

$$a^6 + b^6 + c^6 \le 128R^6 - 419R^3r^3 + 344r^6 \tag{14}$$

is true in any triangle ABC;

3) The best integer constant for (11) in the case $n = 4$ is $k_4 = -96$, thus the inequality

$$a^6 + b^6 + c^6 \le 128R^6 - 96R^4r^3 - 1472r^6 \tag{15}$$

is true in any triangle ABC;

4) Also we proved that the inequality

$$a^6 + b^6 + c^6 \leq 128R^6 - 5269Rr^5 + 7530r^6 \tag{16}$$

is true in any triangle ABC;

5) If we compare the inequalities (13), (14), (15) and (16) we observe that there are not two of them with right-hand side $u(R, r)$ and $v(R, r)$, respectively, such that $u(R, r) \leq v(R, r)$ for each $\dfrac{R}{r} \geq 2$.

Theorem 7. *If $n \in \{1, 2, 3, 4, 5\}$, then in any triangle ABC holds the following inequality*

$$a^6 + b^6 + c^6 \leq 128R^6 + kR^n r^{6-n} - (2^n k + 3008)r^6 \text{ for any } k \in [k_n, \infty),$$

where k_n represents the best constant for (10).

Proof. We consider the function $G : [k_n, \infty) \to \mathbb{R}$,

$$G(k) = 128R^6 - 3008r^6 + kr^{6-n}(R^n - 2^n r^n).$$

Since G is increasing function, from (8) we have that

$$a^6 + b^6 + c^6 \leq G(k_n) \leq G(k) \text{ for any } k \geq k_n.$$

□

Theorem 8. *If $n \in \{1, 2, 3, 4, 5\}$, $\alpha_1, \alpha_2, \ldots, \alpha_7 \in \mathbb{R}$,*

$$\{i_1, i_2, i_3, i_4\} = \{2, 3, 4, 5, 6\} \setminus \{6 - n + 1\}$$

such that $\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}, \alpha_{i_4} \geq 0$, $\alpha_{6-n+1} \geq k_n$, with the property

$$a^6+b^6+c^6 \leq \alpha_1 R^6+\alpha_2 R^5 r+\alpha_3 R^4 r^2+\alpha_4 R^3 r^3+\alpha_5 R^2 r^4+\alpha_6 Rr^5+\alpha_7 r^6, \tag{17}$$

then we have that the inequality

$$128R^6+k_n R^n r^{6-n}-(2^n k_n+3008)r^6 \leq \alpha_1 R^6+\alpha_2 R^5 r+\cdots+\alpha_6 Rr^5+\alpha_7 r^6 \tag{18}$$

is true in any triangle ABC, where k_n represents the best constant for (11).

Proof. In the case of equilateral triangle from (17) we get

$$\alpha_1 R^6 + \alpha_2 R^5 r + \alpha_3 R^4 r^2 + \alpha_4 R^3 r^3 + \alpha_5 R^2 r^4 + \alpha_6 Rr^5 + \alpha_7 r^6 \geq 72^2. \tag{19}$$

In the case of isosceles triangle with the sides $b = c = 1$, $a = 0$, $R = \dfrac{1}{2}$, $r = 0$, from (17) we have

$$\alpha_1 \geq 128. \tag{20}$$

Taking into account (19) and (20), we obtain that

$$(\alpha_1 - 128)R^6 + \alpha_{i_1}R^{6-i_1+1}r^{i_1-1} + \alpha_{i_2}R^{6-i_2+1}r^{i_2-1} + \alpha_{i_3}R^{6-i_3+1}r_{i_3-1}$$
$$+\alpha_{i_4}R^{6-i_4+1}r^{i_4-1} + (\alpha_{6-n+1} - k_n)R^n r^{6-n} + (\alpha_7 + 3008 + 2^n k_n)r^6$$
$$\geq \big[(\alpha_1 - 128)2^6 + \alpha_{i_1} \cdot 2^{6-i_1+1} + \alpha_{i_2} \cdot 2^{6-i_2+1}$$
$$+\alpha_{i_3} \cdot 2^{6-i_3+1} + \alpha_{i_4} \cdot 2^{6-i_4+1} + (\alpha_{6-n+1} - k_n)2^n + \alpha_7 + 3008 + 2^n k_n\big]\, r^6$$
$$= \big[\alpha_1 \cdot 2^6 + \alpha_2 \cdot 2^5 + \alpha_3 \cdot 2^4 + \alpha_4 \cdot 2^3 + \alpha_5 \cdot 2^2 + \alpha_6 \cdot 2 + \alpha_7 - 72^2\big]\, r^6 \geq 0,$$

which yields that

$$\alpha_1 R^6 + \alpha_2 R^5 r + \cdots + \alpha_6 R r^5 + \alpha_7 r^6 \geq 128R^6 + k_n R^n r^{6-n} - (2^n k_n + 3008)r^6.$$

□

REFERENCES

[1] W.J. Blundon, Problem 1935, *The Amer. Math. Monthly* **73**(1966), 1122.
[2] W.J. Blundon, Inequalities associated with triangle, *Canad. Math. Bull.* **8**(1965), 615–626.
[3] M. Drăgan and N. Stanciu, A new proof of the Blundon inequality, *Rec. Mat.* **19**(2017), 100–104.
[4] M. Drăgan, I.V. Maftei, and S. Rădulescu, *Inegalităţi Matematice* (*Extinderi şi generalizări*), E.D.P., 2012.
[5] J.C. Gerretsen, Ongelijkheden in de Driehoek, *Nieuw Tijdschr. Wisk.* **41**(1953), 1–7.
[6] L. Panaitopol, O inegalitate geometrică, *G.M.-B* **87**(1982), 113–115.
[7] S. Rădulescu, M. Drăgan, and I.V. Maftei, Some consequences of W.J. Blundon's inequality, *G.M.-B* **116**(2011), 3–9.
[8] R.A. Satnoianu, General power inequalities between the sides and the circumscribed and inscribed radii related to the fundamental triangle inequality, *Math. Ineq. & Appl.* **5**(2002), 745–751.

Olimpiada de matematică a studenţilor din sud-estul Europei, SEEMOUS 2018[1)]

Gabriel Mincu[2)], Ioana Luca[3)], Cornel Băeţica[4)], Tiberiu Trif[5)]

Abstract. The 12$^{\text{th}}$ South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2018, was hosted by the Gheorghe Asachi Technical University, Iaşi, România, between February 27 and March 4. We present the competition problems and their solutions as given by the corresponding authors. Solutions provided by some of the competing students are also included here.

Keywords: Diagonalizable matrix, rank, change of variable, integrals, series

MSC: Primary 15A03; Secondary 15A21, 26D15.

Introduction

SEEMOUS (South Eastern European Mathematical Olympiad for University Students) este o competiţie anuală de matematică, adresată studenţilor din anii I şi II ai universităţilor din sud-estul Europei. A 12-a ediţie a acestei competiţii a avut loc între 27 februarie şi 4 martie 2018 şi a fost găzduită de către Universitatea Tehnică „Gheorghe Asachi" din Iaşi, România. Au participat 84 de studenţi de la 18 universităţi din Argentina, Bulgaria, FYR Macedonia, Grecia, România, Turkmenistan.

A existat o singură probă de concurs, cu 5 ore ca timp de lucru pentru rezolvarea a patru probleme (problemele 1–4 de mai jos). Acestea au fost selectate de juriu dintre cele 35 de probleme propuse şi au fost considerate ca având diverse grade de dificultate: Problema 1 – grad redus de dificultate, Problemele 2, 3 – dificultate medie, Problema 4 – grad ridicat de dificultate. Pentru studenţi, însă, Problema 3 s-a dovedit a fi cea cu grad ridicat de dificultate.

Au fost acordate 9 medalii de aur, 18 medalii de argint, 29 de medalii de bronz şi o menţiune. Un singur student medaliat cu aur a obţinut punctajul maxim: *Ovidiu Neculai Avădanei* de la Universitatea „Alexandru Ioan Cuza" din Iaşi.

Prezentăm, în continuare, problemele de concurs şi soluţiile acestora, aşa cum au fost indicate de autorii lor. De asemenea, prezentăm şi soluţiile date de către unii studenţi, diferite de soluţiile autorilor.

---

1) http://math.etti.tuiasi.ro/seemous2018/

2) Universitatea din Bucureşti, Bucureşti, România, gamin@fmi.unibuc.ro

3) Universitatea Politehnica Bucureşti, Bucureşti, România, ioana.luca@mathem.pub.ro

4) Universitatea din Bucureşti, Bucureşti, România, cornel.baetica@fmi.unibuc.ro

5) Universitatea Babeş-Bolyai, Cluj-Napoca, Romania, ttrif@math.ubbcluj.ro

Problema 1. Fie $f : [0,1] \to (0,1)$ o funcţie integrabilă Riemann. Arătaţi că

$$\frac{2\displaystyle\int_0^1 xf^2(x)\mathrm{d}x}{\displaystyle\int_0^1 (f(x)^2+1)\mathrm{d}x} < \frac{\displaystyle\int_0^1 f^2(x)\mathrm{d}x}{\displaystyle\int_0^1 f(x)\mathrm{d}x}$$

Vasileios Papadopoulos, Democritus University of Thrace, Grecia

Juriul a considerat că această problemă este simplă. Concurenţii au confirmat în bună măsură această opinie, mulţi dintre ei reuşind să găsească calea spre rezolvare. Juriul a considerat că partea mai delicată a soluţiei este demonstrarea faptului că inegalitatea cerută este strictă, motiv pentru care punctajul maxim pentru această problemă a fost acordat doar acelor studenţi care au argumentat în mod satisfăcător această chestiune. Soluţiile pe care dorim să le propunem diferă în esenţă doar în acest punct. Din acest motiv, în loc să prezentăm mai multe soluţii care coincid la nivelul detaliilor simple, am optat pentru varianta unei singure soluţii, în cadrul căreia chestiunea inegalităţii stricte va fi tranşată într-o lemă pentru care vom prezenta patru demonstraţii, pentru a căror ordonare am ţinut cont de cât de elaborate sunt elementele teoretice utilizate.

Soluţie. Faptul că orice funcţie integrabilă Riemann pe un interval compact ale cărei valori sunt pozitive are integrala pozitivă este o consecinţă imediată a definiţiei integralei Riemann şi se va folosi în mod repetat în cele ce urmează fără vreo menţiune explicită.
Începem prin a proba o lemă care se va dovedi utilă atât pentru a legitima scrierea fracţiilor din enunţ, cât şi pentru a arăta că inegalitatea finală este strictă.

Lemma 1. *Fie $a, b \in \mathbb{R}$, $a < b$, şi $h : [a,b] \to \mathbb{R}$ o funcţie integrabilă Riemann care are toate valorile strict pozitive. Atunci* $\displaystyle\int_a^b h(x)\mathrm{d}x > 0$.

Demonstraţia 1. Presupunem că $\displaystyle\int_a^b h(x)\mathrm{d}x = 0$. Remarcăm că în această situaţie $\displaystyle\int_c^d h(x)\mathrm{d}x = 0$ pentru orice $c, d \in [a,b]$ pentru care $c < d$, căci altminteri am ajunge la contradicţia

$$\int_a^b h(x)\mathrm{d}x = \int_a^c h(x)\mathrm{d}x + \int_c^d h(x)\mathrm{d}x + \int_d^b h(x)\mathrm{d}x > 0.$$

În continuare avem nevoie de

Lemma 2. *Fie $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, $n \in \mathbb{N}^*$ şi $g : [\alpha, \beta] \to \mathbb{R}$ o funcţie integrabilă Riemann care are toate valorile pozitive şi integrala nulă. Atunci* *există intervale închise nedegenerate $I \subset [\alpha, \beta]$ astfel încât $g(x) \leq \frac{1}{n}$ pentru orice $x \in I$.*

Demonstraţia lemei 2. Presupunem contrariul. Atunci orice interval închis nedegenerat inclus în $[\alpha, \beta]$ va conţine elemente x pentru care $g(x) > \frac{1}{n}$. Considerăm şirul de diviziuni $(\Delta_q)_{q \in \mathbb{N}^*}$ ale lui $[\alpha, \beta]$ cu

$$\Delta_q = \left(\alpha, \alpha + \frac{\beta - \alpha}{q}, \alpha + 2\frac{\beta - \alpha}{q}, \ldots, \beta\right)$$

şi sistemele de puncte intermediare $\xi_q = (\xi_{q1}, \xi_{q2}, \ldots, \xi_{qq})$, unde pentru fiecare $q \in \mathbb{N}^*$ şi $k \in \{1, 2, \ldots, q\}$ avem $\xi_{qk} \in \left[\alpha + (k-1)\frac{\beta-\alpha}{q}, \alpha + k\frac{\beta-\alpha}{q}\right]$ şi $g(\xi_{qk}) > \frac{1}{n}$. Pentru fiecare $q \in \mathbb{N}^*$, suma Riemann corespunzătoare lui g, Δ_q şi ξ_q va fi $\sum\limits_{k=1}^{q} \frac{\beta-\alpha}{q} g(\xi_{qk})$, care este mai mare decât $\frac{\beta-\alpha}{n}$, de unde $\int_\alpha^\beta g(x)\mathrm{d}x \geq \frac{\beta-\alpha}{n} > 0$, contradicţie. □

Revenim la demonstraţia lemei 1: Conform lemei 2, există un interval nedegenerat $I_1 = [a_1, b_1] \subset [a, b]$ astfel încât $h(x) \leq 1$ pentru orice $x \in [a_1, b_1]$. Constatăm că restricţia lui h la I_1 se încadrează la rândul-i în ipotezele lemei 2. Continuând inductiv, construim un şir de intervale nedegenerate $(I_n)_{n \in \mathbb{N}^*}$ astfel încât $I_{n+1} = [a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ şi $h(x) \leq \frac{1}{n+1}$ pentru orice $x \in [a_{n+1}, b_{n+1}]$. Atunci $c \overset{\text{not}}{=} \sup\{a_k : k \in \mathbb{N}^*\} \in \bigcap\limits_{n \in \mathbb{N}^*} I_n$, deci $h(c) \leq \frac{1}{n}$ pentru orice $n \in \mathbb{N}^*$, de unde $h(c) = 0$, contradicţie. □

Demonstraţia 2. Întrucât $[0, 1] = \bigcup\limits_{n \in \mathbb{N}^*} h^{-1}\left(\left[\frac{1}{n}, +\infty\right)\right)$, există $m \in \mathbb{N}^*$ pentru care $F_m \overset{\text{not}}{=} h^{-1}\left(\left[\frac{1}{m}, +\infty\right)\right)$ nu este neglijabilă Lebesgue. Există prin urmare un număr $A > 0$ cu proprietatea că F_m nu este conţinută în nicio reuniune de intervale pentru care suma lungimilor nu-l întrece pe A. Din acest motiv, oricum am lua o diviziune $\Delta = (a = x_0, x_1, \ldots, x_n = b)$ a lui $[a, b]$, printre sumele Riemann $\sum\limits_{i=1}^{n} h(\xi_i)(x_i - x_{i-1})$ asociate acesteia se vor număra şi unele care conţin subexpresii $\sum\limits_{i \in L} h(\xi_i)(x_i - x_{i-1})$ satisfăcând condiţiile $L \subset \{1, 2, \ldots, n\}$, $\sum\limits_{i \in L}(x_{i+1} - x_i) > A$ şi $h(\xi_i) \geq \frac{1}{m}$ pentru orice $i \in L$. Aceste sume Riemann vor fi mai mari decât $\frac{A}{m}$, deci $\int_a^b h(x)\mathrm{d}x \geq \frac{A}{m} > 0$, contradicţie. □

Demonstraţia 3. Funcţia h fiind integrabilă Riemann, mulţimea punctelor sale de discontinuitate este neglijabilă Lebesgue. Cum intervalul $[a, b]$ nu este neglijabil, h admite puncte de continuitate pe intervalul (a, b). Fie c un astfel de punct. Atunci există $\delta > 0$ astfel încât $(c - \delta, c + \delta) \subset (a, b)$ şi $h(x) > \frac{h(c)}{2}$ pentru fiece $x \in (c - \delta, c + \delta)$. Se obţine $\int_a^b h(x)\mathrm{d}x \geq \int_{c-\delta}^{c+\delta} h(x)\mathrm{d}x \geq \delta h(c) > 0$, contradicţie. □

Demonstraţia 4. Fiind integrabilă Riemann (pe interval compact), funcţia h este şi integrabilă Lebesgue; cum valorile lui h sunt pozitive şi $\int_a^b h(x)\mathrm{d}x = 0$, deducem că $h(x) = 0$ aproape peste tot pe $[a, b]$, contradicţie. $\square$

Revenind acum la soluţia problemei 1, constatăm, aplicând lema 1, că numitorii fracţiilor din enunţ sunt nenuli, deci aceste fracţii au sens. Avem $xf(x)^2 < f(x)^2$ pentru orice $x \in [0, 1)$; aplicând lema 1 (punând 0 în loc de $1 \cdot f(1)^2$ pentru a ne încadra în condiţiile acesteia), obţinem

$$(0 <) \int_0^1 xf(x)^2\mathrm{d}x < \int_0^1 f(x)^2\mathrm{d}x. \tag{1}$$

Evident,

$$\int_0^1 (f(x)^2 + 1)\mathrm{d}x \geq 2\int_0^1 f(x)\mathrm{d}x, \tag{2}$$

ultima integrală fiind strict pozitivă conform lemei 1. Din relaţia (2) obţinem

$$0 < \frac{1}{\int_0^1 (f(x)^2 + 1)\mathrm{d}x} \leq \frac{1}{2\int_0^1 f(x)\mathrm{d}x}, \tag{3}$$

iar din (1) şi (3) obţinem inegalitatea dorită. $\square$

Observaţii. (1) În cele precedente s-a probat de fapt inegalitatea din enunţ pentru orice funcţie integrabilă Riemann $f : [0, 1] \to (0, +\infty)$.
(2) La această problemă 5 studenţi au obţinut punctaj maxim.

Problema 2. Considerăm numerele naturale $m, n, p, q \geq 1$ şi matricele $A \in \mathcal{M}_{m,n}(\mathbb{R})$, $B \in \mathcal{M}_{n,p}(\mathbb{R})$, $C \in \mathcal{M}_{p,q}(\mathbb{R})$, $D \in \mathcal{M}_{q,m}(\mathbb{R})$, aşa încât

$$A^t = BCD, \quad B^t = CDA, \quad C^t = DAB, \quad D^t = ABC.$$

Demonstraţi că $(ABCD)^2 = ABCD$.

Această problemă nu este originală, fiind dată în anul 2004 la concursul universităţii Taras Shevchenko din Ucraina. Din păcate acest lucru a fost remarcat foarte târziu, iar juriului i-a fost adus la cunoştinţă în dimineaţa concursului, când era dificil ca problema să fie scoasă din listă şi înlocuită.

Soluţia 1. Cu $P \equiv ABCD = AA^t \in \mathcal{M}_m(\mathbb{R})$ avem

$$\begin{aligned} P^3 &= (ABCD)(ABCD)(ABCD) = (ABC)(DAB)(CDA)(BCD) = \\ &= D^tC^tB^t(BCD) = (BCD)^t(BCD) = (A^t)^t(BCD) = ABCD = P. \end{aligned}$$

Apoi,

$$P^3 = P \iff (P^2 - P)(P + I_m) = O_m. \tag{4}$$

Vom demonstra că $\det(P+I_m) \neq 0$. Presupunem că $\det(P+I_m) = 0$. Aceasta implică existenţa lui $X \in \mathcal{M}_{m,1}(\mathbb{R}) \setminus \{O_{m,1}\}$, aşa încât $(P + I_m)X = O_{m,1}$. Atunci,

$$\begin{aligned}(P+I_m)X=O_{m,1} \quad &\Longleftrightarrow \quad (AA^t+I_m)X=O_{m,1}\\ \Longrightarrow X^t(AA^t+I_m)X=0 \quad &\Longleftrightarrow \quad (A^tX)^t(A^tX)=-X^tX. \qquad (5)\end{aligned}$$

Dar, pentru orice $Y=(y_1,y_2,\ldots,y_m)^t\in\mathcal{M}_{m,1}(\mathbb{R})$ avem $Y^tY=\sum_{i=1}^m y_i^2\geq 0$, în timp ce (5), în care $X\neq O_{m,1}$, arată că

$$(A^tX)^t(A^tX)=-X^tX<0.$$

Această contradicţie ne asigură că $\det(P+I_m)\neq 0$, şi astfel relaţia (4) implică $P^2=P$.

Soluţia 2 (**Dragoş Manea**, *student, Facultatea de Matematică, Universitatea Bucureşti*). Matricea $P\equiv ABCD=AA^t\in\mathcal{M}_m(\mathbb{R})$ este simetrică ($(AA^t)^t=(A^t)^tA^t=AA^t$), şi astfel diagonalizabilă (cu valorile proprii $\lambda_1,\ldots,\lambda_m$ reale). Deci, există o matrice inversabilă $S\in\mathcal{M}_m(\mathbb{R})$, aşa încât

$$P=S\,\mathrm{diag}\,(\lambda_1,\ldots,\lambda_m)\,S^{-1}. \qquad (6)$$

Pe de altă parte avem

$$\begin{gathered}P=P^t \quad\Longleftrightarrow\quad ABCD=(ABCD)^t,\\ (ABCD)^t=D^tC^tB^tA^t=(ABC)(DAB)(CDA)(BCD)=(ABCD)^3,\end{gathered}$$

deci $P^3=P$. Astfel, din (6) obţinem

$$\begin{aligned}P^3=P \quad&\Longleftrightarrow\quad \mathrm{diag}\,(\lambda_1^3,\ldots,\lambda_m^3)=\mathrm{diag}\,(\lambda_1,\ldots,\lambda_m)\\ &\Longleftrightarrow\quad \lambda_k^3=\lambda_k,\quad \forall\, k=1,\ldots,m,\end{aligned}$$

ceea ce implică $\lambda_k\in\{-1,0,1\}$, $\forall\ k=1,\ldots,m$. Dacă arătăm că $\lambda_k\neq -1$, vom avea $\lambda_k^2=\lambda_k$, şi cu aceasta justificarea egalităţii $P^2=P$ este încheiată, căci

$$S\,\mathrm{diag}\,(\lambda_1^2,\ldots,\lambda_m^2)\,S^{-1}=S\,\mathrm{diag}\,(\lambda_1,\ldots,\lambda_m)\,S^{-1} \quad\Longleftrightarrow\quad P^2=P.$$

Să demonstrăm, deci, că $\lambda_k\neq -1$; de fapt, vom arăta că $\lambda_k\geq 0$.

Valorile proprii $\lambda_1,\ldots,\lambda_m$ ale matricei P sunt rădăcinile polinomului caracteristic asociat lui P,

$$\det(XI_m-P)=X^m-\sigma_1X^{m-1}+\sigma_2X^{m-2}-\cdots+(-1)^m\sigma_m.$$

Aici, pentru fiecare $k=1,\ldots,m$, coeficientul σ_k este egal cu suma minorilor principali de ordin k ai lui P. Considerăm un astfel de minor, care provine din liniile $j_1,\ldots,j_k$. Remarcăm că, deoarece $P=AA^t$, acest minor se scrie ca

$$\det\begin{pmatrix} \|L_{j_1}\|^2 & \langle L_{j_1}, L_{j_2}\rangle & \dots & \langle L_{j_1}, L_{j_k}\rangle \\ \langle L_{j_2}, L_{j_1}\rangle & \|L_{j_2}\|^2 & \dots & \langle L_{j_2}, L_{j_k}\rangle \\ \vdots & \vdots & & \vdots \\ \langle L_{j_k}, L_{j_1}\rangle & \langle L_{j_k}, L_{j_2}\rangle & \dots & \|L_{j_k}\|^2 \end{pmatrix} = \tag{7}$$

$$= \det\begin{pmatrix} L_{j_1} \\ \vdots \\ L_{j_k} \end{pmatrix}\begin{pmatrix} L_{j_1}^t \dots L_{j_k}^t \end{pmatrix},$$

unde $L_{j_1}, \dots, L_{j_k}$ sunt linii ale matricei A, iar $\langle\,,\,\rangle$ şi $\|\;\|$ reprezintă produsul scalar euclidian, respectiv norma euclidiană în $\mathbb{R}^m$. Folosind faptul că $\det(MM^t) \geq 0$ pentru orice $M \in \mathcal{M}_{m,n}(\mathbb{R})$, din (7) rezultă că minorii principali ai lui P sunt nenegativi, şi cu aceasta $\sigma_k \geq 0$, $k = \overline{1, m}$. Acum, presupunând că polinomul caracteristic are o rădăcină $\lambda < 0$, ajungem la contradicţia

$$\begin{aligned} 0 = \det(\lambda I_m - P) &= \lambda^m - \sigma_1\lambda^{m-1} + \sigma_2\lambda^{m-2} - \dots + (-1)^m\sigma_m = \\ &= (-1)^m\left\{(-\lambda)^m + \sigma_1(-\lambda)^{m-1} + \dots + \sigma_m)\right\} \neq 0\,. \end{aligned}$$

Cu aceasta demonstraţia egalităţii $P^2 = P$ este încheiată.

Observaţii. (1) Justificarea faptului că $\det(I_m + P)$ este nenul, echivalent, matricea $P \equiv AA^t$ nu are ca valoare proprie pe $\lambda = -1$, poate fi făcută remarcând că P este matrice pozitiv semidefinită,

$$\langle AA^t v, v\rangle = \langle A^t v, A^t v\rangle = \|A^t v\|^2 \geq 0\,, \quad v \in \mathcal{M}_{m,1}(\mathbb{R})\,,$$

şi apelând la rezultatul cunoscut conform căruia o astfel de matrice are toate valorile proprii ≥ 0. Această observaţie a fost folosită de către unii studenţi la soluţionarea Problemei 2.
(2) Într-un spaţiu vectorial V cu produs scalar matricea având componentele $\langle v_i, v_j\rangle$, unde $v_1, \dots, v_n \in V$, se numeşte matricea Gram corespunzătoare sistemului de vectori $v_1, \dots, v_n$, iar determinantul acesteia – determinantul Gram. Astfel, dacă $A \in \mathcal{M}_{m,n}(\mathbb{R})$, matricea AA^t este matricea Gram corespunzătoare liniilor lui A, iar determinantul din formula (7) este un determinant Gram. O matrice Gram este pozitiv semidefinită (şi orice matrice pozitiv semidefinită este o matrice Gram), în particular, un determinant Gram (ca produs al valorilor proprii ale matricei corespunzătoare) este nenegativ. Această proprietate este apelată în soluţia 2 a problemei, prin remarca $\det MM^t \geq 0$.
Inegalitatea $\det MM^t \geq 0$ se mai poate demonstra folosind formula Binet-Cauchy pentru calculul determinantului produsului a două matrice.
(3) La această problemă 22 de studenţi au obţinut punctaj maxim.

Problema 3. Fie $A, B \in \mathcal{M}_{2018}(\mathbb{R})$ cu proprietatea că $AB = BA$ şi $A^{2018} = B^{2018} = I$, unde I este matricea unitate. Arătaţi că dacă $\operatorname{tr}(A) = 2018$, atunci $\operatorname{tr}(A) = \operatorname{tr}(B)$.

Vasileios Papadopoulos, Democritus University of Thrace, Grecia

Soluţia 1 (a autorului). Deoarece $A^{2018} = B^{2018} = I$ şi $AB = BA$ avem că $(AB)^{2018} = I$. De aici obţinem că valorile proprii ale matricelor A, B şi AB sunt rădăcini de ordinul 2018 ale unităţii. Din faptul că $\operatorname{tr}(AB) = 2018$ deducem că valorile proprii ale lui AB sunt toate egale cu 1. (Mai general, dacă $z_1, \ldots, z_{2018}$ sunt numere complexe de modul 1 şi $\sum\limits_{i=k}^{2018} z_k = 2018$, atunci $z_1 = \cdots = z_{2018} = 1$. Aceasta rezultă imediat: scriem $z_k = \cos a_k + i \sin a_k$ şi din $\sum\limits_{k=1}^{2018} z_k = 2018$ deducem că $\sum\limits_{k=1}^{2018} \cos a_k = 2018$, deci $\cos a_k = 1$ pentru orice $k = 1, \ldots, 2018$ şi, în consecinţă, $\sin a_k = 0$ pentru orice $k = 1, \ldots, 2018$.) Aşadar polinomul caracteristic al matricei AB este $P_{AB} = (X-1)^{2018}$.
Pe de altă parte, polinomul minimal al matricei AB, μ_{AB}, divide pe $X^{2018}-1$, respectiv pe $(X-1)^{2018}$, deci $\mu_{AB} = X - 1$ iar aceasta înseamnă că $AB = I$. Astfel am obţinut că $B = A^{-1}$.
Se ştie însă că valorile proprii ale inversei unei matrice sunt inversele valorilor proprii ale acelei matrice. În cazul nostru inversele valorilor proprii ale matricei A sunt egale cu conjugatele lor (deoarece au modulul 1), iar de aici deducem că $\operatorname{tr}(B) = \overline{\operatorname{tr}(A)} = \operatorname{tr}(A)$, ultima egalitate având loc pentru că matricea A este reală, deci şi urma sa este tot un număr real.

Soluţia 2. Fie λ_k, $k = 1, \ldots, 2018$, valorile proprii ale matricei A, respectiv μ_k, $k = 1, \ldots, 2018$, valorile proprii ale matricei B. Deoarece $AB = BA$, matricele A şi B sunt simultan triangularizabile (peste $\mathbb{C}$), adică există o matrice inversabilă $U \in M_{2018}(\mathbb{C})$ cu proprietatea că matricele UAU^{-1} şi UBU^{-1} sunt superior triunghiulare. (Acest lucru se demonstrează prin inducţie după dimensiunea matricelor observând că $AB = BA$ implică faptul că cele două matrice au un vector propriu comun.) Aşadar valorile proprii ale matricei AB sunt (eventual după o renumerotare) $\lambda_k \mu_k$, $k = 1, \ldots, 2018$.

(Se putea, de asemenea, argumenta că matricele A şi B sunt diagonalizabile, deoarece valorile lor proprii sunt printre rădăcinile de ordinul 2018 ale unităţii, deci distincte, şi atunci sunt simultan diagonalizabile.)

Din $\operatorname{tr}(AB) = 2018$ se obţine, ca la soluţia 1, că $\lambda_k \mu_k = 1$ pentru orice $k = 1, \ldots, 2018$, adică valorile proprii ale lui B sunt inversele valorilor proprii ale lui A şi ne găsim astfel în situaţia de la soluţia 1.

Observaţii. 1) Soluţiile studenţilor care au rezolvat complet această problemă au fost în spiritul celor două soluţii prezentate mai sus, diferenţele apărând doar la nivel de detalii.
2) La această problemă 15 studenţi au obţinut punctaj maxim.

Problema 4. (a) Fie $f : \mathbb{R} \to \mathbb{R}$ o funcţie polinomială. Să se demonstreze că

$$\int_0^\infty e^{-x} f(x)\,\mathrm{d}x = f(0) + f'(0) + f''(0) + \cdots .$$

(b) Fie $f : \mathbb{R} \to \mathbb{R}$ o funcţie care admite dezvoltare în serie Maclaurin cu raza de convergenţă $R = \infty$. Să se demonstreze că dacă seria $\sum_{n=0}^{\infty} f^{(n)}(0)$ este absolut convergentă, atunci integrala improprie $\int_0^\infty e^{-x} f(x)\,\mathrm{d}x$ este convergentă şi are loc egalitatea

$$\int_0^\infty e^{-x} f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} f^{(n)}(0).$$

Ovidiu Furdui, Universitatea Tehnică, Cluj-Napoca, România

Soluţii şi comentarii. *Toţi concurenţii care au rezolvat punctul* (a) *au dat una dintre următoarele două soluţii. Surprinzător, au fost unii concurenţi (şi chiar dintre studenţii români) care au primit 0 puncte la acest „exerciţiu de seminar".*

(a) **Soluţia 1**. Fie d gradul lui f. Integrând prin părţi, obţinem

$$\begin{aligned}\int_0^\infty e^{-x} f(x)\,\mathrm{d}x &= \int_0^\infty (-e^{-x})' f(x)\,\mathrm{d}x = -e^{-x} f(x)\Big|_0^\infty + \int_0^\infty e^{-x} f'(x)\,\mathrm{d}x \\ &= f(0) + \int_0^\infty e^{-x} f'(x)\,\mathrm{d}x.\end{aligned}$$

Repetând integrarea prin părţi şi ţinând seama că derivatele lui f sunt tot funcţii polinomiale, se obţine

$$\begin{aligned}\int_0^\infty e^{-x} f(x)\,\mathrm{d}x &= f(0) + f'(0) + \int_0^\infty e^{-x} f''(x)\,\mathrm{d}x = \cdots \\ &= f(0) + f'(0) + \cdots + f^{(d)}(0) + \int_0^\infty e^{-x} f^{(d+1)}(x)\,\mathrm{d}x \\ &= f(0) + f'(0) + \cdots + f^{(d)}(0),\end{aligned}$$

deoarece $f^{(d+1)} = 0$. □

Soluţia 2. Deoarece f este funcţie polinomială, conform formulei lui Taylor avem

$$f(x) = \sum_{n=0}^{d} \frac{f^{(n)}(0)}{n!} x^n \quad \text{oricare ar fi } x \in \mathbb{R},$$

unde d este gradul lui f. Drept urmare, avem

$$\begin{aligned}\int_0^\infty e^{-x} f(x)\,\mathrm{d}x &= \sum_{n=0}^{d} \frac{f^{(n)}(0)}{n!} \int_0^\infty e^{-x} x^n \mathrm{d}x = \sum_{n=0}^{d} \frac{f^{(n)}(0)}{n!}\,\Gamma(n+1)\\ &= \sum_{n=0}^{d} f^{(n)}(0).\end{aligned}$$

□

Punctul (b) *s-a dovedit a fi cea mai dificilă problemă din concurs. El a fost rezolvat complet doar de patru studenţi (Ovidiu Neculai Avădanei, George Kotsovolis, Georgios Kampanis şi György Tötös). Soluţiile celor patru studenţi s-au bazat, în esenţă, pe teoremele clasice de convergenţă (teorema convergenţei uniforme, teorema convergenţei monotone sau teorema convergenţei dominate). Prezentăm mai jos soluţia autorului, precum şi două dintre soluţiile date în concurs de studenţi din Grecia.*

(b) **Soluţia 1 (a autorului)**. Pentru orice număr natural n notăm

$$S_n := \sum_{k=0}^{n} f^{(k)}(0) \quad \text{precum şi} \quad T_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\, x^k,$$

cel de-al n-lea polinom Maclaurin al lui f. Cum $T_n^{(k)}(0) = f^{(k)}(0)$ oricare ar fi $k \in \{0, 1, \ldots, n\}$, în baza lui (a) avem

$$\int_0^\infty e^{-x} T_n(x)\,\mathrm{d}x = S_n \quad \text{pentru orice } n \in \mathbb{N}.$$

Pentru a dovedi că integrala improprie $\int_0^\infty e^{-x} f(x)\,\mathrm{d}x$ este convergentă, vom arăta că limita $\lim\limits_{v\to\infty} \int_0^v e^{-x} f(x)\,\mathrm{d}x$ există şi este finită. Folosim teorema lui Bolzano. Fie $\varepsilon > 0$ arbitrar. Deoarece seria $\sum\limits_{n=0}^{\infty} \left|f^{(n)}(0)\right|$ este convergentă, există un $n_0 \in \mathbb{N}$ astfel ca

$$\sum_{n=n_0+1}^{\infty} \left|f^{(n)}(0)\right| < \frac{\varepsilon}{2}\,.$$

Cum integrala improprie $\int_0^\infty e^{-x} T_{n_0}(x)\,\mathrm{d}x$ este convergentă (conform punctului (a)), limita $\lim\limits_{v\to\infty} \int_0^v e^{-x} T_{n_0}(x)\,\mathrm{d}x$ există şi este finită. În baza părţii de necesitate a teoremei lui Bolzano, există un $\delta > 0$ în aşa fel încât pentru orice $v, v' \in (\delta, \infty)$ să avem

$$\left|\int_v^{v'} e^{-x} T_{n_0}(x)\,\mathrm{d}x\right| < \frac{\varepsilon}{2}\,.$$

Atunci pentru orice $v, v' \in (\delta, \infty)$ cu $v < v'$, avem

$$\begin{aligned}
\left|\int_v^{v'} e^{-x} f(x)\,\mathrm{d}x\right| &\le \left|\int_v^{v'} e^{-x} T_{n_0}(x)\,\mathrm{d}x\right| + \int_v^{v'} e^{-x} |f(x) - T_{n_0}(x)|\,\mathrm{d}x < \\
&< \frac{\varepsilon}{2} + \int_v^{v'} e^{-x} \left|\sum_{n=n_0+1}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n\right| \mathrm{d}x \le \\
&\le \frac{\varepsilon}{2} + \sum_{n=n_0+1}^{\infty} \frac{\left|f^{(n)}(0)\right|}{n!} \int_v^{v'} e^{-x} x^n \mathrm{d}x \le \\
&\le \frac{\varepsilon}{2} + \sum_{n=n_0+1}^{\infty} \frac{\left|f^{(n)}(0)\right|}{n!}\, \Gamma(n+1) = \\
&= \frac{\varepsilon}{2} + \sum_{n=n_0+1}^{\infty} \left|f^{(n)}(0)\right| < \\
&< \varepsilon.
\end{aligned}$$

Cum ε a fost arbitrar, în baza părţii de suficienţă a teoremei lui Bolzano rezultă că limita $\lim\limits_{v\to\infty} \int_0^v e^{-x} f(x)\,\mathrm{d}x$ există şi este finită, adică integrala improprie $\int_0^\infty e^{-x} f(x)\,\mathrm{d}x$ este convergentă.

Pentru orice $n \in \mathbb{N}$ avem

$$\begin{aligned}
\left|\int_0^{\infty} e^{-x} f(x)\,\mathrm{d}x - S_n\right| &= \left|\int_0^{\infty} e^{-x} \big(f(x) - T_n(x)\big)\,\mathrm{d}x\right| \le \\
&\le \int_0^{\infty} e^{-x} \big|f(x) - T_n(x)\big|\,\mathrm{d}x \le \\
&\le \int_0^{\infty} e^{-x} \sum_{k=n+1}^{\infty} \frac{\left|f^{(k)}(0)\right|}{k!}\, x^k \mathrm{d}x \le \\
&\le \sum_{k=n+1}^{\infty} \left|f^{(k)}(0)\right|.
\end{aligned}$$

Convergenţa seriei $\sum\limits_{k=0}^{\infty} \left|f^{(k)}(0)\right|$ implică faptul că şirul (S_n) este convergent, având limita $\int_0^\infty e^{-x} f(x)\,\mathrm{d}x$. □

(b) **Soluţia 2** (*dată în concurs de un student din Grecia*). Fie şirul de funcţii $g_n : [0, \infty) \to \mathbb{R}$ $(n \ge 1)$, definite prin

$$g_n(x) := e^{-x} \sum_{k=0}^{n} \frac{\left|f^{(k)}(0)\right|}{k!}\, x^k.$$

Evident, şirul $(g_n(x))$ este crescător pentru orice $x \in [0, \infty)$ fixat. Mai mult, notând $S := \sum\limits_{k=0}^{\infty} \left|f^{(k)}(0)\right|$, avem

$$0 \leq g_n(x) \leq Se^{-x} \sum_{k=0}^{n} \frac{x^k}{k!} \leq S \quad \text{pentru orice } n \geq 1 \text{ şi orice } x \in [0, \infty).$$

Prin urmare, există $\lim\limits_{n\to\infty} g_n(x) \in \mathbb{R}$ oricare ar fi $x \in [0, \infty)$. Fie funcţia $g : [0, \infty) \to \mathbb{R}$, definită prin $g(x) := \lim\limits_{n\to\infty} g_n(x)$. Deoarece şirul de funcţii (g_n) converge uniform pe compacte, rezultă că g este local integrabilă Riemann. În baza teoremei convergenţei monotone, avem

$$\int_0^{\infty} g(x)\,\mathrm{d}x = \int_0^{\infty} \lim_{n\to\infty} g_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_0^{\infty} g_n(x)\,\mathrm{d}x.$$

Dar

$$\int_0^{\infty} g_n(x)\,\mathrm{d}x = \sum_{k=0}^{n} \left|f^{(k)}(0)\right| \quad \text{oricare ar fi } n \geq 1,$$

conform celor demonstrate la (a). Deducem de aici că

$$\int_0^{\infty} g(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{k=0}^{n} \left|f^{(k)}(0)\right| = S,$$

deci $\int_0^\infty g(x)\,\mathrm{d}x$ converge.

Fie acum şirul de funcţii $f_n : [0, \infty) \to \mathbb{R}$ $(n \geq 1)$, definite prin

$$f_n(x) := e^{-x} \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\, x^k.$$

Deoarece f este dezvoltabilă în serie Maclaurin pe $\mathbb{R}$, rezultă că

$$\lim_{n\to\infty} f_n(x) = e^{-x} f(x) \quad \text{oricare ar fi } x \in [0, \infty).$$

Avem şi

$$f_n(x) \leq g_n(x) \leq g(x) \quad \text{pentru orice } n \geq 1 \text{ şi orice } x \in [0, \infty).$$

Aplicând teorema convergenţei dominate, deducem că

$$\int_0^{\infty} e^{-x} f(x)\,\mathrm{d}x = \int_0^{\infty} \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_0^{\infty} f_n(x)\,\mathrm{d}x.$$

Dar

$$\int_0^{\infty} f_n(x)\,\mathrm{d}x = \sum_{k=0}^{n} f^{(k)}(0) \quad \text{oricare ar fi } n \geq 1,$$

conform celor demonstrate la (a). Drept urmare, avem

$$\int_0^{\infty} e^{-x} f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{k=0}^{n} f^{(k)}(0) = \sum_{k=0}^{\infty} f^{(k)}(0).$$

$\square$

(b) **Soluţia 3** (*dată în concurs de un student din Grecia*). Se aplică şirului de funcţii $f_n : [0, \infty) \to \mathbb{R}$ ($n \geq 1$), definite prin

$$f_n(x) := e^{-x} \frac{f^{(n)}(0)}{n!} x^n,$$

următoarea variantă a teoremei lui Tonelli (a se vedea R. Gelca şi T. Andreescu, *Putnam and Beyond*. Springer, 2007, p. 178): dacă

$$\int_0^\infty \sum_{n=0}^\infty |f_n(x)| \,\mathrm{d}x < \infty \quad \text{sau} \quad \sum_{n=0}^\infty \int_0^\infty |f_n(x)| \,\mathrm{d}x < \infty,$$

atunci are loc egalitatea

$$\int_0^\infty \sum_{n=0}^\infty f_n(x) \,\mathrm{d}x = \sum_{n=0}^\infty \int_0^\infty f_n(x) \,\mathrm{d}x.$$

Rămâne doar să notăm că, în ipotezele problemei, avem

$$\sum_{n=0}^\infty \int_0^\infty |f_n(x)| \,\mathrm{d}x = \sum_{n=0}^\infty \left| f^{(n)}(0) \right| < \infty,$$

$$\int_0^\infty \sum_{n=0}^\infty f_n(x) \,\mathrm{d}x = \int_0^\infty e^{-x} f(x) \,\mathrm{d}x,$$

precum şi

$$\sum_{n=0}^\infty \int_0^\infty f_n(x) \,\mathrm{d}x = \sum_{n=0}^\infty f^{(n)}(0).$$

$\square$

Observaţie. Pentru Problema 4 punctajul maxim (10 puncte) a fost obţinut de 9 dintre studenţi.

NOTE MATEMATICE

A happy case of mathematical (mis)induction[1)]

ÁRPÁD BÉNYI[2)], IOAN CAŞU[3)]

Abstract. From a missed attempt to mathematical induction and through a journey in calculus, we arrive to a stronger inequality concerning partial sums of the harmonic series.

Keywords: Inequalities, mathematical induction, sequences, series

MSC: Primary 26A06, 26D06, 26D15; Secondary 40A05.

The topic of an exercise in [1] is the following inequality: for all $n \in \mathbb{N}$, $n \geq 2$, we have

$$\sum_{k=2}^{n} \frac{1}{k} < n\Big(1 - \frac{1}{\sqrt[n]{n}}\Big). \tag{1}$$

This follows straightforwardly from the AM-GM inequality if we re-write it as

$$\frac{1 + \sum\limits_{k=2}^{n} \dfrac{k-1}{k}}{n} > \sqrt[n]{\frac{1}{n}}.$$

However, as with many of the statements that contain the phrase "for all $n \in \mathbb{N}$...", lots of our students will be tempted into using the Principle of Mathematical Induction to prove it. It does not quite work for this statement for reasons that will become clear soon, but our story has a happy ending triggered by a series of fortunate calculus events.

If we denote by $\mathcal{P}(n)$ the statement in (1), one easily checks first that $\mathcal{P}(2)$ is true (being equivalent to $\sqrt{2} < 1.5$). Showing that $\mathcal{P}(k) \Rightarrow \mathcal{P}(k+1)$ is at the core of the method and, in this case, it comes down to proving that for all $k \in \mathbb{N}$:

$$\frac{k+1}{\sqrt[k+1]{k+1}} - \frac{k}{\sqrt[k]{k}} < \frac{k}{k+1}. \tag{2}$$

The statement expressed in (2) is quite strong in the sense that the difference between the left and right-hand sides is getting smaller as k gets larger. This is already apparent for small values of k: when $k = 1$, (2) is $\sqrt{2} - 1 \approx 0.41 < 0.5 = 1/2$, while when $k = 2$, (2) is $\sqrt[3]{9} - \sqrt{2} \approx 0.665 < 0.666 < 2/3$. A computer algebra system such as `Mathematica` easily verifies that (2) holds

1) This work is partially supported by a grant from the Simons Foundation (No. 246024 to Árpád Bényi).

2) Department of Mathematics, 516 High St, Western Washington University, Bellingham, WA 98225, USA, arpad.benyi@wwu.edu

3) Department of Mathematics, West University of Timişoara, Bd. Vasile Pârvan, no. 4, 300223 Timişoara, Romania, ioan.casu@e-uvt.ro

for other randomly chosen natural values of k, so...(2) must be true?! Yet, it is not an obvious inequality, and the considerations above do not amount to a rigorous proof of it. We extrapolate that a typical student would conclude at this point that showing (1) by induction is either not possible or, if possible, not an easily accomplishable task.

The moral of our tale is that doing mathematics is a fun roller coaster and perseverance pays off in the end. We will indicate below how basic ideas from calculus combine to prove (2). In what follows, we will assume that $k \in \mathbb{N}$ and $k \geq 20$. This is just a technicality that will simplify some of the calculations; as mentioned already, the fact that (2) holds for $k \in \{1, 2, \ldots, 19\}$ can be checked directly by a computing device.

Let $f : [1, \infty), f(x) = \frac{x}{\sqrt[x]{x}}$. The inequality we want to prove can be re-written as

$$f(k+1) - f(k) < \frac{k}{k+1}, \ \forall\ k \geq 20. \tag{3}$$

First of all, this inequality is meaningful in the sense that the expression on the left is always strictly positive due to the fact that f is strictly increasing. Indeed, since $\ln(f(x)) = \left(1 - \frac{1}{x}\right) \ln x$, by implicit differentiation we find that, for all $x \geq 1$, $f'(x) = x^{-1-1/x}(x - 1 + \ln x) > 0$. Moreover, by the Mean Value Theorem we see that proving (3) is equivalent to proving that for some $\theta \in (0, 1)$ we have $f'(k + \theta) < \frac{k}{k+1}$. Now, a tedious calculation further shows that, for all $x \geq 1$,

$$\begin{aligned} f''(x) &= x^{-3-1/x}((\ln x)^2 - 2 \ln x + x + 1) \\ &= x^{-3-1/x}[((\ln x)^2 - \ln x + 1) + (x - \ln x)] > 0. \end{aligned}$$

Thus, f is concave up (that is, f' is increasing), and so it suffices to show that for all $k \in \mathbb{N}$ and $k \geq 20$,

$$f'(k+1) \leq \frac{k}{k+1}. \tag{4}$$

This last inequality can be rewritten as

$$(k+1)^{-1-1/(k+1)}(k + \ln(k+1)) < \frac{k}{k+1} \Leftrightarrow \ln(k+1) < k\big((k+1)^{1/(k+1)} - 1\big).$$

Denote

$$a_k = \sqrt[k]{k} - 1 > 0 \Leftrightarrow k = (1 + a_k)^k \Leftrightarrow k = \frac{\ln(k)}{\ln(1 + a_k)}.$$

With this notation, (4) is equivalent to

$$(k+1)\ln(1 + a_{k+1}) < k a_{k+1} \Leftrightarrow \frac{\ln(1 + a_{k+1})}{a_{k+1}} < \frac{k}{k+1}. \tag{5}$$

Intuitively, we see that (5) is just a quantitative version of the well-known fact that, assuming we have $\lim\limits_{k \to \infty} a_k = 0$, $\lim\limits_{k \to \infty} \frac{\ln(1+a_{k+1})}{a_{k+1}} = 1$. In the remainder of this note we show that (5) holds for $k \in \mathbb{N}$ with $k \geq 20$. Our argument will be based on two simple observations.

The first observation is that, indeed, $\lim_{k\to\infty} a_k = 0$, and, more importantly, we can express this convergence in a quantitative way. We can show this by using the Binomial Theorem, which, in particular, gives that for all $x \geq 0$ and $k \in \mathbb{N}$ we have

$$(1+x)^k \geq 1 + kx + \frac{k(k-1)}{2}x^2.$$

If we plug in $x = 2/\sqrt{k}$ in the last displayed inequality, we see that

$$\left(1 + \frac{2}{\sqrt{k}}\right)^k > 1 + \frac{k(k-1)}{2}\frac{4}{k} \geq k,$$

which gives

$$0 \leq \sqrt[k]{k} - 1 < \frac{2}{\sqrt{k}} \Leftrightarrow a_k < \frac{2}{\sqrt{k}}; \tag{6}$$

by the Squeeze Principle for sequences, we obtain from here $\lim_{k\to\infty} a_k = 0$.

Our second observation is that, for all $x \geq 0$, we have

$$\ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}. \tag{7}$$

The student that has encountered series in his or her studies will recognize that the right hand-side of (7) is just a partial sum of the series expansion of $\ln(1+x)$. Regardless of this, we can prove (7) easily as follows. Let $h : [0, \infty) \to \mathbb{R}$, $h(0) = 0$, and, for $x > 0$,

$$h(x) = \ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3}.$$

Since

$$h'(x) = \frac{1}{1+x} - 1 + x - x^2 = -\frac{x^3}{x+1} \leq 0,$$

h is decreasing and hence $h(x) < 0$ for $x > 0$.

We are ready to implement our two observations in obtaining (5). Applying (7) to $x = a_{k+1}$ we get

$$\frac{\ln(1 + a_{k+1})}{a_{k+1}} < 1 - \frac{a_{k+1}}{2} + \frac{a_{k+1}^2}{3}.$$

Thus, it suffices to show that

$$1 - \frac{a_{k+1}}{2} + \frac{a_{k+1}^2}{3} < 1 - \frac{1}{k+1}$$

that is

$$(k+1)a_{k+1}\left(\frac{1}{2} - \frac{a_{k+1}}{3}\right) > 1.$$

Notice now that, by (6) and for $k \geq 20$, we have

$$\frac{1}{2} - \frac{a_{k+1}}{3} > \frac{1}{2} - \frac{2}{3\sqrt{k+1}} > \frac{1}{3}.$$

We are left to show that for $n \geq 21$ we have $na_n > 3$ ($n = k+1$ above). Rewriting, we have

$$na_n > 3 \Leftrightarrow a_n > \frac{3}{n} \Leftrightarrow n > \left(1 + \frac{3}{n}\right)^n.$$

But since $(1+y)^{1/y} < e$ for all $y > 0$, letting $y = 3/n$, we get

$$\left(1 + \frac{3}{n}\right)^n < e^3 < n$$

for $n \geq 21$. The proof of (5) is complete.

An immediate corollary of (2) is that, as observed at the beginning of this note, we have

$$\lim_{k\to\infty} \left(\frac{k+1}{\sqrt[k+1]{k+1}} - \frac{k}{\sqrt[k]{k}} - \frac{k}{k+1} \right) = 0,$$

or, equivalently,

$$\lim_{k\to\infty} \left(\frac{k+1}{\sqrt[k+1]{k+1}} - \frac{k}{\sqrt[k]{k}} \right) = 1.$$

We note first that for all $k \in \mathbb{N}$, $(k+1)^{1/(k+1)} < k^{1/k}$ or $\frac{\ln(k+1)}{k+1} < \frac{\ln k}{k}$, which in turn follows easily from the fact that the function $g : [3, \infty), g(x) = \frac{\ln x}{x}$ is decreasing; since $g'(x) = \frac{1 - \ln x}{x^2} < 0$ for $x \geq 3$. Now, by (2) and since $(\sqrt[k]{k})$ is a decreasing sequence, we have

$$1 > \frac{k+1}{\sqrt[k+1]{k+1}} - \frac{k}{\sqrt[k]{k}} > \frac{k+1}{\sqrt[k]{k}} - \frac{k}{\sqrt[k]{k}} = \frac{1}{\sqrt[k]{k}},$$

and the conclusion follows from the fact observed before that $\lim\limits_{k\to\infty} \sqrt[k]{k} = 1$ and the Squeeze Principle for sequences.

References

[1] T. Zvonaru and N. Stanciu, *Problema 27269*, Gazeta Matematică (Seria B), no. 9 (2016).

The monotony in hazard rate sense of some families of multidimensional distributions

Luigi-Ionuţ Catana[1)]

Abstract. In this note we will prove that the monotony in hazard rate sense holds for some families of multidimensional distributions.

Keywords: Stochastic orders, risk theory, hazard rate, distribution
MSC: 60E15.

Introduction

For a random vector $X : \Omega \to \mathbb{R}^d$ we consider its distribution $\mu(B) = P(X \in B)$, its distribution function $F(x) = P(X \leq x)$, and $F^*(x) = P(X \geq x, X \neq x)$. In this article, the distribution function for another random vector Y will be denoted by G.

In this note we are using the notation and some well known results from [1] and [2].

Definition 1. *Let X and Y be two random vectors. We say that X is smaller than Y in hazard rate sense (and we denote $X \prec_{\mathrm{hr}} Y$) if*

$$F^*(x)G^*(y) \leq F^*(x \wedge y)G^*(x \vee y), \quad \forall x, y \in \mathbb{R}^d.$$

Definition 2. *Let X and Y be two random vectors. We say that X is smaller than Y in weak hazard rate sense (and we denote $X \prec_{\mathrm{whr}} Y$) if $\frac{G^*}{F^*}$ is increasing on $Supp(G^*)$.*

Theorem 3. *Let X, Y be two random vectors. If $X \prec_{\mathrm{hr}} Y$ then $X \prec_{\mathrm{whr}} Y$.*

Theorem 4. *Let $X = (X_i)_{i=\overline{1,d}}$ and $Y = (Y_i)_{i=\overline{1,d}}$ be two random vectors.*
If $X \prec_{\mathrm{hr}} Y$ then $X_i \prec_{\mathrm{hr}} Y_i$, $i = \overline{1,d}$.

Theorem 5. *Let $(X_i)_{i=\overline{1,d}}$ and $(Y_i)_{i=\overline{1,d}}$ be random variables.*
If $X_i \prec_{\mathrm{hr}} Y_i$, $i = \overline{1,d}$, then $\otimes_{i=1}^d X_i \prec_{\mathrm{hr}} \otimes_{i=1}^d Y_i$.

For the sake of completeness, let us recall the definition for some multidimensional distributions.

The *multivariate uniform* distribution $\mathrm{Unif}(I)$ on some product of intervals $I = I_1 \times \cdots \times I_d \subset \mathbb{R}^d$ has the density function $f(x) = \frac{1_I(x)}{\lambda^d(I)}$. Its marginals are $\mathrm{Unif}(I_i)$.

1) Departament of Mathematics, Faculty of Mathematics and Informatics, University of Bucharest, Romania, luigi_catana@yahoo.com

The *multivariate Poisson* distribution of $\lambda = (\lambda_d, \lambda_{d-1}, \ldots, \lambda_1, \lambda_0)$, $\text{Poisson}(\lambda)$, where each λ_i is nonnegative, has the density function

$$f(x) = e^{-\sum\limits_{k=1}^{d} \lambda_k} \cdot \prod_{k=1}^{d} \frac{\lambda_k^{x_k}}{x_k!} \cdot \sum_{i=0}^{\min(x_1,\ldots,x_d)} \prod_{j=1}^{d} C_{x_j}^{i} i! \left(\frac{\lambda_0}{\prod\limits_{j=1}^{d} \lambda_j} \right)^{i}.$$

The marginals are $\text{Poisson}(\lambda_i + \lambda_0)$.

The *multivariate normal* distribution $N(\mu, \sum)$ has the density function

$$f(x) = \left(\det\left(2\pi \cdot \sum\right)\right)^{-\frac{1}{2}} \cdot e^{-\frac{1}{2}(x-\mu)^T \sum^{-1}(x-\mu)},$$

where $\mu = (EX_1, EX_2, \ldots, EX_d)$ and $\sum$ denotes the covariance matrix.

The marginals are $N(\mu_i, \sum\limits_{ii})$.

The *bivariate Bernoulli* distribution $B(1,p)$ has density

$$P(X = a_i) = p_i,$$

where $a_i \in \{(0,0), (1,0), (0,1), (1,1)\}$ and p_i are four nonnegative real numbers with sum one.

1. Main Results

Proposition 6. *Let* $a, b \in \mathbb{R}_+^d$. *Then* $\text{Unif}([0,a]) \prec_{\text{hr}} \text{Unif}([0,b])$ *if and only if* $a \le b$.

Proof. We have $\text{Unif}(I) \prec_{\text{hr}} \text{Unif}(J) \Rightarrow \text{Unif}([0,a_i]) \prec_{\text{hr}} \text{Unif}([0,b_i]), i = \overline{1,d} \Rightarrow a_i \le b_i \Rightarrow a \le b$.

Conversely, assume $a \le b$. Then $\text{Unif}([0,a_i]) \prec_{\text{hr}} \text{Unif}([0,b_i]), i = \overline{1,d}$. It follows that $\otimes_{i=1}^{d} \text{Unif}([0,a_i]) \prec_{\text{hr}} \otimes_{i=1}^{d} \text{Unif}([0,b_i])$, in other words, we have $\text{Unif}([0,a]) \prec_{\text{hr}} \text{Unif}([0,b])$. □

Proposition 7. *Let* $\alpha, \beta \in \mathbb{R}_+^{d+1}$. *If* $\text{Poisson}(\alpha) \prec_{\text{hr}} \text{Poisson}(\beta)$, *then* $\alpha_i + \alpha_0 \le \beta_i + \beta_0,\ i = \overline{1,d}$. *The converse holds if the marginals are independent.*

Proof. Suppose that $\text{Poisson}(\alpha) \prec_{\text{hr}} \text{Poisson}(\beta)$. Then for $i = \overline{1,d}$ one has $\text{Poisson}(\alpha_i+\alpha_0) \prec_{\text{hr}} \text{Poisson}(\beta_i+\beta_0)$, and therefore $\alpha_i+\alpha_0 \le \beta_i+\beta_0$, $i = \overline{1,d}$.

For the converse, if for each $i = \overline{1,d}$ it holds $\alpha_i + \alpha_0 \le \beta_i + \beta_0$ then $\text{Poisson}(\alpha_i+\alpha_0) \prec_{\text{hr}} \text{Poisson}(\beta_i+\beta_0)$. Assuming moreover that both random vectors have independent marginals, it follows that

$$\bigotimes_{i=1}^{d} \text{Poisson}(\alpha_i + \lambda_0) \prec_{\text{hr}} \bigotimes_{i=1}^{d} \text{Poisson}(\beta_i + \lambda_0),$$

that is, $\text{Poisson}(\alpha) \prec_{\text{hr}} \text{Poisson}(\beta)$. □

Proposition 8. *Let be* $\mu, \mu' \in \mathbb{R}^d$ *and* $\sum \geq 0$. *If* $N(\mu, \sum) \prec_{\mathrm{hr}} N(\mu', \sum)$ *then* $\mu \leq \mu'$. *The converse holds if both distributions have independent marginals.*

Proof. From $N(\mu, \sum) \prec_{\mathrm{hr}} N(\mu', \sum)$ it follows $N(\mu_i, \sum\limits_{ii}) \prec_{\mathrm{hr}} N(\mu_i', \sum\limits_{ii})$, which in turn implies that $\mu_i \leq \mu_i'$ for all $i = \overline{1, d}$, so that $\mu \leq \mu'$.

Conversely, $\mu \leq \mu'$ means $\mu_i \leq \mu_i'$ for each $i = \overline{1, d}$. Therefore, one has $N(\mu_i, \sum\limits_{ii}) \prec_{\mathrm{hr}} N(\mu_i', \sum\limits_{ii})$, and hence $\otimes_{i=1}^d N(\mu_i, \sum\limits_{ii}) \prec_{\mathrm{hr}} \otimes_{i=1}^d N(\mu_i', \sum\limits_{ii})$, in other words $N(\mu, \sum) \prec_{\mathrm{hr}} N(\mu', \sum)$. $\square$

Proposition 9. *We have* $B(1, p) \prec_{\mathrm{hr}} B(1, q) \Leftrightarrow \frac{q_4}{p_4} \geq \max\left(1, \frac{q_2}{p_2}, \frac{q_3}{p_3}, \frac{q_2+q_3}{p_2+p_3}\right)$.

Proof. Let X and Y be the random vector for $B(1,p)$ and $B(1,q)$, respectively. If $X \prec_{\mathrm{hr}} Y$ then $X \prec_{\mathrm{whr}} Y$, so that $\frac{G^*}{F^*}$ is increasing on $\mathrm{Supp}(G^*)$, or $\frac{G^*}{F^*}(x) \leq \frac{G^*}{F^*}(y)$, for all $x, y \in \mathrm{Supp}(G^*)$ with $x \leq y$. We specialize the vectors x, y to convenient values.

For $x = (-\infty, -\infty), y = \left(\frac{1}{2}, \frac{1}{2}\right)$ we get that $\frac{q_4}{p_4} \geq 1$.

For $x = \left(-\frac{1}{4}, \frac{1}{4}\right), y = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $x = \left(\frac{1}{4}, -\frac{1}{4}\right), y = \left(\frac{1}{2}, \frac{1}{2}\right)$ we get that $\frac{q_4}{p_4} \geq \frac{q_i+q_4}{p_i+p_4}, i = 2, 3$. But it is easy to verify the following implication:

$$\frac{q_4}{p_4} \geq \frac{q_i + q_4}{p_i + p_4}, i = 2, 3 \Rightarrow \frac{q_4}{p_4} \geq \frac{q_i}{p_i}, i = 2, 3.$$

For $x = (0, 0), y = \left(\frac{1}{2}, \frac{1}{2}\right)$ we obtain $\frac{q_4}{p_4} \geq \frac{q_2+q_3+q_4}{p_2+p_3+p_4}$, and this implies $\frac{q_4}{p_4} \geq \frac{q_2+q_3}{p_2+p_3}$.

So $\frac{q_4}{p_4} \geq \max\left(1, \frac{q_2}{p_2}, \frac{q_3}{p_3}, \frac{q_2+q_3}{p_2+p_3}\right)$.

Conversely, it is easy to verify that $F^*(x)G^*(y) \leq F^*(x \wedge y)G^*(x \vee y)$, $\forall x, y \in \mathbb{R}^d$. $\square$

References

[1] M. Shaked, J. G. Shanthikumar, *Stochastic Orders*, Springer, New York, 2007.

[2] Gh. Zbăganu, *Metode Matematice în Teoria Riscului şi Actuariat*, Ed. Univ. Bucureşti, 2004.

PROBLEMS

Authors should submit proposed problems to `gmaproblems@rms.unibuc.ro`. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of May 2019**.

PROPOSED PROBLEMS

472. Let $a, b, c \in [0, \frac{\pi}{2}]$ such that $a+b+c = \pi$. Prove the following inequality:

$$\sin a + \sin b + \sin c \geq 2 + 4\left|\sin\left(\frac{a-b}{2}\right)\sin\left(\frac{b-c}{2}\right)\sin\left(\frac{c-a}{2}\right)\right|.$$

Proposed by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania and Jiahao He, South China University of Technology, People's Republic of China.

473. Let $e_1, \ldots, e_n$ be the elementary symmetric polynomials in the variables $X_1, \ldots, X_n$,

$$e_k(X_1, \ldots, X_n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} X_{i_1} \cdots X_{i_k},$$

and let S be the ideal generated by $e_1, \ldots, e_n$ in $\mathbb{R}[X_1, \ldots, X_n]$.

Then every monomial $X_1^{m_1} \cdots X_n^{m_n}$ with the degree $m = m_1 + \cdots + m_n$ strictly greater than $\binom{n}{k}$ belongs to S. On the other hand, there exists a monomial of degree $\binom{n}{k}$ which does not belong to S.

Proposed by George Stoica, New Brunswick, Canada.

474. Calculate $\displaystyle\sum_{n=1}^{\infty}(2n-1)\left[\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots\right)^2 - \frac{1}{n^2}\right]$.

Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

475. We say that a function $f : \mathbb{R} \to \mathbb{R}$ has the property (P) if it is continous and

$$2f(f(x)) = 3f(x) - x \quad \text{for all } x \in \mathbb{R}.$$

a) Prove that if f has property (P) then $M = \{x \in \mathbb{R} : f(x) = x\}$ is a nonempty interval.
b) Find all functions with property (P).

Proposed by Dan Moldovan and Bogdan Moldovan, Cluj-Napoca, Romania.

476. Calculate the integral
$$\int_0^\infty \frac{\operatorname{arctg} x}{\sqrt{x^4+1}} \mathrm{d}x.$$

Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.

477. For every complex matrix A we denote by A^* its adjoint, i.e., the transposed of its conjugate, $A^* = \bar{A}^T$. If A is square and $A = A^*$ we say that A is self-adjoint (or Hermitian). In this case for every complex vector x we have $x^*Ax \in \mathbb{R}$. If A, B are self-adjoint we say that $A \geq B$ if $x^*Ax \geq x^*Bx$ for every complex vector x.

For a complex matrix A we denote $|A|^2 = AA^*$. Note that $|A|^2$ is self-adjoint and ≥ 0. (If $A^*x = y = (y_1, \dots, y_n)^T$ then $x^*|A|^2x = y^*y = (\overline{y}_1, \dots, \overline{y}_n)(y_1, \dots, y_n)^T = |y_1|^2 + \dots + |y_n|^2 \geq 0$.)

Let A be a square matrix with complex coefficients and I the identity matrix of the same order. Then the following statements are equivalent:

(i) $|I + zA|^2 = |I - zA|^2$ for all $z \in \mathbb{C}$;

(ii) $|I + zA|^2 \geq I$ for all $z \in \mathbb{C}$;

(iii) $A = 0$.

Are these statements still equivalent if we replace "complex" by "real" throughout?

Proposed by George Stoica, New Brunswick, Canada.

478. Determine the largest positive constant k such that for every $a, b, c \geq 0$ with $a^2 + b^2 + c^2 = 3$ we have
$$(a + b + c)^2 + k|(a - b)(b - c)(c - a)| \leq 9.$$

Proposed by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania.

479. Let p be an odd prime number and $A \in \mathcal{M}_p(\mathbb{Q})$ a matrix such that $\det(A^p + I_p) = 0$ and $\det(A + I_p) \neq 0$. Prove that:

a) $\operatorname{Tr}(A)$ is an eigenvalue of $A + I_p$.

b) $\det(A + I_p) - \det(A - I_p) = (p - 1) \operatorname{Tr}(A) + 2$.

Proposed by Vlad Mihaly, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

480. Let k, n be natural numbers, $x_1, x_2, \dots, x_k$ be distinct complex numbers and the matrix $A \in \mathcal{M}_n(\mathbb{C})$ such that $(A - x_1I_n)(A - x_2I_n) \cdots (A - x_kI_n) = O_n$. Prove that $\operatorname{rank}(A - x_1I_n) + \operatorname{rank}(A - x_2I_n) + \dots + \operatorname{rank}(A - x_kI_n) = n(k - 1)$.

Proposed by Dan Moldovan and Vasile Pop, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

481. Let K be a field and let $n \geq 1$. Let $A, B \in M_n(K)$ such that $[A, B]$ commutes with A or B.

If $\operatorname{char} K = 0$ or $\operatorname{char} K > n$ then it is known that $[A, B]$ is nilpotent, i.e., $[A, B]^n = 0$.

Prove that this result no longer holds if $0 < \operatorname{char} K \leq n$.

(Here by $[\cdot, \cdot]$ we mean the commutator, $[A, B] = AB - BA$.)

Constantin-Nicolae Beli, IMAR, Bucharest, Romania.

SOLUTIONS

455. Let $n \geq 2$ be an integer. Determine the largest number of real solutions the equation $a_1\sqrt{x + b_1} + \cdots + a_n\sqrt{x + b_n} = 0$ can have. Here $a_1, \ldots, a_n$ are real numbers, not all zero, and $b_1, \ldots, b_n$ are mutually distinct numbers.

Proposed by Marius Cavachi, Ovidius University, Constanţa, Romania.

Editor's note. Unfortunately, the solution provided by the author is wrong. A reformulation of the problem is the following:

Given $n \geq 1$ we want to find the largest m for which there are mutually distinct $b_1, \ldots, b_n$, mutually distinct $x_1, \ldots, x_m$ and not all zero $a_1, \ldots, a_n$ such that $\sum_{j=1}^{n} a_j \sqrt{x_i + b_j} = 0$ for $1 \leq i \leq m$. Equivalently, if $A \in M_{m,n}(\mathbb{R})$, $A = (\sqrt{x_i + b_j})_{i,j}$, then the equation $AX = 0$ has a non-trivial real solution, viz., $X = (a_1, \ldots, a_n)^T$. For this to happen one needs that $\operatorname{rank} A \leq n - 1$. Hence m is the largest number such that there are mutually distinct $b_1, \ldots, b_n$ and mutually distinct $x_1, \ldots, x_m$ such that the rank of the $m \times n$ matrix $A = (\sqrt{x_i + b_j})_{i,j}$ is $\leq n - 1$. Obviously, when $m \leq n - 1$, regardless of the values of x_i and b_j, we have $\operatorname{rank} A \leq m \leq n - 1$, so it is possible to have m solutions. The author claims that when $m = n$, i.e., A is a square matrix, we always have $\det A \neq 0$ so $\operatorname{rank} A = n$, so it is not possible to have n solutions. Therefore the required maximum is $n - 1$.

More precisely, if we rearrange the x_i's and b_j's such that $x_1 < \cdots < x_n$ and $b_1 < \cdots < b_n$, the claim is that $(-1)^{n-1} \det A > 0$. The proof is based on the fact that the function $f : [0, \infty) \to [0, \infty)$, $f(x) = \sqrt{x}$, is strictly concave. Unfortunately, while the proof is very ingenious, it contains an error. It seems that the result is true when $n \leq 3$. We don't know what happens when $n \geq 4$.

456. Let f, g, h be non-negative continuous functions on $[0, 1]$ satisfying the inequality $f(tx + (1 - t)y) \geq g^t(x)h^{1-t}(y)$ for all $x, y \in [0, 1]$ and some (fixed)

$t \in (0,1)$. If, in additon, we have $\displaystyle\int_0^1 g(x)\mathrm{d}x = \int_0^1 h(x)\mathrm{d}x = 1$, prove that $\displaystyle\int_0^1 f(x)\mathrm{d}x \geq 1$.

`Proposed by George Stoica, University of New Brunswick, Saint John, New Brunswick, Canada.`

Solution by the author. We assume first that g, h are positive, not merely non-negative on $[0,1]$. Then $y \mapsto \int_0^y g(s)\mathrm{d}s$ and $y \mapsto \int_0^y h(s)\mathrm{d}s$ are strictly increasing and differentiable functions sending $[0,1]$ to $[0,1]$. We denote by $u, v : [0,1] \to [0,1]$ their inverses. Then u, v are strictly increasing and differentiable with $u(0) = v(0) = 0$ and $u(1) = v(1) = 1$.

We have

$$\int_0^{u(x)} g(s)\mathrm{d}s = \int_0^{v(x)} h(s)\mathrm{d}s = x.$$

By differentiating we get

$$u'(x)g\big(u(x)\big) = v'(x)h\big(v(x)\big) = 1. \tag{1}$$

Define $w(x) = tu(x) + (1-t)v(x)$. Along with u and v, $w : [0,1] \to [0,1]$ is strictly increasing and differentiable with $w(0) = 0$, $w(1) = 1$. Using the inequality between the arithmetic and geometric means, we have

$$w'(x) = tu'(x) + (1-t)v'(x) \geq (u'(x))^t (v'(x))^{1-t}. \tag{2}$$

From (1) and (2), the hypotheses and the change of variables $s = w(x)$, it follows that

$$\begin{aligned}
\int_0^1 f(s)\mathrm{d}s &\geq \int_0^1 f(w(x)) \cdot w'(x)\mathrm{d}x \\
&\geq \int_0^1 g^t(u(x))h^{1-t}(v(x))(u'(x))^t(v'(x))^{1-t}\mathrm{d}x \\
&= \int_0^1 \left(u'(x)g(u(x))\right)^t \left(u'(x)g(u(x))\right)^{1-t} \mathrm{d}x = \int_0^1 1\mathrm{d}x = 1.
\end{aligned}$$

If g, h are only non-negative then for any $0 < \delta < 1$ we define $g_\delta(x) = (1-\delta)g(x) + \delta$ and $h_\delta(x) = (1-\delta)h(x) + \delta$. Same as for g, h, we have $\displaystyle\int_0^1 g_\delta(x)\mathrm{d}x = \int_0^1 h_\delta(x)\mathrm{d}x = 1$. However, g_δ, h_δ are positive, not merely non-negative. Then, by continuity and the compactness of $[0,1] \times [0,1]$, from $f(tx + (1-t)y) \geq g^t(x)h^{1-t}(y)$ we get that for any $\varepsilon > 0$ there is some $0 < \delta < 1$ small enough such that $f(tx + (1-t)y) + \varepsilon \geq g_\delta^t(x)h_\delta^{1-t}(y)$ $\forall 0 \leq x, y \leq 1$. Then, by applying the case we have just proved to g_δ, h_δ and $f_\varepsilon := f + \varepsilon$, we get $\int_0^1 (f(x) + \varepsilon)\mathrm{d}x \geq 1$, i.e., $\int_0^1 f(x)\mathrm{d}x + \varepsilon \geq 1$. Since this happens for every $\varepsilon > 0$, we get our result. $\square$

457. Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be so that $A^2 + B^2 + A - B = 2(AB + I_n)$. Prove the following equalities:

a) $\operatorname{Tr}\big((A-B)(A-B+I_n)\big) = 2n$,

b) $\det\big((A-B)(A-B+I_n)\big) = 2^n$.

`Proposed by Vasile Pop, Tehnical University of Cluj-Napoca, Cluj-Napoca, Romania.`

Solution by the author. We use the notation $[\cdot,\cdot]$ for the commutator $[X,Y] = XY - YX$.

The matrix $M = (A-B)(A-B+I_n) = (A-B)^2 + A - B$ writes as

$$M = A^2 + B^2 - AB - BA + A - B = (AB - BA) + 2I_n = [A,B] + 2I_n.$$

Since $\operatorname{Tr}[A,B] = 0$ we get $\operatorname{Tr} M = \operatorname{Tr}(2I_n) = 2n$, i.e., we have a).

Let $C = A - B$, so that $M = C(C + I_n)$. Note that $[C,B] = [A-B,B] = [A,B]$. Then we have

$$C(C+I_n) = [C,B] + 2I_n \tag{3}$$

and equivalently

$$[C,B] = C^2 + C - 2I_n. \tag{4}$$

If we note $D = [C,B] = \overline{C^2 + C + I_n}$ then D commutes with C. We prove that $\operatorname{Tr}(D^k) = 0$, $k = \overline{1,n}$. Indeed, we have

$$\begin{aligned} D^k &= D^{k-1}\cdot D = D^{k-1}(CB - BC) \\ &= D^{k-1}CB - D^{k-1}BC = C(D^{k-1}B) - (D^{k-1}B)C, \end{aligned}$$

so that $D^k = \left[C, D^{k-1}B\right]$, which implies $\operatorname{Tr}\left(D^k\right) = 0$. In conclusion

$$\operatorname{Tr}(D) = \operatorname{Tr}\left(D^2\right) = \cdots = \operatorname{Tr}\left(D^n\right) = 0,$$

so that, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the matrix D, from the system

$$\begin{aligned} \lambda_1 + \lambda_2 + \cdots + \lambda_n &= 0, \\ \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2 &= 0, \\ &\vdots \\ \lambda_1^n + \lambda_2^n + \cdots + \lambda_n^n &= 0, \end{aligned}$$

we obtain $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$ (from the Newton formulas the polynomial f_D with the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ is unique, more precisely $f_D(\lambda) = \lambda^n$.) From the equality (3) it results that the eigenvalues of the matrix $C \cdot (C + I_n) = D + 2I_n$ are $\mu_1 = \mu_2 = \cdots = \mu_n = 2$ and their product is

$$\det(C(C+I_n)) = 2^n,$$

i.e., the equality b).

Note. The method used here to prove that $\lambda_1 = \cdots = \lambda_n = 0$, i.e., that D is nilpotent, can be used to prove a more general result: If X, Y are square matrices of the same dimension n and $[X, Y]$ commutes with X then $[X, Y]$ is nilpotent. In our case $X = C$, $Y = B$ and we have that $[C, B] = C^2 + C - 2I_n$ commutes with C. The result holds, with the same proof, over every field K of characteristic 0. If the characteristic is $p > 0$ then it holds only if $n < p$. See problem 481 from the current issue of GMA.

458. (**Corrected**) For a continuous and non-negative function f on $[0, 1]$ we define the Hausdorff moments

$$\mu_n := \int_0^1 x^n f(x)\mathrm{d}x, \quad n = 0, 1, 2, \ldots.$$

Prove that

$$\mu_{n+2} t\mu_0^2 + \mu_n\mu_1^2 \geq 2\mu_{n+1}\mu_1\mu_0, \quad n = 0, 1, 2, \ldots.$$

Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, USA

Solution by the author. Let us recall the celebrated Schur's inequality

$$x^n(x-y)(x-z) + y^n(y-z)(y-x) + z^n(z-x)(z-y) \geq 0 \tag{5}$$

for all $x, y, z \geq 0$ and $n \geq 1$. Then we have

$$J := \int_0^1\int_0^1\int_0^1 (X+Y+Z)f(x)f(y)f(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \geq 0,$$

where $X = x^n(x-y)(x-z)$, $Y = y^n(y-z)(y-x)$ and $Z = z^n(z-x)(z-y)$. We have $J = J_X + J_Y + J_Z$, where $J_X = \int_0^1\int_0^1\int_0^1 Xf(x)f(y)f(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$ and similarly for J_Y, J_Z. But, by symmetry reasons, $J_X = J_Y = J_Z$. Thus $3J_X = J \geq 0$, so $J_X \geq 0$. We have

$$\begin{aligned}
J_X &= \int_0^1\int_0^1\int_0^1 (x^{n+2} + x^n yz - x^{n+1}y - x^{n+1}z)f(x)f(y)f(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_0^1 x^{n+2}f(x)\mathrm{d}x \int_0^1 f(y)\mathrm{d}y \int_0^1 f(z)\mathrm{d}z \\
&\quad + \int_0^1 x^n f(x)\mathrm{d}x \int_0^1 yf(y)\mathrm{d}y \int_0^1 zf(z)\mathrm{d}z \\
&\quad - \int_0^1 x^{n+1}f(x)\mathrm{d}x \int_0^1 yf(y)\mathrm{d}y \int_0^1 f(z)\mathrm{d}z \\
&\quad - \int_0^1 x^{n+1}f(x)\mathrm{d}x \int_0^1 f(y)\mathrm{d}y \int_0^1 zf(z)\mathrm{d}z \\
&= \mu_{n+2}\mu_0\mu_0 + \mu_n\mu_1\mu_1 - \mu_{n+1}\mu_1\mu_0 - \mu_{n+1}\mu_0\mu_1,
\end{aligned}$$

so $\mu_{n+2}\mu_0^2 + \mu_n\mu_1^2 - 2\mu_{n+1}\mu_0\mu_1 = J_X \geq 0$. Hence the conclusion. $\square$

Solution by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania. If $f \equiv 0$ then we have nothing to prove. So we assume that $f \not\equiv 0$, which implies that all μ_i are positive.

The relation to prove writes as

$$\frac{\mu_{n+2}}{\mu_{n+1}} \cdot \frac{\mu_0}{\mu_1} + \frac{\mu_n}{\mu_{n+1}} \cdot \frac{\mu_1}{\mu_0} \geq 2.$$

By the Arithmetic Mean–Geometric Mean inequality we have

$$\frac{\mu_{n+2}}{\mu_{n+1}} \cdot \frac{\mu_0}{\mu_1} + \frac{\mu_n}{\mu_{n+1}} \cdot \frac{\mu_1}{\mu_0} \geq 2\sqrt{\frac{\mu_{n+2}\mu_n}{\mu_{n+1}^2}}.$$

Hence it suffices to prove that $\mu_{n+2}\mu_n \geq \mu_{n+1}^2$. But this follows from the Cauchy-Bunyakovsky-Schwarz inequality:

$$\begin{aligned}\mu_{n+2}\mu_n &= \left(\int_0^1 x^{n+2} f(x)\mathrm{d}x\right)\left(\int_0^1 x^n f(x)\mathrm{d}x\right)\\ &\geq \left(\int_0^1 \sqrt{x^{n+2}f(x)x^n f(x)}\mathrm{d}x\right)^2 = \left(\int_0^1 x^{n+1} f(x)\mathrm{d}x\right)^2 = \mu_{n+1}^2.\end{aligned}$$

The proof is complete. □

459. The faces of an icosahedron are colored with blue or white such that a blue face cannot be adjacent to more than two other blue faces. What is the largest number of blue faces that can be obtained following this rule?

(Two faces are considered adjacent if they share an edge.)

Proposed by Eugen J. Ionaşcu, Columbus State University, Columbus, GA, USA.

Solution by the author. We prove that the answer is 14 and the coloring that gives this maximum is the one shown in Figure 1(a), which is a Schlegel diagram of the icosahedron in which the projection is done from a point close to a blue face. (That is, there is an extra blue face that cannot be seen.)

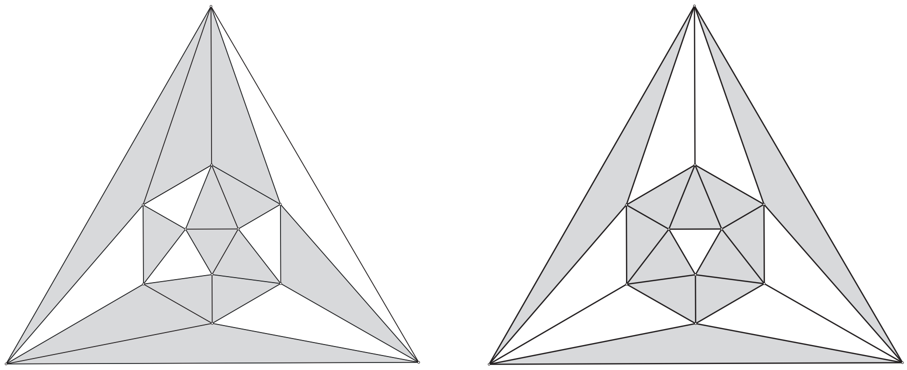

Figure 1(a) Figure 1(b)

Figure 1. Icosahedral graph

Indeed, for the each of the 20 faces $T_1, T_2, \ldots, T_{20}$ we assign a Boolean variable x_i, $i = 1, \ldots, 20$, which is equal to 1 if the face T_i is blue or 0 if the face T_i is white, given a certain coloring. Let us denote by N_i the indices j for which face T_j is adjacent to T_i. The condition we require is then written as

$$x_i + \sum_{j \in N(i)} x_j \leq 3, \; i = 1, \ldots, 20. \tag{6}$$

Clearly, we want to maximize $S = \sum_{1}^{20} x_i$. After summing all these inequalities up over $i = 1, \ldots, 20$, we obtain $S + 3S \leq 60$. Hence, it follows that $S \leq 15$. Since we have arrangement that accomplishes 14 blue faces we only need to prove that 15 blue faces are not possible to be arranged without violating one of our requirements.

So, by way of contradiction, let us assume that it is possible to have 15 blue faces satisfying the requirement of a 2-dependence set (as in the text of our problem). Then all the inequalities in (6) have to be equalities. So, for every face that is white we need to have exactly three blue around it and for every face that is blue we need to have exactly two around it. Then we are forced to have a coloring as in Figure 1(b) if we start with a white face (the one in the middle) and then we end up with too many white faces (at least 7), which is in contradiction to the number of faces left possible (i.e., $20 - 15 = 5$). $\square$

460. Let X be a set with at least two elements, and fix $a, b \in X$, $a \neq b$. We define the function $f : X^3 \to X$ by

$$f(x, y, z) = \begin{cases} a \text{ if } x, y, z \neq a, \\ b \text{ if } a \in \{x, y, z\}. \end{cases}$$

Is there a binary operation $* : X^2 \to X$ such that $f(x, y, z) = (x * y) * z$ for all $x, y, z \in X$?

`Proposed by George Stoica, University of New Brunswick, Saint John, New Brunswick, Canada.`

Solution by the author. The answer is no. Indeed, let us assume that, for every $x, y \in X$, $x, y \neq a$, we have $x * y \neq a$. Then, for every $x, y, z \in X$, $x, y, z \neq a$, we have

$$a = f(x, y, z) = (x * y) * z \neq a,$$

a contradiction. Hence, there exists $u, v \in X$, $u, v \neq a$, such that $u * v = a$. From here it follows that

$$a * a = (u * v) * a = f(u, v, a) = b$$

and then, for every $x \in X$ we have

$$b * x = (a * a) * x = f(a, a, x) = b.$$

Finally,

$$a = f(b, u, v) = (b * u) * v = b * v = b,$$

a contradiction with $a \neq b$.

461. Let $A, B \in \mathcal{M}_n(\mathbb{R})$ so that $A^2 = A$, $B^2 = B$, and $\det(2A + B) = 0$. Prove that $\det(A + 2B) = 0$.

`Proposed by Vasile Pop, Tehnical University of Cluj-Napoca, Cluj-Napoca, Romania.`

Solution by Francisco Perdomo and Ángel Plaza, Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, España.

If $\det(2A+B) = 0$ then there exists $v \neq 0$ in $\mathbb{R}^n$ such that $(2A+B)v = 0$. From $(I - A)(2A + B)v = 0$ and $(I - A)A = 0$ we get $(I - A)Bv = 0$ and so $Bv = ABv$. Similarly, from $(I - B)B = 0$ and $(I - B)(2A + B)v = 0$ we get $Av = BAv$. Then $(A + 2B)^2v = A^2v + 2ABv + 2BAv + 4B^2v = Av + 2Bv + 2Av + 4Bv = (3A + 6B)v$ and so $(A + 2B)(A + 2B - 3I)v = 0$, where I is the identity matrix.

If $(A+2B-3I)v = 0$, then $(A+2B)v = 3v$. Together with $(2A+B)v = 0$, this implies that $Bv = 2v$. But B is idempotent so its only possible eigenvalues are 0 and 1. Hence $(A + 2B - 3I)v \neq 0$. Thus $\det(A + 2B) = 0$, because $(A+2B)w = 0$ has the non trivial solution $(A+2B-3I)v$ in $\mathbb{R}^n$. □

The *author's solution* is essentially the same up to the relation $(A + 2B)^2v = 3(A + 2B)v = 0$. Here he assumes that $\det(A + 2B) \neq 0$ and concludes that $(A + 2B)v = 3v$. Together with $(2A + B)v = 0$, this implies that $Av = -v$. Hence $A^2v = -Av = v$. But $A^2v = Av = -v$, so $v = 0$. Contradiction.

Solution by Moubinool Omarjee, Lycée Henri IV, Paris, France.

We prove that if $A + 2B \in GL_n(\mathbb{R})$ then $\ker(2A + B) = 0$.

Take $x \in \ker(2A + B)$. We have $2Ax + Bx = 0$, i.e., $Bx = -2Ax$. When we multiply to the left by A we get $ABx = -2A^2x = -2Ax = Bx$. When we multiply it to the left by B we get that $B^2x = -2BAx$. But $B^2x = Bx = -2AX$, so $Bx = -2BAx$ and $Ax = BAx$.

Then $(A+2B)^2x = (A^2+2AB+2BA+4B^2)x = Ax+2Bx+2Ax+4Bx = 3(A+2B)x$. Since $A+2B \in GL_n(\mathbb{R})$ we have $(A+2B)x = 3x$. We multiply to the left by A and we get $(A + 2AB)x = 3Ax$, whence $Ax = ABx = Bx$. Since also $2Ax + Bx = 0$, we have $Ax = Bx = 0$ and so $3x = 2Ax + Bx = 0$.

So we proved that if $A + 2B \in GL_n(\mathbb{R})$ then $\ker(2A + B) = \{0\}$, i.e., $2A + B \in GL_n(\mathbb{R})$.

By contraposition, if $\det(2A + B) = 0$ then $\det(A + 2B) = 0$. □

462. If $f:[0,1]\to\mathbb{R}$ is a convex function with $f(0)=0$ then prove that

$$\frac{1}{6}\left(\int_{1/2}^{1} f(x)\mathrm{d}x-\int_{0}^{1/2} f(x)\mathrm{d}x\right)\geq\int_{0}^{1/2} xf(x)\mathrm{d}x.$$

`Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.`

Solution by the author. We need two well known properties of convex functions.

Lemma 1. If I is an interval and $f:I\to\mathbb{R}$ is a convex function then for every $a\in I$ the function $r_a:I\setminus\{a\}\to\mathbb{R}$, $r_a(x)=\frac{f(x)-f(a)}{x-a}$, is increasing.

Lemma 2. If I is an interval and $f:I\to\mathbb{R}$ is a convex function then for every $x,y,z,t\in I$ with $x<y\leq z<t$ we have

$$\frac{f(y)-f(x)}{y-x}\leq\frac{f(t)-f(z)}{t-z}.$$

By Lemma 1 with $a=0$ the function $\frac{f(x)}{x}$ is increasing. It follows that for every $x\in(0,1/2]$, $y\in[0,1/2]$ we have $\frac{f(x)}{x}\leq\frac{f(x+y)}{x+y}$, so

$$xf(x)+yf(x)\leq xf(x+y)\quad\forall x,y\in[0,1/2].$$

(When $x=0$ we have equality, $0=0$.)

We integrate this inequality first with respect to y, then with respect to x. We get

$$\frac{1}{2}xf(x)+\frac{1}{8}f(x)\leq x\int_{0}^{1/2} f(x+y)\mathrm{d}y=x\int_{x}^{x+1/2} f(t)\mathrm{d}t\quad\forall x\in\left[0,\frac{1}{2}\right],$$

so

$$\frac{1}{2}\int_{0}^{1/2} xf(x)\mathrm{d}x+\frac{1}{8}\int_{0}^{1/2} f(x)\mathrm{d}x\leq\int_{0}^{1/2}\left(\frac{1}{2}x^2\right)'\left(\int_{x}^{x+1/2} f(t)\mathrm{d}t\right)\mathrm{d}x$$
$$=\frac{1}{8}\int_{1/2}^{1} f(x)\mathrm{d}x-\frac{1}{2}\int_{0}^{1/2} x^2\left(f\left(x+\frac{1}{2}\right)-f(x)\right)\mathrm{d}x.$$

Here in the right hand side of the inequality we used integration by parts and the fact that the derivative of $x\mapsto\int_x^{x+1/2} f(t)\mathrm{d}t$ is $f(x+1/2)-f(x)$.

We now use Lemma 2. If $x,y\in[0,1/2]$ with $x<y$ then $0\leq x<y\leq x+1/2<y+1/2\leq 1$, so

$$\frac{f(y)-f(x)}{y-x}\leq\frac{f\left(y+\frac{1}{2}\right)-f\left(x+\frac{1}{2}\right)}{\left(y+\frac{1}{2}\right)-\left(x+\frac{1}{2}\right)},$$

which implies that $f(x+1/2)-f(x)\leq f(y+1/2)-f(y)$. Hence the function $x\mapsto f(x+1/2)-f(x)$, $x\in[0,1/2]$, is increasing. Then, by applying the

Chebyshev inequality ($\int_a^b f(t)g(t)\mathrm{d}t \geq \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \int_a^b g(t)\mathrm{d}t$ if f and g have the same monotony) we get

$$\begin{aligned}\int_0^{1/2} x^2\left(f\left(x+\frac{1}{2}\right)-f(x)\right)\mathrm{d}x &\geq 2\int_0^{1/2} x^2\mathrm{d}x\int_0^{1/2}\left(f\left(x+\frac{1}{2}\right)-f(x)\right)\mathrm{d}x\\ &= \frac{1}{12}\left(\int_{1/2}^1 f(x)\mathrm{d}x - \int_0^{1/2} f(x)\mathrm{d}x\right).\end{aligned}$$

In conclusion, we get

$$\begin{aligned}\frac{1}{2}\int_0^{1/2} xf(x)\mathrm{d}x + \frac{1}{8}\int_0^{1/2} f(x)\mathrm{d}x \leq &\frac{1}{8}\int_{1/2}^1 f(x)\mathrm{d}x\\ &-\frac{1}{24}\left(\int_{1/2}^1 f(x)\mathrm{d}x - \int_0^{1/2} f(x)\mathrm{d}x\right)\end{aligned}$$

and equivalently

$$12\int_0^{1/2} xf(x)\mathrm{d}x \leq 2\int_{1/2}^1 f(x)\mathrm{d}x - 2\int_0^{1/2} f(x)\mathrm{d}x.$$

When we divide by 12 we get the required result. $\square$

Solution by the editor. There is an alternative approach using a general property of convex functions. If $a < b$ we denote by $C([a,b])$ the set of all continuous functions on $[a,b]$ with values in $\mathbb{R}$. Then $C([a,b])$ is a vector space over $\mathbb{R}$ and it has a metric topology, with the distance given by the norm $||\cdot||$, where $||f|| = \max_{x\in[a,b]} |f(x)|$.

Lemma. If $T : C([a,b]) \to \mathbb{R}$ is a linear continuous operator then $T(f) \geq 0$ for every convex function $f : [a,b] \to \mathbb{R}$ if and only if the following hold:

(1) $T(1) = T(x) = 0$. Equivalently, $T(f) = 0$ for any affine function f, i.e., when f has the form $f(x) = mx + c$.

(2) $T(f_c) \geq 0$ $\forall c \in (a,b)$, where

$$f_c(x) = \begin{cases} 0, & x \in [a,c],\\ x-c, & x \in [c,b].\end{cases}$$

This result is already known but we couldn't provide a reference. For the sake of self-containment we sketch a proof here.

For the necessity note that if f is affine so is $-f$. Hence f, $-f$ are both convex and we have $T(f) \geq 0$ and $-T(f) = T(-f) \geq 0$, i.e., $T(f) = 0$. Since 1 and x form a basis for all affine functions, in order that $T(f) = 0$ for every affine function on $[a,b]$ it suffices to have $T(1) = T(x) = 0$. Since f_c are convex we also have $T(f_c) \geq 0$ $\forall c \in (a,b)$.

Before proving the reverse implication, note that for every $c \in (a,b)$, f_c is continuous and it is differentiable everywhere except at c and we have $f'(x) = 0$ if $x < c$ and $f'(x) = 1$ if $x > c$. Also note that $f_c(a) = 0$.

Assume first that f is a convex function and its graph is a broken line with the vertices at $(c_0, d_0), \dots, (c_s, d_s)$, where $a = c_0 < c_1 < \cdots < c_s = b$ is a division of the interval $[a,b]$. Let $m_i = \frac{d_i - d_{i-1}}{c_i - c_{i-1}}$ be the slope of the graph on the interval $[c_{i-1}, c_i]$. Since f is convex we have $m_1 \le \cdots \le m_s$. The function f is continuous on $[a,b]$ and differentiable everywhere except at $c_0, \dots, c_s$. Namely, if $1 \le i \le s$ then $f'(x) = m_i \ \forall x \in (c_{i-1}, c_i)$. We claim that

$$f(x) = m_1(x-a) + d_0 + (m_2 - m_1) f_{c_1}(x) + \cdots + (m_s - m_{s-1}) f_{c_{s-1}}(x).$$

If we denote by $g(x)$ the right side of the above equality then the easiest way to verify that $f = g$ is to prove that g has the same properties with f: g is continuous, $g(a) = f(a)$ and $g'(x) = f'(x) \ \forall x \in [a,b]$, $x \neq c_0, \dots, c_s$. The continuity is obvious since each f_c is continuous. Since $f_c(a) = 0 \ \forall c \in (a,b)$ we have $g(a) = m_1(a-a) + d_0 = d_0 = f(c_0) = f(a)$. For the third condition let $x \in [a,b]$, $x \neq c_0, \dots, c_s$. Then $c_{i-1} < x < c_i$ for some $1 \le i \le s$, so $f'(x) = m_i$. We have

$$f'(x) = m_1 + (m_2 - m_1) f'_{c_1}(x) + \cdots + (m_s - m_{s-1}) f'_{c_{s-1}}(x).$$

If $j \le i-1$ then $c_j \le c_{i-1} < x$, so $f'_{c_j}(x) = 1$. If $j \ge i$ then $c_j \ge c_i > x$, so $f'_{c_j}(x) = 0$. It follows that

$$g'(x) = m_1 + (m_2 - m_1) + \cdots + (m_i - m_{i-1}) = m_i = f'(x).$$

By the linearity of T we get

$$T(f) = T(m_0(x-a) + d_0) + (m_2 - m_1) T(f_{c_1}) + \cdots + (m_s - m_{s-1}) T(f_{c_{s-1}}).$$

But $m_0(x-a)+d_0$ is affine, so by property (1) $T(m_0(x-a)+d_0) = 0$, and by property (2) $T(f_{c_1}) \ge 0$ for $1 \le i \le s-1$. But we also have $m_1 \le \cdots \le m_s$, so each $m_i - m_{i-1}$ is non-negative. We conclude that $T(f) \ge 0$.

Suppose now that $f : [a,b] \to \mathbb{R}$ is convex arbitrary. For each $n \ge 1$ we define the function $f_n : [a,b] \to \mathbb{R}$ whose graph is the broken line with vertices at $(a + \frac{i}{n}(b-a), f(a + \frac{i}{n}(b-a)))$ with $0 \le i \le n$. Note that $a + \frac{i}{n}(b-a)$ with $0 \le i \le n$ make a partition of $[a,b]$ into n equal intervals. Also note that $(a + \frac{i}{n}(b-a), f(a + \frac{i}{n}(b-a)))$ are points on the graph of f, which is convex, so f_n will be convex, as well. Since the graph of f_n is a broken line, by the particular case we have just proved, we have $T(f_n) \ge 0$.

Since f is continuous on a compact interval, it is uniformly continuous. Then for any $\varepsilon > 0$ there is n_ε such that for any $n \ge n_\varepsilon$ we have

$$|f(x) - f(y)| < \varepsilon \quad \forall x, y \in [a,b] \quad \text{with} \quad |x - y| \le \frac{1}{n}(b-a).$$

Let $x \in [a,b]$. Then $x \in [a+\frac{i-1}{n}(b-a), a+\frac{i}{n}(b-a)]$ for some $1 \le i \le n$. Since the length of $[a+\frac{i-1}{n}(b-a), a+\frac{i}{n}(b-a)]$ is $\frac{1}{n}(b-a)$ we have $|x-(a+\frac{i-1}{n}(b-a))|, |x-(a+\frac{i}{n}(b-a))| \le \frac{1}{n}(b-a)$, so $|f(x)-f(a+\frac{i-1}{n}(b-a))|, |f(x)-f(a+\frac{i}{n}(b-a))| < \varepsilon$. It follows that $f(a+\frac{i-1}{n}(b-a)), f(a+\frac{i}{n}(b-a)) \in (f(x)-\varepsilon, f(x)+\varepsilon)$. But x belongs to the interval $[a+\frac{i-1}{n}(b-a), a+\frac{i}{n}(b-a)]$, where f_n is affine. It follows that $f_n(x)$ takes an intermediate value between $f_n(a+\frac{i-1}{n}(b-a)) = f(a+\frac{i-1}{n}(b-a))$ and $f_n(a+\frac{i}{n}(b-a)) = f(a+\frac{i}{n}(b-a))$. Since $f(a+\frac{i-1}{n}(b-a)), f(a+\frac{i}{n}(b-a)) \in (f(x)-\varepsilon, f(x)+\varepsilon)$, this implies $f_n(x) \in (f(x)-\varepsilon, f(x)+\varepsilon)$, i.e., $|f_n(x)-f(x)| < \varepsilon$. In conclusion $||f_n - f|| = \max\limits_{x\in[a,b]} |f_n(x)-f(x)| < \varepsilon$.

Since $||f_n - f|| < \varepsilon$, for $n \ge n_\varepsilon$ we have $\lim\limits_{n\to\infty} f_n = f$. Since T is continuous, this implies $\lim\limits_{n\to\infty} T(f_n) = T(f)$. But $T(f_n) \ge 0$, $\forall n$, so $T(f) \ge 0$. $\square$

We now return to our problem. We must prove that $T(f) \ge 0$ for every convex function $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$, where

$$T(f) = \frac{1}{6}\left(\int_{1/2}^{1} f(x)\mathrm{d}x - \int_0^{1/2} f(x)\mathrm{d}x\right) - \int_0^{1/2} xf(x)\mathrm{d}x.$$

If we try to prove that $T(f) \ge 0$ for all convex functions f we see that this is not true. We have $T(1) = -\frac{1}{8} \ne 0$, so the condition (1) of the Lemma is not satisfied. Therefore we need the information that $f(0) = 0$. The idea is to find a $\lambda \in \mathbb{R}$ such that $\overline{T}(f) \ge 0$ for every convex function $f : [0,1] \to \mathbb{R}$, where $\overline{T}(f) = T(f) + \lambda f(0)$. If we prove that $\overline{T}$ has this property then for a convex function f that has the additional property that $f(0) = 0$ we have $\overline{T}(f) = T(f) + \lambda f(0) = T(f)$, so $\overline{T}(f) \ge 0$ implies $T(f) \ge 0$ and we are done.

When we take $f \equiv 1$ we get $\overline{T}(1) = T(1) + \lambda \cdot 1 = -\frac{1}{8} + \lambda$, so in order that $T'(1) = 0$ we need to take $\lambda = \frac{1}{8}$, so $\overline{T}(f) = T(f) + \frac{1}{8}f(0)$. We have:

$$\overline{T}(1) = \frac{1}{6}\left(\int_{1/2}^{1} 1\mathrm{d}x - \int_0^{1/2} 1\mathrm{d}x\right) - \int_0^{1/2} x\mathrm{d}x + \frac{1}{8}\cdot 1 = 0,$$
$$\overline{T}(x) = \frac{1}{6}\left(\int_{1/2}^{1} x\mathrm{d}x - \int_0^{1/2} x\mathrm{d}x\right) - \int_0^{1/2} x^2\mathrm{d}x + \frac{1}{8}\cdot 0 = 0,$$

so the condition (1) of the Lemma is satisfied.

We now prove the condition (2), i.e., that $\overline{T}(f_c) \ge 0$ $\forall c \in (0,1)$, where $f_c(x) = 0$ if $x \le c$ and $f_c(x) = x - c$ if $x \ge c$. First note that $f_c(0) = 0$, so $\overline{T}(f_c) = T(f_c) + \frac{1}{8}f_c(0) = T(f_c)$ $\forall c \in (0,1)$. We have two cases:

Case 1. $c \le 1/2$. Then on the interval $[1/2, 1]$ we have $f_c(x) = x - c$, while on the interval $[0, 1/2]$ we have $f_c(x) = 0$ for $x \in [0,c]$ and $f_c(x) = x - c$

for $x \in [c, 1/2]$. It follows that

$$\overline{T}(f_c) = T(f_c) = \frac{1}{6}\left(\int_{1/2}^{1}(x-c)\mathrm{d}x - \int_c^{1/2}(x-c)\mathrm{d}x\right) - \int_c^{1/2} x(x-c)\mathrm{d}x$$
$$= -\frac{c^3}{6} - \frac{c^2}{12} + \frac{c}{8} = -\frac{c}{24}(4c^2+2c-3).$$

But we have $0 < c \le 1/2$, so $-\frac{c}{24} < 0$ and $4c^2+2c-3 \le 4(1/2)^2+2(1/2)-3 = -1 < 0$. Thus $T'(f_c) = -\frac{c}{24}(4c^2+2c-3) > 0$.

Case 2. $c \ge 1/2$. Then on $[0,1/2]$ we have $f_c(x) = 0$, while on $[1/2,1]$ we have $f_c(x) = 0$ if $x \in [1/2, c]$ and $f_c(x) = x - c$ if $x \in [c,1]$. We get

$$\overline{T}(f_c) = T(f_c) = \frac{1}{6}\left(\int_c^1 (x-c)\mathrm{d}x - 0\right) - 0 = \frac{1}{6}\left(\frac{c^2}{2} - c + \frac{1}{2}\right) = \frac{(c-1)^2}{12} > 0.$$

Hence $\overline{T}$ satisfies the conditions (1) and (2) of the Lemma, which implies that $\overline{T}(f) \ge 0$ for every convex $f : [0,1] \to \mathbb{R}$, and we are done. $\square$

463. Prove that there exists $n_0 \in \mathbb{N}$ such that $\forall n \ge n_0$ the equation

$$\frac{1}{1+x} + \frac{1}{2+x} + \cdots + \frac{1}{n+x} = \ln n$$

has a unique solution in the interval $(0, \infty)$, denoted by x_n, and that

$$\lim_{n\to\infty} x_n = a,$$

where $a \in (0,1)$ is the unique solution in the interval $(0,1)$ of the equation $x \sum\limits_{i=1}^{\infty} \frac{1}{i(i+x)} = \gamma$, where γ is the Euler constant.

`Proposed by Dumitru Popa, Ovidius University, Constanţa, Romania.`

Solution by the author. For every natural number n let us define $\gamma_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$. As it is well-known, $0 < \gamma_n < 1$, $\forall n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} \gamma_n = \gamma \in (0,1)$. For every natural number n let $h_n : [0,\infty) \to \mathbb{R}$, $h_n(x) = \frac{1}{1+x} + \frac{1}{2+x} + \cdots + \frac{1}{n+x} - \ln n$. Let us note that all h_n are continuous and strictly decreasing on $[0,\infty)$. We have $h_n(0) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n = \gamma_n > 0$, $h_n(1) = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} - \ln n = \gamma_n - 1 + \frac{1}{n+1}$. Since $\lim\limits_{n\to\infty} h_n(1) = \gamma - 1 < 0$, it follows that there exists $n_0 \in \mathbb{N}$ such that $\forall n \ge n_0$ we have $h_n(1) < 0$. Thus there exists $n_0 \in \mathbb{N}$ such that $\forall n \ge n_0$ the equation $h_n(x) = 0$ has a unique solution in the interval $(0,\infty)$ and this solution, denoted by x_n, is in the interval $(0,1)$, that is $0 < x_n < 1$, $\forall n \ge n_0$. For every $n \ge n_0$ we have $h_n(x_n) = 0$, $\frac{1}{1+x_n} + \frac{1}{2+x_n} + \cdots + \frac{1}{n+x_n} = \ln n$, or

$\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\left(\frac{1}{1+x_n}+\frac{1}{2+x_n}+\cdots+\frac{1}{n+x_n}\right)=\gamma_n$, and thus

$$x_n\sum_{i=1}^{n}\frac{1}{i(i+x_n)}=\gamma_n \quad \forall n\geq n_0. \tag{7}$$

For every natural number n define $\varphi_n:[0,1]\to\mathbb{R}$ by

$$\varphi_n(x)=x\sum_{i=1}^{n}\frac{1}{i(i+x)}-\gamma_n=\sum_{i=1}^{n}\left(\frac{1}{i}-\frac{1}{i+x}\right)-\gamma_n.$$

Define also $\varphi:[0,1]\to\mathbb{R}$ by

$$\varphi(x)=x\sum_{i=1}^{\infty}\frac{1}{i(i+x)}-\gamma=\sum_{i=1}^{\infty}\left(\frac{1}{i}-\frac{1}{i+x}\right)-\gamma.$$

Let us observe that $\forall x\in[0,1]$, $\forall i\in\mathbb{N}$ $0\leq\frac{x}{i(i+x)}\leq\frac{1}{i^2}$ and the series $\sum\limits_{i=1}^{\infty}\frac{1}{i^2}$ is convergent, by the Weierstrass criterion, so the series $\sum\limits_{i=1}^{\infty}\frac{x}{i(i+x)}$ is uniformly convergent on $[0,1]$ and its sum is a continuous function, that is, φ is a continuous function. Also φ is a strictly increasing function. Since $\varphi(0)=-\gamma<0$ and $\varphi(1)=\sum\limits_{i=1}^{\infty}\frac{1}{i(i+1)}-\gamma=1-\gamma>0$, it follows that the equation $\varphi(x)=0$ has a unique solution in $(0,1)$, which is denoted in the sequel by $a\in(0,1)$. Moreover, since for every $n\in\mathbb{N}$ and $x\in[0,1]$ it holds

$$0\leq x\sum_{i=1}^{\infty}\frac{1}{i(i+x)}-x\sum_{i=1}^{n}\frac{1}{i(i+x)}=x\sum_{i=n+1}^{\infty}\frac{1}{i\,(i+x)}\leq\sum_{i=n+1}^{\infty}\frac{1}{i^2},$$

it follows that $\lim\limits_{n\to\infty}\varphi_n(x)=\varphi(x)$ uniformly with respect to $x\in[0,1]$. From the uniform convergence, by a general result, which we will prove in the end of the proof, it follows that $\lim\limits_{n\to\infty}[\varphi_n(x_n)-\varphi(x_n)]=0$. Since by (7) $\forall n\geq n_0$ $\varphi_n(x_n)=0$, we deduce that $\varphi(x_n)\to 0=\varphi(a)$. Since φ is strictly increasing it follows that $x_n\to a$. Indeed, let $\varepsilon>0$. Take $0<\eta<\min(\varepsilon,a,1-a)$; such a number exists since $0<a<1$. Then $0<a-\eta<a<a+\eta<1$ and, since φ is strictly increasing, $\varphi(a-\eta)<\varphi(a)=0<\varphi(a+\eta)$. From $\varphi(x_n)\to 0$ it follows that there exists $n_\varepsilon\geq n_0$ such that $\forall n\geq n_\varepsilon$, $\varphi(a-\eta)<\varphi(x_n)<\varphi(a+\eta)$, that is, since φ is strictly increasing, $\forall n\geq n_\varepsilon$, $a-\eta<x_n<a+\eta$, $|x_n-a|<\eta<\varepsilon$, i.e., $\lim\limits_{n\to\infty}x_n=a$.

It remains to show that if $\lim\limits_{n\to\infty}\varphi_n(x)=\varphi(x)$ uniformly with respect to $x\in[0,1]$, then for every sequence $(x_n)_{n\in\mathbb{N}}\subset[0,1]$ it follows that $\lim\limits_{n\to\infty}[\varphi_n(x_n)-\varphi(x_n)]=0$. Indeed, we have: $\forall\varepsilon>0$, $\exists n_\varepsilon\in\mathbb{N}$ such that $\forall n\geq n_\varepsilon$ and $\forall x\in[0,1]$ we have $|\varphi_n(x)-\varphi(x)|<\varepsilon$. In particular, for

$x = x_n$ we deduce that $\forall n \geq n_\varepsilon$ we have $|\varphi_n(x_n) - \varphi(x_n)| < \varepsilon$, i.e., $\lim\limits_{n\to\infty} [\varphi_n(x_n) - \varphi(x_n)] = 0$. $\square$

Instrucţiuni pentru autori

Materialele trimise redacţiei spre publicare vor fi redactate într-una dintre limbile română, engleză sau franceză. Articolele nu vor depăşi 8-10 pagini, iar notele matematice şi cele metodice 2-3 pagini. Autorii sunt rugaţi să fragmenteze ei înşişi materialele de dimensiuni mai mari pentru a se încadra în limitele indicate mai sus. Fiecare articol va fi însoţit de un scurt rezumat al lucrării, nedepăşind 15-20 de rânduri tehnoredactate în limbile franceză sau engleză, de indicii de clasificare AMS şi de câteva cuvinte cheie. Notele de subsol vor fi incluse în pagina unde se face trimiterea.

Referinţele bibliografice se vor trece la sfârşitul lucrării, conform normelor standard internaţionale, prescurtările admise fiind cele uzuale, folosite în Mathematical Reviews.

Orice manuscris va fi însoţit de adresa exactă a autorului.

Manuscrisele vor fi, de regulă, tehnoredactate în limbajul LaTeX. Fişierul sursă şi cel pdf vor fi transmise prin e-mail la adresa `office@rms.unibuc.ro` sau expediate la adresa redacţiei.

Orice material se va publica pe răspunderea autorului, redacţia neasumându-şi fondul opiniilor exprimate. Manuscrisele nepublicate nu se restituie autorilor.

Pentru lucrările cu caracter original, copyright-ul aparţine editorului. Orice reproducere integrală sau parţială a unui material deja publicat se va face numai cu acordul scris al Societăţii şi cu specificarea coordonatelor exacte ale primei apariţii.

Autorii pot comanda la redacţie, contra cost, extrase din revistă conţinând articolele publicate.

Instructions for authors

The articles submitted to the editorial office should be written in Romanian, English or French. The articles will not exceed 8-10 pages and the mathematical class-room notes 2-3 pages. If the articles are larger than the above limits, the authors are required to divide the articles by themselves in order to comply with the above. Each article should be preceded by a short summary of its contents in English or French, not exceeding 15-20 typed rows. Keywords and AMS classification should also be attached.

Footnotes should be included at the bottom of the same page.

Bibliographical references should be given at the end of the article by using the international standard norms and the admitted abbreviations according to Mathematical Reviews.

Any manuscript should be accompanied by the exact address of the author.

The manuscript should be processed in LaTeX and the source file and the pdf file should be mailed to the editorial office by usual mail or to `office@rms.unibuc.ro`.

Any material is published in the author's own responsibility. The editorial board will assume no responsibility for the contents of any published items.

Unpublished manuscripts are not to be returned to the authors.

For any original paper the copyright belongs to the publisher – all rights reserved. Any material already published may be reproduced, in whole or in part, only with the written permission of the Society and be mentioning the exact dates of the first edition.

The authors can order, by payment, reprints of their articles.