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On second-order differential equations associated with gradients of pseudoconvex functions

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Abstract. In this note we are concerned with the equation $u''(t) = \nabla\phi(u(t))$, $t \in [0,\infty)$, in a real Hilbert space H, subject to $u(0) = u_0$, $\sup_{t\geq 0} \|u(t)\| < \infty$, where $\phi : H \to \mathbb{R}$ is assumed to be a differentiable pseudoconvex function. Using a strategy based on an existence result for the Cauchy problem associated with the above equation, we discuss in detail two examples. A conjecture on existence and uniqueness for the above problem is also formulated.

Keywords: Second-order differential equation on the positive half-line, pseudoconvex function, existence, uniqueness

MSC: 34A34, 34B15.

1. Introduction

Let H be a real Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $\phi : H \to \mathbb{R}$ be a (Fréchet) differentiable function, whose gradient at any $x \in H$ is denoted $\nabla\phi(x)$. Consider the problem

$$u''(t) = \nabla\phi(u(t)), \quad t \in [0,\infty), \tag{1.1}$$

$$u(0) = u_0, \quad \sup_{t\geq 0} \|u(t)\| < \infty, \tag{1.2}$$

where $u_0 \in H$ is a given vector. A vector function $u : [0,\infty) \to H$ is said to be a solution of this problem if u is twice differentiable and satisfies (1.1) (for all $t \geq 0$) and (1.2).

In the case when ϕ is a convex function problem (1.1), (1.2) has been solved long ago by V. Barbu in a more general context (see [1, p. 315]; see also [2], [3] and the references therein for more general equations); more precisely,

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in this case, for all $u_0 \in H$ there exists a unique solution of this problem. Recently, my colleague H. Khatibzadeh inquired me whether existence for problem (1.1), (1.2) still holds if ϕ is a pseudoconvex function. To the best of my knowledge this is an open problem even in the case $H = \mathbb{R}$.

In this note we discuss some examples in the case $H = \mathbb{R}$ which confirm the conjecture and might be appealing to people interested in differential equations.

Let us first recall the definition of a pseudoconvex function: a differentiable function $\phi : H \to \mathbb{R}$ is said to be *pseudoconvex* if

$$\forall x, y \in H,\ (\nabla\phi(x), y - x) \geq 0 \implies \phi(x) \leq \phi(z),\ \forall z \in [x, y],$$

where $[x, y]$ is the segment connecting x and y (i.e., $[x, y] = \{\alpha x + (1-\alpha)y : \alpha \in [0, 1]\}$). In other words, ϕ is increasing in any direction where its derivative (gradient) is positive. It is easily seen that any convex differentiable function $\phi : H \to \mathbb{R}$ is pseudoconvex (since $\phi(z) - \phi(x) \geq (\nabla\phi(x), z - x)$), but the converse implication is not true in general. A nice property of any differentiable pseudoconvex function ϕ is the following: x^* is a local minimizer of $\phi \iff \nabla\phi(x^*) = 0$. This property is very useful in optimization.

2. The Cauchy problem associated with equation (1.1)

In order to discuss problem (1.1), (1.2) we need the following

Proposition 1. *Assume that $\phi : H \to \mathbb{R}$ is differentiable and $\nabla\phi$ is a Lipschitz operator on H. Then for all $u_0, v_0 \in H$ there exists a unique function $u \in C^2([0, \infty); H)$ satisfying (1.1) and $u(0) = u_0$, $u'(0) = v_0$.*

Proof. As usual the Cauchy problem

$$u''(t) = \nabla\phi(u(t)), \quad t \in [0, \infty), \tag{2.3}$$

$$u(0) = u_0, \quad u'(0) = v_0, \tag{2.4}$$

can be expressed as a Cauchy problem for a first-order differential system as follows

$$\frac{d}{dt}(u, v) = (v, \nabla\phi(u)), \tag{2.5}$$

$$(u, v)(0) = (u_0, v_0), \tag{2.6}$$

where the operator in the right-hand side of (2.5) is Lipschitz on the product space $H \times H$. So for any $T > 0$ it follows by the Banach Fixed Point Principle that there exists a unique solution $(u, v) \in C^1([0, T]; H \times H)$ of (2.5) on $[0, T]$ satisfying (2.6). Since T was arbitrarily chosen, we infer that problem (2.5), (2.6) has a unique solution $(u, v) \in C^1([0, \infty); H \times H)$. Therefore, the first component u belongs to $C^2([0, \infty); H)$ and is the unique solution of problem (2.3), (2.4). □

In order to solve problem (1.1), (1.2) one can fix $u_0 \in H$ and look for some appropriate v_0 so that the corresponding Cauchy problem (2.3), (2.4) have a bounded solution. This idea will be exploited in the next section.

3. Examples

We illustrate the previous considerations with the following two examples.

Example 1. Let $\phi : H = \mathbb{R} \to \mathbb{R}$, $\phi(r) = \frac{r^2}{2(1+r^2)}$. We have

$$\nabla\phi(r) = \phi'(r) = \frac{r}{(1+r^2)^2}\,, \quad \phi''(r) = \frac{1-3r^2}{(1+r^2)^3}\,, \quad \forall r \in \mathbb{R}.$$

Obviously, ϕ is not convex, but is pseudoconvex. In this case, problem (1.1), (1.2) reads

$$u''(t) = \frac{u(t)}{(1+u(t)^2)^2}\,, \quad t \in [0,\infty), \tag{3.7}$$

$$u(0) = u_0, \quad \sup_{t\geq 0} |u(t)| < \infty. \tag{3.8}$$

In what follows we will prove the following result.

Proposition 2. *For all $u_0 \in \mathbb{R}$ problem* (3.7), (3.8) *has a unique solution $u \in C^\infty[0,\infty) := C^\infty([0,\infty); \mathbb{R})$.*

Proof. First of all, note that $\phi \in C^\infty(\mathbb{R})$ and ϕ' is a Lipschitz continuous function. Therefore Proposition 1 is applicable to (3.7) and every solution of this equation belongs to $C^\infty[0,\infty)$. Note also that if u is a solution of (3.7) then $-u$ is also a solution of (3.7). So we can assume $u_0 \geq 0$. In what follows we discuss all possible cases.

Case 1: $u_0 = 0$**,** $u'(0) =: v_0 = 0$. Obviously, the null function is a solution to problem (3.7), (3.8). According to Proposition 1, this is the unique solution.

Case 2: $u_0 = 0$**,** $v_0 > 0$. Since $v_0 > 0$, it follows that the derivative of corresponding solution u is positive in a small interval $[0,\delta]$, hence $u(\delta) > 0$, $u'(\delta) > 0$. Since the equation (3.7) is autonomous, one can consider that these conditions are satisfied at $t = 0$ and so Case 2 is similar to Case 5 below.

Case 3: $u_0 = 0$**,** $v_0 < 0$. Substituting u with $-u$ we obtain Case 2 (hence Case 5).

Case 4: $u_0 > 0$**,** $v_0 = 0$. Since $u_0 > 0$ it follows from (3.7) that $u''(0) > 0$. So $u'' > 0$ on a small interval $[0,\delta]$ and hence u' is strictly increasing in $[0,\delta]$. Therefore, $u(\delta) > 0$, $u'(\delta) > 0$. This case is similar to Case 5 below (since the equation (3.7) is autonomous).

Case 5: $u_0 > 0$, $v_0 > 0$. Let $[0, a)$ be the maximal interval where $u > 0$. By (2.5) it follows that $u'' > 0$ in $[0, a)$, hence u' is strictly increasing in $[0, a)$. In particular,

$$u'(t) \geq u'(0) = v_0, \quad \forall t \in [0, a). \tag{3.9}$$

Integrating (3.9) over $[0, t]$ leads to

$$u(t) \geq u_0 + v_0 t, \quad \forall t \in [0, a). \tag{3.10}$$

Note that a cannot be finite. Indeed, if we assume by contradiction that $a < \infty$, we derive from (3.10) $u(a) \geq u_0 + v_0 a > 0$, hence $u > 0$ in $[0, a + \varepsilon]$ for a small $\varepsilon > 0$. This contradicts the maximality of a, so $a = \infty$. Then, it follows from (3.10) that $u(t) \to +\infty$, as $t \to +\infty$, i.e., the boundedness condition on u fails to hold.

Case 6: $u_0 > 0$, $v_0 < 0$. Note that if we multiply (1.1) by $u'(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u'(t)\|^2 = \frac{d}{dt}\phi(u(t)), \quad t \geq 0. \tag{3.11}$$

Integrating (3.11) over $[0, t]$ leads to

$$\|u'(t)\|^2 = 2\phi(u(t)) + \|v_0\|^2 - 2\phi(u_0), \quad t \geq 0. \tag{3.12}$$

In our specific case (3.12) reads

$$(u')^2 = \frac{u^2}{1 + u^2} + C, \quad t \geq 0, \tag{3.13}$$

where $C := v_0^2 - u_0^2/(1 + u_0^2)$. In what follows we will discuss three subcases of Case 6.

Subcase 6.1: $u_0 > 0$, $v_0 < 0$, $C > 0$. Let $[0, a)$ be the maximal interval where $u > 0$. Since $C > 0$, it follows from (3.13) that u' cannot vanish, so $u'(t) < 0$, $\forall t \in [0, a)$. We derive from (3.13)

$$u' = -\sqrt{\frac{u^2}{1 + u^2} + C}, \quad \forall t \in [0, a), \tag{3.14}$$

which yields

$$\int_{u(t)}^{u_0} \sqrt{\frac{1 + r^2}{(1 + C)r^2 + C}}\, \mathrm{d}r = t, \quad \forall t \in [0, a). \tag{3.15}$$

Since $u(t) > 0$, $\forall t \in [0, a)$, we deduce from (3.15) that $a < \infty$. It follows that $u(a) = 0$, hence (cf. (3.13)) $u'(a)^2 = C \Longrightarrow u'(a) = -\sqrt{C} < 0$. So we are in the case $u(a) = 0$, $u'(a) < 0$, which is similar to Case 3, hence $u(t)$ is unbounded.

Subcase 6.2: $u_0 > 0$, $v_0 < 0$, $C < 0$. Consider again the maximal interval $[0, a)$ where $u > 0$. Assume that u' vanishes at some point $t = b \in$ $\in (0, a)$. Since (cf. (3.7)) $u'' > 0$ in $(0, a)$, hence u' is strictly increasing in this interval, we have $u(b + \delta) > 0$, $u'(b + \delta) > 0$ for a small $\delta > 0$. This implies that $u(t)$ is unbounded (cf. Case 5). Because we are looking for a bounded solution of (3.7), we need to see what happens if $u'(t) < 0, \ \ \forall t \in [0, a)$. Now, if we suppose that a is finite, then $u(a) = 0$. $u'(a)$ cannot be zero, because u would be the null solution of equation (3.7), which is not the case since $u(0) = u_0 \neq 0$. Hence $u'(a) < 0$ and hence $u(t)$ is unbounded (see Cases 3 and 5).

So we need to investigate the situation $a = \infty$ and $u > 0, u' < 0$ in $[0, \infty)$. Note first that $C > -u_0^2/(1 + u_0^2)$, hence $C + 1 > 0$. Since $u' < 0$ in $[0, \infty)$, it follows from (3.13) that $C + u^2/(1 + u^2) > 0$, hence $u^2 > -C/(1 + C) > 0$ in $[0, \infty)$. Integrating over $[0, t]$ the equation (3.14) (which is now valid in $[0, \infty)$), we infer that

$$t = \int_{u(t)}^{u_0} \sqrt{\frac{1 + r^2}{(1 + C)r^2 + C}}\, \mathrm{d}r \leq \int_{\sqrt{-C/(1+C)}}^{u_0} \sqrt{\frac{1 + r^2}{(1 + C)r^2 + C}}\, \mathrm{d}r, \quad \forall t \geq 0.$$

This is impossible since the last integral is finite.

It remains to investigate the following

Subcase 6.3: $u_0 > 0$**,** $v_0 < 0$**,** $C = 0$. It follows from (3.13) that

$$u'(t) = -\frac{u(t)}{\sqrt{1 + u(t)^2}} \tag{3.16}$$

for t in the maximal interval $[0, a)$ where $u > 0$. In fact $a = +\infty$, otherwise $u(a) = 0$ and so u would be the null solution, which is impossible since $u(0) = u_0 \neq 0$. Since $u' < 0$ in $[0, \infty)$, it follows that u is strictly decreasing and therefore $0 < u(t) < u_0$ for all $t \in (0, \infty)$.

Summarizing, we see that only in Case 1 and Subcase 6.3 we obtain bounded solutions. More precisely, problem (3.7), (3.8) admits a unique positive solution u for each $u_0 > 0$ and (changing u to $-u$) the same result holds true for each $u_0 < 0$, while for $u_0 = 0$ the unique solution is the null function. □

Remark 1. Moreover, for every $u_0 > 0$ ($u_0 < 0$) the corresponding solution u of problem (3.7), (3.8) strictly decreases (respectively increases) to zero as $t \to \infty$. It suffices to examine the case $u_0 > 0$ (the conclusion in the case $u_0 < 0$ follows by changing u to $-u$). We already know that $u(t)$ is strictly decreasing and its values belong to $(0, u_0)$. Therefore, there exists $\lim_{t\to\infty} u(t) = u*$, with $u^* \in [0, u_0)$. In general, if this is the case for a solution of the equation $u' = f(u)$, where $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f(u^*) = 0$. Indeed, by the mean value theorem, $u(n + 1) - u(n) = u'(t_n) = f(u(t_n)),\ t_n \in (n, n + 1)$, which implies the assertion. In our specific case, see (3.16), $f(u^*) = 0$ implies $u^* = 0$. One can also derive this result by integrating (3.16) and then analyzing the solution starting from u_0.

Example 2. Let $\phi : H = \mathbb{R} \to \mathbb{R}$,

$$\phi(r) = \begin{cases} \dfrac{r^2}{2(1+r^2)}, & r \geq 0, \\ \dfrac{r^2}{2}, & r < 0. \end{cases}$$

We have

$$\phi'(r) = \begin{cases} \dfrac{r}{(1+r^2)^2}, & r \geq 0, \\ r, & r < 0. \end{cases}$$

Obviously, ϕ is pseudoconvex, but not convex. In this case problem (1.1), (1.2) reads

$$u''(t) = \phi'(u(t)), \quad t \in [0, \infty), \tag{3.17}$$

$$u(0) = u_0, \quad \sup_{t \geq 0} |u(t)| < \infty. \tag{3.18}$$

We have

Proposition 3. *For all* $u_0 \in \mathbb{R}$ *problem* (3.17), (3.18) *has a unique solution* $u \in C^\infty[0, \infty) := C^\infty([0, \infty); \mathbb{R})$.

Proof. It is easily seen that $\phi \in C^3(\mathbb{R})$ and $\sup_{r \in \mathbb{R}} |\phi''(r)| < \infty$, hence ϕ' is a Lipschitz continuous function. Therefore Proposition 1 is applicable to this problem.

Denote again $v_0 = u'(0)$. Assume that $u_0 < 0$. Then, for any $v_0 \in \mathbb{R}$, the equation (3.17) with the initial conditions $u(0) = u_0$ and $u'(0) = v_0$ has a unique solution u, which is given by

$$u(t) = \frac{u_0 + v_0}{2}\left(e^t - e^{-t}\right) + u_0 e^{-t}, \quad \forall t \in [0, a), \tag{3.19}$$

where $[0, a)$ is the maximal interval where $u < 0$. We also have

$$u'(t) = \frac{u_0 + v_0}{2} e^t + \frac{v_0 - u_0}{2} e^{-t}, \quad \forall t \in [0, a). \tag{3.20}$$

In what follows we discuss three cases corresponding to $u_0 < 0$.

Case I: $u_0 < 0$**,** $u_0 + v_0 < 0$. The function u given by (3.19) satisfies $u = u'' < 0$ in $[0, \infty)$. So $a = \infty$ and u is the unique solution of (3.17) satisfying $u_0(0) = u_0,\ u'(0) = v_0$. Obviously, $u(t) \to -\infty$ as $t \to \infty$.

Case II: $u_0 < 0$**,** $u_0 + v_0 > 0$. We have $v_0 > 0$ and u' given by (3.20) is positive in $[0, \infty)$, so u given by (3.19) is strictly increasing and $u(t) \to \infty$ as $t \to \infty$. It follows that there exists an $a \in (0, \infty)$ such that $u(a) = 0$ and $u'(a) > 0$. Therefore $u(t) > 0,\ u'(t) > 0$ in a small interval $(a, a + \delta)$, where u satisfies now the equation (3.7) (see Example 1). According to Case 5 of Example 1 above, $u(t) \to \infty$, as $t \to \infty$.

Case III: $u_0 < 0$**,** $u_0 + v_0 = 0$. In this case the corresponding solution is $u(t) = u_0 e^{-t} < 0,\ t \in [0, \infty)$, which is bounded (so this is a favorable case).

Next we discuss all the other possible cases.

Case IV: $u_0 = 0$**,** $v_0 = 0$. $u(t) = 0, \quad \forall t \geq 0$ (so this case is also favorable).

Case V: $u_0 = 0$**,** $v_0 > 0$. For a small $\delta > 0$ we have $u(\delta) > 0$, $u'(\delta) > 0$, so u is unbounded (cf. Case 5, Example 1).

Case VI: $u_0 = 0$**,** $v_0 < 0$. For a small $\delta > 0$ we have $u(\delta) < 0$ and $u'(\delta) < 0$, so u is unbounded (cf. Case I above).

Case VII: $u_0 > 0$**,** $v_0 > 0$. u is unbounded (cf. Case 5, Example 1).

Case VIII: $u_0 > 0$**,** $v_0 = 0$. Since $u'' > 0$ is positive in a neighbourhood of $t = 0$, u' is strictly increasing in this neighbourhood, so $u'(\delta) > 0$, $u(\delta) > 0$ for a small $\delta > 0$. Therefore, again u is unbounded (see Case VII or Case 5, Example 1).

Case IX: $u_0 > 0$**,** $v_0 < 0$. This case is also a favorable one: the corresponding (unique) solution u is bounded (cf. Case 6, Example 1).

The fact that $u \in C^\infty[0, \infty)$ for all $u_0 \in \mathbb{R}$ is obvious from what we have done so far. □

Remark 2. From the above proof it follows that for $u_0 > 0$ ($u_0 < 0$) the unique solution of problem (3.17), (3.18) strictly decreases (exponentially increases, respectively) to zero.

It seems that problem (1.1), (1.2) has a (unique?) solution, provided that $\phi : H \to \mathbb{R}$ is differentiable, pseudoconvex (not convex) and satisfies reasonable additional conditions (e.g., the Lipschitz condition for $\nabla\phi$).

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Determining upper and lower bounds of a series of numbers that involve arithmetic functions

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Abstract. In this article we determine bounds for series written as Euler products along verticals $x = \sigma + it$, σ = constant. Use of a form of Kronecker's theorem will be decisive.

Keywords: Arithmetic functions, series and products of numbers, uniform convergence

MSC: 30B30, 11A25, 11M06

Introduction

Many series in which appear arithmetic functions can be written as Euler products. This article aims to determine the lower and upper bounds of such series (see Theorems 6–8). Another result of this article, viz., Theorem 9, helps us to determine a lower bound for the Riemann zeta function $\zeta(s)$ with $\operatorname{Re} s > 1$ (see Proposition 10). Some inequalities are derived from previous results in Theorems 12 and 13.

The arithmetic functions that occur are:

- Möbius function

$$\mu(n) = \begin{cases} 1, & n = 1 \\ (-1)^k, & n = p_1 p_2 \cdots p_k;\ p_1, p_2, \cdots, p_k \text{ distinct primes} \\ 0, & \text{otherwise} \end{cases}$$

- Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}},\ \sigma > 1$$

- sums of divisors

$$\sigma(n) = \sum_{d|n} d;\quad \sigma_\alpha(n) = \sum_{d|n} d^\alpha;\quad \tau(n) = \sum_{d|n} 1;\quad \alpha \text{ real or complex}$$

- Euler totient function

  $\varphi(n) = \sum_{k=1}^{n}{}' 1$; ($'$ indicates that the sum is extended over those k relatively prime to n)

- Liouville's function

$$\lambda(n) = \begin{cases} 1, & n = 1 \\ (-1)^{\alpha_1 + \alpha_2 + \cdots + \alpha_k}, & n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} \end{cases}$$

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- Dirichlet L-functions

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s},\ \sigma>1,\quad \text{where } \chi \text{ is a Dirichlet character.}$$

1. PRELIMINARIES

In the proof of main results we will use the following lemmas:

Lemma 1. ([1]) *If* $\operatorname{Re} s>1$ *then*

$$\sum_{n\geq 1}\frac{\mu(n)\tau(n)}{n^s}=\prod_{p\ \text{prime}}\left(1-\frac{2}{p^s}\right),$$

$$\sum_{n\geq 1}\frac{|\mu(n)|\tau(n)}{n^s}=\prod_{p\ \text{prime}}\left(1+\frac{2}{p^s}\right),$$

$$\zeta^2(s)=\sum_{n\geq 1}\frac{\tau(n)}{n^s}.$$

Lemma 2. ([1]) *If* $\operatorname{Re} s>2$ *then*

$$\sum_{n\geq 1}\frac{\mu(n)\sigma(n)}{n^s}=\prod_{p\ \text{prime}}\left(1-\frac{p+1}{p^s}\right),$$

$$\sum_{n\geq 1}\frac{|\mu(n)|\sigma(n)}{n^s}=\prod_{p\ \text{prime}}\left(1+\frac{p+1}{p^s}\right),$$

$$\zeta(s)\cdot\zeta(s-1)=\sum_{n\geq 1}\frac{\sigma(n)}{n^s}.$$

Lemma 3. ([1]) *If* $\operatorname{Re} s>2$ *then*

$$\sum_{n\geq 1}\frac{\mu(n)\varphi(n)}{n^s}=\prod_{p\ \text{prime}}\left(1-\frac{p-1}{p^s}\right),$$

$$\sum_{n\geq 1}\frac{|\mu(n)|\varphi(n)}{n^s}=\prod_{p\ \text{prime}}\left(1+\frac{p-1}{p^s}\right).$$

Lemma 4. ([2]) *We have the following Euler products:*

$$\frac{\zeta(s-1)}{\zeta(s)}=\sum_{n\geq 1}\frac{\varphi(n)}{n^s}=\prod_{p}\frac{1-p^{-s}}{1-p^{1-s}},\ \ \textit{if}\ \sigma>2.$$

$$\zeta(s)\zeta(s-\alpha)=\sum_{n\geq 1}\frac{\sigma_\alpha(n)}{n^s}=\prod_{p}\frac{1}{(1-p^{-s})(1-p^{\alpha-s})},\ \ \textit{if}\ \sigma>\max\{1,1+\operatorname{Re}\alpha\}.$$

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n\geq 1} \frac{\lambda(n)}{n^s} = \prod_p \frac{1}{1+p^{-s}}, \ \textit{if } \sigma > 1.$$

$$L(s,\xi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}}, \ \textit{if } \sigma > 1.$$

Lemma 5. ([3])(Kronecker) *If $\theta_1, \theta_2, \ldots, \theta_n$ are linearly independent, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arbitrary and T and ε are positive, then there exist a real number t and integers $h_1, h_2, \ldots, h_n$ such that $t > T$ and*

$$|t\theta_k - h_k - \alpha_k| < \varepsilon, \quad \textit{for } k = 1, 2, \ldots, n.$$

2. Main results

Theorem 6. *If $\sigma > 1$ is a constant and $s = \sigma + it$, then*

$$\sup_{t\in\mathbb{R}} \left| \sum_{n\geq 1} \frac{\mu(n)\tau(n)}{n^s} \right| = \sup_{t\in\mathbb{R}} \left| \sum_{n\geq 1} \frac{|\mu(n)|\tau(n)}{n^s} \right| = \prod_{p \text{ prime}} (1 + 2p^{-\sigma}),$$

and

$$\inf_{t\in\mathbb{R}} \left| \sum_{n\geq 1} \frac{\mu(n)\tau(n)}{n^s} \right| = \inf_{t\in\mathbb{R}} \left| \sum_{n\geq 1} \frac{|\mu(n)|\tau(n)}{n^s} \right| = \prod_{p \text{ prime}} (1 - 2p^{-\sigma}).$$

Proof. From Lemma 1,

$$\left| \sum_{n\geq 1} \frac{\mu(n)\tau(n)}{n^s} \right| = \prod_{p \text{ prime}} |1 - 2p^{-s}|,$$

with

$$1 - 2p^{-s} = 1 - 2p^{-\sigma}p^{-it} = 1 - 2p^{-\sigma}e^{-it\ln p} = 1 + 2p^{-\sigma}e^{i(-t\ln p + \pi)}.$$

We choose the numbers $\theta_k = -\frac{1}{2\pi}\ln p_k$, $k = 1, 2, \ldots, n$, where $p_1, p_2, \ldots, p_n$ are the first n primes. It is well-known that θ_k are linearly independent over the integers. If $\varepsilon \in (0, \frac{\pi}{2})$, we take $\alpha_1 = \alpha_2 = \cdots = \alpha_n = -\frac{1}{2}$. From Lemma 5 it follows that there exist a real t and integers $h_1, h_2, \ldots, h_n$ such that

$$|t\theta_k - h_k - \alpha_k| < \tfrac{\varepsilon}{2\pi}, \ k = \overline{1,n},$$

that is,

$$|-t\ln p_k + \pi - 2\pi h_k| < \varepsilon. \tag{2.1}$$

For this t we have

$$1 - 2p_k^{-s} = 1 + 2p_k^{-\sigma}\mathrm{e}^{i(-t\ln p_k + \pi)} = 1 + 2p_k^{-\sigma}\cos(-t\ln p_k + \pi) + 2p_k^{-\sigma} i \sin(-t\ln p_k + \pi),$$

so that

$$|1 - 2p_k^{-s}| \geq |1 + 2p_k^{-\sigma}\cos(-t\ln p_k + \pi)| = 1 + 2p_k^{-\sigma}\cos(-t\ln p_k + \pi).$$

The inequality (2.1) implies
$$\cos|-t\ln p_k+\pi| = \cos|-t\ln p_k+\pi-2\pi h_k| > \cos\varepsilon,$$
so
$$\left|1-2p_k^{-s}\right| \geq 1+2p_k^{-\sigma}\cos\varepsilon.$$
For a given ε and n there exists a real t (depending on ε and on n) such that
$$\prod_{k=1}^{n}|1-2p_k^{-s}| \geq \prod_{k=1}^{n}(1+2p_k^{-\sigma}\cos\varepsilon). \tag{2.2}$$
As $\sum 2p_k^{-\sigma}\cos\varepsilon \leq 2\sum n^{-\sigma} = 2\zeta(\sigma)$, the series converges uniformly on $[0,\frac{\pi}{2}]$, hence the product $\prod\limits_{k\geq1}(1+2p_k^{-\sigma}\cos\varepsilon)$ converges uniformly on this set. Since $\sum\limits_{k\geq1} 2p_k^{-s}$ converges absolutely, it follows that the product $\prod\limits_{k\geq1}(1-2p_k^{-s})$ converges. Hence, the product $\prod\limits_{k\geq1}|1-2p_k^{-s}|$ converges. By Cauchy condition for convergent products, there is a positive integer n_0 such that for $n\geq n_0$ one has
$$1-\varepsilon < \prod_{k\geq n+1}|1-2p_k^{-s}| < 1+\varepsilon,$$
and considering (2.2) we obtain
$$\left|\sum_{n\geq1}\frac{\mu(n)\tau(n)}{n^s}\right| = \prod_{k=1}^{n}|1-2p^{-s}|\prod_{k\geq n+1}|1-2p_k^{-s}| \geq (1-\varepsilon)\prod_{k=1}^{n}(1+2p_k^{-\sigma}\cos\varepsilon). \tag{2.3}$$
It follows $\sup\limits_{t\in\mathbb{R}}\left|\sum\limits_{n\geq1}\frac{\mu(n)\tau(n)}{n^s}\right| \geq (1-\varepsilon)\prod\limits_{k\geq1}(1+2p_k^{-\sigma}\cos\varepsilon)$, and since the product $\prod\limits_{k\geq1}(1+2p_k^{-\sigma}\cos\varepsilon)$ converges uniformly on $[0,\frac{\pi}{2}]$, we can let $\varepsilon\to0$ and pass to the limit term by term to obtain
$$\sup_{t\in\mathbb{R}}\left|\sum_{n\geq1}\frac{\mu(n)\tau(n)}{n^s}\right| \geq \prod_p(1+2p^{-\sigma}). \tag{2.4}$$
Lemma 1 gives
$$\left|\sum_{n\geq1}\frac{\mu(n)\tau(n)}{n^s}\right| = \prod_p\left|1-\frac{2}{p^s}\right| \leq \prod_p(1+2p^{-\sigma}). \tag{2.5}$$
From (2.4) and (2.5) we find
$$\sup_{t\in\mathbb{R}}\left|\sum_{n\geq1}\frac{\mu(n)\tau(n)}{n^s}\right| = \prod_p(1+2p^{-\sigma}).$$

On the other hand, Lemma 1 also yields

$$\left|\sum_{n\geq 1}\frac{|\mu(n)|\tau(n)}{n^s}\right| = \left|\prod_p (1+2p^{-s})\right| \leq \prod_p (1+2p^{-\sigma}). \tag{2.6}$$

We choose the numbers $\theta_k = -\frac{1}{2\pi}\ln p_k$, $k = \overline{1,n}$, linearly independent over the integers and $\varepsilon \in \left(0, \frac{\pi}{2}\right)$, $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. From Lemma 5 it follows that there exist a real t and integers $h_1, h_2, \ldots, h_n$ such that $|t\theta_k - h_k - \alpha_k| < \frac{\varepsilon}{2\pi}$, that is,

$$|-t\ln p_k - 2\pi h_k| < \varepsilon,\ \ k = \overline{1,n}. \tag{2.7}$$

For this t, we have

$$1+2p_k^{-s} = 1+2p_k^{-\sigma}\mathrm{e}^{-it\ln p_k} = 1+2p_k^{-\sigma}\cos(-t\ln p_k) + 2p_k^{-\sigma} i \sin(-t\ln p_k),$$

$$|1+2p_k^{-s}| \geq |1+2p_k^{-\sigma}\cos(-t\ln p_k)| = 1+2p_k^{-\sigma}\cos(-t\ln p_k).$$

Note that (2.7) implies $\cos|-t\ln p_k| = \cos|-t\ln p_k - 2\pi h_k| > \cos\varepsilon$, so

$$\prod_{k=1}^{n} |1+2p_k^{-s}| \geq \prod_{k=1}^{n}(1+2p_k^{-\sigma}\cos\varepsilon). \tag{2.8}$$

By Cauchy condition for the product $\prod\limits_p |1+2p^{-s}|$, there is an n_0 such that $n \geq n_0$ implies

$$1-\varepsilon < \prod_{k\geq n+1} |1+2p_k^{-s}| < 1+\varepsilon,$$

and considering (2.8) we obtain

$$\left|\sum_{n\geq 1}\frac{|\mu(n)|\tau(n)}{n^s}\right| = \prod_{k=1}^{n}|1+2p_k^{-s}| \prod_{k\geq n+1}|1+2p_k^{-s}| \geq$$

$$\geq (1-\varepsilon)\prod_{k\geq n+1}(1+2p_k^{-\sigma}\cos\varepsilon),$$

whence, just like above,

$$\sup_{t\in\mathbb{R}}\left|\sum_{n\geq 1}\frac{|\mu(n)|\tau(n)}{n^s}\right| \geq \prod_p (1+2p^{-\sigma}). \tag{2.9}$$

From (2.6) and (2.9) it results

$$\sup_{t\in\mathbb{R}}\left|\sum_{n\geq 1}\frac{|\mu(n)|\tau(n)}{n^s}\right| = \prod_p (1+2p^{-\sigma}).$$

From $||z_1| - |z_2|| \le |z_1 - z_2|$ and Lemma 1 it follows

$$\left|\sum_{n\ge1}\frac{\mu(n)\tau(n)}{n^s}\right| = \prod_p\left|1-\frac{2}{p^s}\right| \ge \prod_p\left|1-\frac{2}{p^\sigma}\right| = \prod_p\left(1-\frac{2}{p^\sigma}\right).$$

Hence

$$\inf_{t\in\mathbb{R}}\left|\sum_{n\ge1}\frac{\mu(n)\tau(n)}{n^s}\right| = \inf_{t\in\mathbb{R}}\prod_p\left|1-\frac{2}{p^s}\right| \ge \prod_p\left(1-\frac{2}{p^\sigma}\right).$$

The last product is obtained for $s=\sigma$, so

$$\inf_{t\in\mathbb{R}}\left|\sum_{n\ge1}\frac{\mu(n)\tau(n)}{n^s}\right| = \prod_p\left(1-\frac{2}{p^\sigma}\right).$$

From $||z_1| - |z_2|| \le |z_1 + z_2|$ and Lemma 1 it follows

$$\left|\sum_{n\ge1}\frac{|\mu(n)|\tau(n)}{n^s}\right| = \prod_p\left|1+\frac{2}{p^s}\right| \ge \prod_p\left|1-\frac{2}{p^\sigma}\right| = \prod_p\left(1-\frac{2}{p^\sigma}\right),$$

hence

$$\inf_{t\in\mathbb{R}}\left|\sum_{n\ge1}\frac{|\mu(n)|\tau(n)}{n^s}\right| \ge \prod_p\left(1-\frac{2}{p^\sigma}\right). \tag{2.10}$$

We retain from the equalities

$$\begin{aligned}|1+2p_k^{-s}| &= |1+2p_k^{-\sigma}\mathrm{e}^{-it\ln p_k}| = |1-2p_k^{-\sigma}\mathrm{e}^{i(-t\ln p_k-\pi)}| = \\ &= |1-2p_k^{-\sigma}\cos(-t\ln p_k-\pi) - 2p_k^{-\sigma}i\sin(-t\ln p_k-\pi)| = \\ &= \sqrt{1-4p_k^{-\sigma}\cos(-t\ln p_k-\pi)+4p_k^{-2\sigma}}\end{aligned}$$

that one has

$$|1+2p_k^{-s}| = \sqrt{1-4p_k^{-\sigma}\cos(-t\ln p_k-\pi)+4p_k^{-2\sigma}}. \tag{2.11}$$

Considering again the numbers $\theta_k = -\frac{1}{2\pi}\ln p_k$ $(k=\overline{1,n})$, $\varepsilon\in(0,\frac{\pi}{2})$, $\alpha_1=\alpha_2=\cdots=\alpha_n=\frac{1}{2}$, from Lemma 5 it follows that there exist a real t and integers $h_1,h_2,\ldots,h_n$ such that $|-t\ln p_k-\pi-2\pi h_k|<\varepsilon$ $(k=\overline{1,n})$, and $\cos|-t\ln p_k-\pi| = \cos|-t\ln p_k-\pi-2\pi h_k| > \cos\varepsilon$. From this relation and (2.11), we have

$$|1+2p_k^{-\sigma}| \le \sqrt{1-4p_k^{-\sigma}\cos\varepsilon+4p_k^{-2\sigma}}, \quad k=\overline{1,n},$$

whence

$$\left|\sum_{n\geq 1}\frac{|\mu(n)|\tau(n)}{n^s}\right| = \prod_p \left|1+\frac{2}{p^s}\right| \leq$$
$$\leq \prod_{k=1}^{n}\sqrt{1-4p_k^{-\sigma}\cos\varepsilon+4p_k^{-2\sigma}}\prod_{k\geq n+1}\left|1+\frac{2}{p_k^s}\right|. \tag{2.12}$$

Just like above we find that the product $\prod\limits_p |1+2p^{-s}|$ converges. By Cauchy condition, there is an n_0 such that $n \geq n_0$ implies

$$1-\varepsilon < \prod_{k\geq n+1}\left|1+\frac{2}{p_k^s}\right| < 1+\varepsilon.$$

From here and (2.12) it results

$$\left|\sum_{n\geq 1}\frac{|\mu(n)|\tau(n)}{n^s}\right| \leq \sqrt{\prod_{k=1}^{n}\left(1-4p_k^{-\sigma}\cos\varepsilon+4p_k^{-2\sigma}\right)}(1+\varepsilon). \tag{2.13}$$

Since $|-4p_k^{-\sigma}\cos\varepsilon+4p_k^{-2\sigma}| = 4p_k^{-\sigma}|-\cos\varepsilon+p_k^{-\sigma}| \leq 8p_k^{-\sigma}$, we have

$$\left|\sum_{k\geq 1}(-4p_k^{-\sigma}\cos\varepsilon+4p_k^{-2\sigma})\right| \leq 8\sum_{k\geq 1}p_k^{-\sigma} \leq 8\zeta(\sigma),$$

so the series $\sum\limits_{k\geq 1}(-4p_k^{-\sigma}\cos\varepsilon+4p_k^{-2\sigma})$ converges uniformly on $[0,\frac{\pi}{2}]$, hence the product $\prod\limits_{k=1}^{n}(1-4p_k^{-\sigma}\cos\varepsilon+4p_k^{-2\sigma})$ converges uniformly on this set. From (2.13) it results

$$\inf_{t\in\mathbb{R}}\left|\sum_{n\geq 1}\frac{|\mu(n)|\tau(n)}{n^s}\right| \leq \sqrt{\prod_{k=1}^{n}\left(1-4p_k^{-\sigma}\cos\varepsilon+4p_k^{-2\sigma}\right)}(1+\varepsilon).$$

We can let $\varepsilon \to 0$ and pass to the limits term by term to obtain

$$\inf_{t\in\mathbb{R}}\left|\sum_{n\geq 1}\frac{|\mu(n)|\tau(n)}{n^s}\right| \leq \sqrt{\prod_{k\geq 1}(1-2p_k^{-\sigma})^2} = \prod_{k\geq 1}(1-2p_k^{-\sigma}). \tag{2.14}$$

From (2.10) and (2.14) we find

$$\inf_{t\in\mathbb{R}}\left|\sum_{n\geq 1}\frac{|\mu(n)|\tau(n)}{n^s}\right| = \prod_p(1-2p^{-\sigma}).$$

□

Theorem 7. *If $\sigma > 2$ is a constant and $s = \sigma + it$, then*

$$\sup_{t\in\mathbb{R}} \left| \sum_{n\geq 1} \frac{\mu(n)\sigma(n)}{n^s} \right| = \sup_{t\in\mathbb{R}} \left| \sum_{n\geq 1} \frac{|\mu(n)|\sigma(n)}{n^s} \right| = \prod_{p \text{ prime}} \left(1 + \frac{p+1}{p^\sigma}\right),$$

and

$$\inf_{t\in\mathbb{R}} \left| \sum_{n\geq 1} \frac{\mu(n)\sigma(n)}{n^s} \right| = \inf_{t\in\mathbb{R}} \left| \sum_{n\geq 1} \frac{|\mu(n)|\sigma(n)}{n^s} \right| = \prod_{p \text{ prime}} \left(1 - \frac{p+1}{p^\sigma}\right).$$

Proof. Since

$$\left| \sum_p \frac{p+1}{p^s} \right| \leq \sum_p \left| \frac{1}{p^{s-1}} \right| + \sum_p \left| \frac{1}{p^s} \right| = \sum_p \frac{1}{p^{\sigma-1}} + \sum_p \frac{1}{p^\sigma} \leq \zeta(\sigma-1) + \zeta(\sigma),$$

$\sum\limits_p \frac{p+1}{p^s}$ is absolutely convergent. It results that $\prod\limits_{k\geq 1} \left(1 + \frac{p_k+1}{p_k^s}\right)$ as well as $\prod\limits_{k\geq 1} \left(1 - \frac{p_k+1}{p_k^s}\right)$ converge, hence $\prod\limits_{k\geq 1} \left|1 + \frac{p_k+1}{p_k^s}\right|$ and $\prod\limits_{k\geq 1} \left|1 - \frac{p_k+1}{p_k^s}\right|$ converge, too.

Taking into account these observations, the proof proceeds analogously to the previous one. □

Similarly one can show the following

Theorem 8. *If $\sigma > 2$ is a constant and $s = \sigma + it$, then*

$$\sup_{t\in\mathbb{R}} \left| \sum_{n\geq 1} \frac{\mu(n)\varphi(n)}{n^s} \right| = \sup_{t\in\mathbb{R}} \left| \sum_{n\geq 1} \frac{|\mu(n)|\varphi(n)}{n^s} \right| = \prod_{p \text{ prime}} \left(1 + \frac{p-1}{p^\sigma}\right),$$

and

$$\inf_{t\in\mathbb{R}} \left| \sum_{n\geq 1} \frac{\mu(n)\varphi(n)}{n^s} \right| = \inf_{t\in\mathbb{R}} \left| \sum_{n\geq 1} \frac{|\mu(n)|\varphi(n)}{n^s} \right| = \prod_{p \text{ prime}} \left(1 - \frac{p-1}{p^\sigma}\right).$$

Theorem 9. *If $(a_n)_{n\geq 1}$ is an infinite sequence of coprime integers larger than 1, and $s = \sigma + it$, $P(s) = \prod\limits_{k\geq 1} (1 - a_k^{-s})$, with σ constant, $\sigma > 1$, then*

$$\inf_{t\in\mathbb{R}} |P(s)|^{-1} = \prod_{k\geq 1} (1 + a_k^{-\sigma})^{-1}.$$

Proof. We choose the numbers $\theta_k = -\frac{1}{2\pi} \ln a_k$, $k = 1, 2, \ldots, n$. Since a_n are pairwise coprime, θ_k are linearly independent over the integers. For any $\varepsilon \in \left(0, \frac{\pi}{2}\right)$, we take $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \frac{1}{2}$. From Lemma 5 it follows that there exist a real t and integers $h_1, h_2, \ldots, h_n$ such that

$$|t\theta_k - \alpha_k - h_k| < \frac{\varepsilon}{2\pi}, \quad k = \overline{1, n},$$

that is, $|-t\ln a_k-\pi-2\pi h_k|<\varepsilon$.

Since

$$1-a_k^{-s}=1-a_k^{-\sigma-it}=1-a_k^{-\sigma}a_k^{-it}=1-a_k^{-\sigma}\mathrm{e}^{-it\ln a_k}=1+a_k^{-\sigma}\mathrm{e}^{i(-t\ln a_k-\pi)},$$

one has

$$\begin{aligned}|1-a_k^{-s}|&=|1+a_k^{-\sigma}\cos(-t\ln a_k-\pi)+ia_k^{-\sigma}\sin(-t\ln a_k-\pi)|\\&\geq|1+a_k^{-\sigma}\cos(-t\ln a_k-\pi)|=1+a_k^{-\sigma}\cos(-t\ln a_k-\pi-2\pi h_k)\\&\geq 1+a_k^{-\sigma}\cos\varepsilon,\end{aligned}$$

so that

$$|1-a_k^{-s}|\geq 1+a_k^{-\sigma}\cos\varepsilon \tag{2.15}$$

and

$$\left|\sum_{k\geq1}a_k^{-s}\right|\leq\sum_{k\geq1}|a_k^{-s}|=\sum_{k\geq1}a_k^{-\sigma}\leq\sum_{k\geq1}k^{-\sigma}=\zeta(\sigma),\quad \sigma>1.$$

It results that $\sum\limits_{k\geq1}a_k^{-s}$ is absolutely convergent, therefore $\prod\limits_{k\geq1}(1-a_k^{-s})$ is convergent. Hence the product $\prod\limits_{k\geq1}|1-a_k^{-s}|$ converges. By Cauchy condition for convergent products, there is an n_0 such that $n\geq n_0$ implies

$$\left|\prod_{k\geq n+1}|1-a_k^{-s}|-1\right|<\varepsilon,$$

which is equivalent to

$$1-\varepsilon<\prod_{k\geq n+1}|1-a_k^{-s}|<1+\varepsilon.$$

Taking into accoount (2.15), we obtain

$$\prod_{k\geq1}|1-a_k^{-s}|=\prod_{k=1}^{n}|1-a_k^{-s}|\cdot\prod_{k\geq n+1}|1-a_k^{-s}|\geq(1-\varepsilon)\prod_{k=1}^{n}(1+a_k^{-\sigma}\cos\varepsilon). \tag{2.16}$$

From $\sum\limits_{k\geq1}a_k^{-\sigma}\cos\varepsilon\leq\zeta(\sigma)$ it follows that the series $\sum\limits_{k\geq1}a_k^{-\sigma}\cos\varepsilon$ converges uniformly on $\left[0,\frac{\pi}{2}\right]$, hence the product $\prod\limits_{k\geq1}^{n}(1+a_k^{-\sigma}\cos\varepsilon)$ converges uniformly on this set. From (2.16) it results

$$\prod_{k\geq1}\frac{1}{|1-a_k^{-s}|}\leq\frac{1}{(1-\varepsilon)\prod\limits_{k=1}^{n}(1+a_k^{-\sigma}\cos\varepsilon)}$$

for t previously determined. Therefore

$$\inf_{t\in\mathbb{R}} \frac{1}{|P(s)|} \leq \frac{1}{(1-\varepsilon)\prod\limits_{k=1}^{n}(1+a_k^{-\sigma}\cos\varepsilon)},$$

and since the product $\prod\limits_{k\geq 1}^{n}(1+a_k^{-\sigma}\cos\varepsilon)$ converges uniformly, we can let $\varepsilon \to 0$ and pass to the limit term by term to obtain

$$\inf_{t\in\mathbb{R}} \frac{1}{|P(s)|} \leq \frac{1}{\prod\limits_{k\geq 1}(1+a_k^{-\sigma})}.$$

On the other hand, $|1-a_k^{-s}| \leq 1 + a_k^{-\sigma}$, so

$$\inf_{t\in\mathbb{R}} \frac{1}{|P(s)|} \geq \frac{1}{\prod\limits_{k\geq 1}(1+a_k^{-\sigma})}.$$

In conclusion,

$$\inf_{t\in\mathbb{R}} \frac{1}{|P(s)|} = \frac{1}{\prod\limits_{k\geq 1}(1+a_k^{-\sigma})}.$$

□

Proposition 10. *For σ = constant, $\sigma > 1$ one has*

$$\inf_{t\in\mathbb{R}} |\zeta(s)| = \frac{\zeta(2\sigma)}{\zeta(\sigma)}.$$

Proof. Consider $a_k = p_k$, the kth prime number. So, for $\sigma > 1$, constant, we have

$$\inf_{t\in\mathbb{R}} |\zeta(s)| = \inf_{t\in\mathbb{R}} \prod_{k\geq 1} \frac{1}{|1-p_k^{-s}|} = \frac{1}{\prod\limits_{k\geq 1}(1+p_k^{-\sigma})} = \frac{\prod\limits_{k\geq 1}(1-p_k^{-\sigma})}{\prod\limits_{k\geq 1}(1-p_k^{-2\sigma})} = \frac{\zeta(2\sigma)}{\zeta(\sigma)}.$$

□

Remark 11. $|\zeta(s)| \leq \zeta(\sigma)$, $s = \sigma + it$ and $\zeta(s)$ is obtained for $s = \sigma + 0 \cdot t$, so $\sup\limits_{t\in\mathbb{R}} |\zeta(s)| = \zeta(\sigma)$, where σ = constant, $\sigma > 1$.

Theorem 12. *If $s = \sigma + it$ then the following inequalities hold:*

a) $\frac{\zeta(2\sigma-2)}{\zeta(\sigma-1)\zeta(\sigma)} \leq \left|\frac{\zeta(s-1)}{\zeta(s)}\right| = \left|\sum\limits_{n\geq 1} \frac{\varphi(n)}{n^s}\right| \leq \frac{\zeta(\sigma-1)\zeta(\sigma)}{\zeta(2\sigma)}$, *if $\sigma > 2$;*

b) $\frac{\zeta(2\sigma)}{\zeta(\sigma)} \cdot \frac{\zeta(2\sigma-2\operatorname{Re}\alpha)}{\zeta(\sigma-\operatorname{Re}\alpha)} \leq |\zeta(s)\zeta(s-\alpha)| = \left|\sum\limits_{n\geq 1} \frac{\sigma_\alpha(n)}{n^s}\right| \leq \zeta(\sigma)\zeta(\sigma-\operatorname{Re}\alpha)$, *if $\sigma > \max\{1, 1+\operatorname{Re}\alpha\}$;*

c) $\frac{\zeta(4\sigma)}{\zeta(2\sigma)\zeta(\sigma)} \leq \left|\frac{\zeta(2s)}{\zeta(s)}\right| = \left|\sum_{n\geq 1} \frac{\lambda(n)}{n^s}\right| \leq \zeta(\sigma)$, *if* $\sigma > 1$;

d) $\frac{\zeta^2(2\sigma)}{\zeta^2(\sigma)} \leq \left|\sum_{n\geq 1} \frac{\tau(n)}{n^s}\right| \leq \zeta^2(\sigma)$, *if* $\sigma > 1$.

Proof. If we denote $m_\sigma = \inf_{t\in\mathbb{R}} |\zeta(\sigma+it)|$ and $M_\sigma = \sup_{t\in\mathbb{R}} |\zeta(\sigma+it)|$, for $\sigma > 1$, σ constant, from Proposition 10 we have $m_\sigma = \frac{\zeta(2\sigma)}{\zeta(\sigma)}$ and $M_\sigma = \zeta(\sigma)$.

For $\sigma > 2$, arbitrary and fixed,

$$\left|\frac{\zeta(s-1)}{\zeta(s)}\right| \leq \frac{M_{\sigma-1}}{m_\sigma} = \frac{\zeta(\sigma-1)\zeta(\sigma)}{\zeta(2\sigma)},$$

and

$$\left|\frac{\zeta(s-1)}{\zeta(s)}\right| \geq \frac{m_{\sigma-1}}{M_\sigma} = \frac{\zeta(2\sigma-2))}{\zeta(\sigma-1)\zeta(\sigma)}.$$

From these inequalities and Lemma 4 it results a).

For $\sigma > \max\{1; 1+\operatorname{Re}\alpha\}$ arbitrary and fixed,

$$|\zeta(s)\zeta(s-\alpha)| \leq M_\sigma \cdot M_{\sigma-\operatorname{Re}\alpha} = \zeta(\sigma)\zeta(\sigma-\operatorname{Re}\alpha),$$

$$|\zeta(s)\zeta(s-\alpha)| \geq m_\sigma \cdot m_{\sigma-\operatorname{Re}\alpha} = \frac{\zeta(2\sigma)}{\zeta(\sigma)} \cdot \frac{\zeta(2\sigma-2\operatorname{Re}\alpha)}{\zeta(\sigma-\operatorname{Re}\alpha)},$$

so from Lemma 4 it results b).

For $\sigma > 1$, arbitrary and fixed,

$$\left|\frac{\zeta(2s)}{\zeta(s)}\right| \geq \frac{m_{2\sigma}}{M_\sigma} = \frac{\zeta(4\sigma)}{\zeta(2\sigma)\zeta(\sigma)},$$

$$\left|\frac{\zeta(2s)}{\zeta(s)}\right| \leq \frac{M_{2\sigma}}{m_\sigma} = \frac{\zeta(2\sigma)}{\zeta(2\sigma)/\zeta(\sigma)} = \zeta(\sigma),$$

hence, from Lemma 4, it results c). Moreover, as for fixed $\sigma > 1$ one has

$$\frac{\zeta^2(2\sigma)}{\zeta^2(\sigma)} = m_\sigma^2 \leq |\zeta(s)|^2 \leq M_\sigma^2 = \zeta^2(\sigma),$$

and from Lemma 1 it results d). □

Theorem 13. *If* $\sigma > 1$ *is a constant,* $s = \sigma + it$, *and* $\operatorname{Re}\chi(p_k) \geq 0$ *if* $k \geq 1$, *then*

$$\frac{\zeta(2\sigma)}{\zeta(\sigma)} \leq \inf_{t\in\mathbb{R}} \left|\sum_{n\geq 1} \frac{\chi(n)}{n^s}\right| \leq \prod_{p \text{ prime}} \frac{1}{1+p^{-\sigma}\operatorname{Re}\chi(p)}.$$

Proof. It is known that $\chi(n)^{\varphi(k)} = 1$ if $(n,k) = 1$ and $\chi(n) = 0$ if $(n,k) > 1$, so that $|\chi(n)| = 1$ if $(n,k) = 1$. From $|1-\chi(p)p^{-s}| \leq 1 + p^{-\sigma}$ it then results

$$\left|\prod_p \frac{1}{1-\chi(p)p^{-s}}\right| \geq \prod_p \frac{1}{1+p^{-\sigma}} = \frac{\zeta(2\sigma)}{\zeta(\sigma)}.$$

From Lemma 4 it results the left inequality.

As a consequence of

$$\begin{aligned} 1 - \frac{\chi(p)}{p^s} &= 1 - \chi(p)p^{-\sigma}\mathrm{e}^{-it\ln p} = 1 + \chi(p)p^{-\sigma}\mathrm{e}^{i(-t\ln p - \pi)} = \\ &= 1 + \operatorname{Re}\chi(p)p^{-\sigma}\cos(-t\ln p - \pi) - \operatorname{Im}\chi(p)p^{-\sigma}\sin(-t\ln p-\pi) + \\ &\quad + i\big(\operatorname{Im}\chi(p)p^{-\sigma}\cos(-t\ln p - \pi) + \operatorname{Re}\chi(p)p^{-\sigma}\sin(-t\ln p - \pi)\big) \end{aligned}$$

we have

$$|1-\chi(p)| \geq \left|1 + p^{-\sigma}\big(\operatorname{Re}\chi(p)\cos(-t\ln p-\pi) - \operatorname{Im}\chi(p)\sin(-t\ln p - \pi)\big)\right|.$$

Since

$$\begin{aligned} &|\operatorname{Re}\chi(p)\cos(-t\ln p - \pi) - \operatorname{Im}\chi(p)\sin(-t\ln p-\pi)| \\ &\qquad \leq \sqrt{(\operatorname{Re}\chi(p))^2 + (\operatorname{Im}\chi(p))^2} = 1 \end{aligned}$$

we obtain

$$|1-\chi(p)| \geq 1 + p^{-\sigma}\big(\operatorname{Re}\chi(p)\cos(-t\ln p - \pi) - \operatorname{Im}\chi(p)\sin(-t\ln p - \pi)\big). \quad (2.17)$$

We choose $\theta_k = -\frac{1}{2\pi}\ln p_k$, $\alpha_k = \frac{1}{2}$ $(k = \overline{1,n})$.

From Lemma 5 it results that there exist a real t and integers $h_1, h_2, \ldots, h_n$ such that $|-t\ln p_k - \pi - 2\pi h_k| < \varepsilon$ for $0 < \varepsilon < \frac{\pi}{2}$.

Note that $\cos(-t\ln p_k - \pi) = \cos(-t\ln p_k - \pi - 2\pi h_k) > \cos\varepsilon$ and

$$-\sin\varepsilon < \sin(-t\ln p_k - \pi) = \sin(-t\ln p_k - \pi - 2\pi h_k) < \sin\varepsilon$$

because $-\frac{\pi}{2} < -\varepsilon < -t\ln p_k - \pi - 2\pi h_k < \varepsilon < \frac{\pi}{2}$, so it results that

$$|1-\chi(p)p^{-s}| \geq 1 + p^{-\sigma}\left(\operatorname{Re}\chi(p)\cos\varepsilon - |\operatorname{Im}\chi(p)|\sin\varepsilon\right) > 0$$

and

$$\frac{1}{|1-\chi(p)p^{-s}|} \leq \frac{1}{1 + p^{-\sigma}\left(\operatorname{Re}\chi(p)\cos\varepsilon - |\operatorname{Im}\chi(p)|\sin\varepsilon\right)}.$$

Since $\sum\limits_{k\geq 1}\chi(p_k)p_k^{-s}$ converges absolutely, it follows that $\prod\limits_{k\geq 1}\big(1 - \chi(p_k)p_k^{-s}\big)$ converges, so $\prod\limits_{k\geq 1}|1-\chi(p_k)p_k^{-s}|$ converges, too. It results that $\prod\limits_{k\geq 1}\frac{1}{|1-\chi(p_k)p_k^{-s}|}$ converges. By Cauchy condition, there is an n_0 such that $n \geq n_0$ implies

$$1 - \varepsilon < \prod_{k\geq n+1}\frac{1}{|1-\chi(p_k)p_k^{-s}|} < 1 + \varepsilon.$$

We obtain, for n and ε given, a t such that

$$\prod_{k\geq 1}\frac{1}{|1-\chi(p_k)p_k^{-s}|}=\prod_{k=1}^{n}\frac{1}{|1-\chi(p_k)p_k^{-s}|}\cdot\prod_{k\geq n+1}\frac{1}{|1-\chi(p_k)p_k^{-s}|}\leq$$
$$\leq(1+\varepsilon)\prod_{k=1}^{n}\frac{1}{1+p_k^{-\sigma}\left(\operatorname{Re}\chi(p_k)\cos\varepsilon-|\operatorname{Im}\chi(p_k)|\sin\varepsilon\right)},$$

so that

$$\inf_{t\in\mathbb{R}}\left|\prod_{k\geq 1}\frac{1}{1-\chi(p_k)p_k^{-s}}\right|\leq(1+\varepsilon)\prod_{k=1}^{n}\frac{1}{1+p_k^{-\sigma}[\operatorname{Re}\chi(p_k)\cos\varepsilon-|\operatorname{Im}\chi(p_k)|\sin\varepsilon]}.$$

From

$$\left|\sum_{k\geq 1}p_k^{-\sigma}\big(\operatorname{Re}\chi(p_k)\cos\varepsilon-|\operatorname{Im}\chi(p_k)|\sin\varepsilon\big)\right|\leq$$
$$\leq\sum_{k\geq 1}p_k^{-\sigma}\sqrt{|\operatorname{Re}\chi(p_k)|^2+|\operatorname{Im}\chi(p)|^2}=\sum_{k\geq 1}p_k^{-\sigma}\leq\zeta(\sigma)$$

it results that the product $\prod_{k\geq 1}(1+p_k^{-\sigma}(\operatorname{Re}\chi(p_k)\cos\varepsilon-|\operatorname{Im}\chi(p_k)|\sin\varepsilon)$ converges uniformly on $\left[0,\frac{\pi}{2}\right]$. We can let $\varepsilon\to 0$ and pass to the limit term by term to obtain

$$\inf_{t\in\mathbb{R}}\left|\prod_{k\geq 1}\frac{1}{1-\chi(p_k)p_k^{-s}}\right|\leq\prod_{k\geq 1}\frac{1}{1+p_k^{-\sigma}\operatorname{Re}\chi(p_k)}.$$

□

Remark 14. χ is completely multiplicative, therefore χ^m is completely multiplicative, so, from [2, Theorem 11.7] it results that for $\sigma>1$ one has

$$L(s,\chi^m)=\sum_{n\geq 1}\frac{\chi^m(n)}{n^s}=\prod_{p\ prime}\frac{1}{1-\chi^m(p)p^{-s}}.$$

Just like before it results that, when $k\geq 1$ and m is positive integer such that $\operatorname{Re}\chi^m(p_k)\geq 0$, one has

$$\frac{\zeta(2\sigma)}{\zeta(\sigma)}\leq\inf_{t\in\mathbb{R}}\left|\sum_{n\geq 1}\frac{\chi^m(n)}{n^s}\right|\leq\prod_{p}\frac{1}{1+p^{-\sigma}\operatorname{Re}\chi^m(p)},$$

with σ constant.

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Twice Fréchet differentiability of some functions on $\mathbb{R}^n$

Dumitru Popa[1)]

Abstract. We give necessary and sufficient conditions for the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x_1, \dots, x_n) = \begin{cases} \dfrac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{\prod\limits_{i=1}^{m} \left(x_1^{2\beta_i} + \cdots + x_n^{2\beta_i} \right)^{\gamma_i}} & \text{if } (x_1, \dots, x_n) \neq (0, \dots, 0) \\ 0 & \text{if } (x_1, \dots, x_n) = (0, \dots, 0) \end{cases}$$

to be twice Fréchet differentiable at $(0, \dots, 0)$.

Keywords: twice Fréchet differentiable, mixed partial derivative
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The Fréchet differentiability is a fundamental concept in analysis, see [1]. There are standard examples of functions $f : \mathbb{R}^2 \to \mathbb{R}$ which are continuous at $(0, 0)$, for which there exist the derivatives $\frac{\partial f}{\partial x}(0, 0)$, $\frac{\partial f}{\partial y}(0, 0)$, but which are not Fréchet differentiable at $(0, 0)$. As the author knows, in the literature there does not seem to exist a standard example of function $f : \mathbb{R}^2 \to \mathbb{R}$ for which the derivatives $\frac{\partial^2 f}{\partial x^2}(0, 0)$, $\frac{\partial^2 f}{\partial x \partial y}(0, 0) = \frac{\partial^2 f}{\partial y \partial x}(0, 0)$, $\frac{\partial^2 f}{\partial y^2}(0, 0)$ exist but f is not twice Fréchet differentiable at $(0, 0)$. For different examples see [2]. The main purpose of this paper is to find necessary and sufficient conditions for some functions on $\mathbb{R}^n$ to be twice Fréchet differentiable at 0.

1. Preliminary results

Let n be a positive integer. If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $1 \leq p < \infty$, we consider $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$ and $\|x\|_\infty = \max\limits_{1 \leq i \leq n} |x_i|$. We will use sometimes the notation 0 to denote the vector $(0, \dots, 0) \in \mathbb{R}^n$. By definition, a function $f : \mathbb{R}^n \to \mathbb{R}$ is twice Fréchet differentiable at 0 if and only if there exist $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} : \mathbb{R}^n \to \mathbb{R}$ and the mapping $f' : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f'(x) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$ is Fréchet differentiable at 0, which is equivalent to saying that there exist all $\frac{\partial^2 f}{\partial x_i \partial x_j}(0)$, $1 \leq i, j \leq n$, and

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$\lim\limits_{x\to 0}\frac{\|f'(x)-f'(0)-f''(0)(x)\|_2}{\|x\|_2}=0$, where $f''(0)=\left(\frac{\partial^2 f}{\partial x_i\partial x_j}(0)\right)_{1\le i,j\le n}$ is the Hessian matrix, see [1]. All notations used in this paper are standard, see [1].

Proposition 1. *Let $n\geq 2$, m be positive integers, $1\leq p_1,\dots,p_m\leq\infty$ and a, a_1, …, a_m be real numbers. If $a>a_1+\cdots+a_m+1$, then*

$$\lim_{x\to 0}\frac{\left(\|x\|_{p_1}\cdots\|x\|_{p_m}\right)^{\frac{a}{m}}}{\|x\|_{p_1}^{a_1}\cdots\|x\|_{p_m}^{a_m}\|x\|_2}=0.$$

Proof. Let $x\in\mathbb{R}^n$, $x\neq 0$. We have

$$\frac{\left(\|x\|_{p_1}\cdots\|x\|_{p_m}\right)^{\frac{a}{m}}}{\|x\|_{p_1}^{a_1}\cdots\|x\|_{p_m}^{a_m}\|x\|_2}=$$
$$=\left(\frac{\|x\|_{p_1}}{\|x\|_2}\cdots\frac{\|x\|_{p_m}}{\|x\|_2}\right)^{\frac{a}{m}}\cdot\left(\frac{\|x\|_2}{\|x\|_{p_1}}\right)^{a_1}\cdots\left(\frac{\|x\|_2}{\|x\|_{p_m}}\right)^{a_m}\cdot\|x\|_2^{a-(a_1+\cdots+a_m+1)}.$$

But, for $1\leq p\leq\infty$ there exist $A_n(p),B_n(p)>0$ such that $A_n(p)\leq\frac{\|x\|_p}{\|x\|_2}\leq$ $\leq B_n(p)$ (this is a standard result, see [1]; this follows from the well known result that on $\mathbb{R}^n$ two norms are equivalent). Then $\left(\frac{\|x\|_p}{\|x\|_2}\right)^{\alpha}\leq C_n(\alpha,p)$, where

$$C_n(\alpha,p)=\begin{cases}[B_n(p)]^{\alpha} & \text{if } \alpha>0\\ 1 & \text{if } \alpha=0\\ [A_n(p)]^{\alpha} & \text{if } \alpha<0\end{cases}.$$

Then

$$\frac{\left(\|x\|_{p_1}\cdots\|x\|_{p_m}\right)^{\frac{a}{m}}}{\|x\|_{p_1}^{a_1}\cdots\|x\|_{p_m}^{a_m}\|x\|_2}\leq\left[\prod_{i=1}^{m}C_n\left(\frac{a}{m},p_i\right)C_n(-a_i,p_i)\right]\|x\|_2^{a-(a_1+\cdots+a_m+1)}. \tag{1.1}$$

Since $a>a_1+\cdots+a_m+1$ it follows that $\lim\limits_{x\to 0}\|x\|_2^{a-(a_1+\cdots+a_m+1)}=0$, which combined to (1.1) proves Proposition 1. ◻

Proposition 2. *Let n, m, α_1, …, α_n, β_1, …, β_m, $n\geq 2$ be positive integers, γ_1, …, γ_m be positive real numbers, $\psi_i:\mathbb{R}^n\to\mathbb{R}$ $(1\leq i\leq n)$*

$$\psi_i(x_1,\dots,x_n)=x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}}x_i^{\alpha_i-1}x_{i+1}^{\alpha_{i+1}}\cdots x_n^{\alpha_n}.$$

Then

$$|\psi_i(x_1,\dots,x_n)|\leq\left(\|x\|_{2\beta_1}\cdots\|x\|_{2\beta_m}\right)^{\frac{\alpha_1+\cdots+\alpha_n-1}{m}}.$$

Proof. Let $x=(x_1,\dots,x_n)\in\mathbb{R}^n$. We have

$$|\psi_i(x_1,\dots,x_n)|=|x_1|^{\alpha_1}\cdots|x_{i-1}|^{\alpha_{i-1}}|x_i|^{\alpha_i-1}|x_{i+1}|^{\alpha_{i+1}}\cdots|x_i|^{\alpha_n}$$

and let us note that $|x_1| \leq \|x\|_{2\beta_1}, \ldots, |x_n| \leq \|x\|_{2\beta_1}$. If $\alpha_i - 1 \neq 0$ we have that

$$|\psi_i(x_1, \ldots, x_n)| \leq \|x\|_{2\beta_1}^{\alpha_1+\cdots+\alpha_{i-1}+(\alpha_i-1)+\alpha_{i+1}+\cdots+\alpha_n} = \|x\|_{2\beta_1}^{\alpha_1+\cdots+\alpha_n-1}$$

and if $\alpha_i - 1 = 0$

$$\begin{aligned} |\psi_i(x_1, \ldots, x_n)| &= |x_1|^{\alpha_1} \cdots |x_{i-1}|^{\alpha_{i-1}} |x_{i+1}|^{\alpha_{i+1}} \cdots |x_i|^{\alpha_n} \\ &\leq \|x\|_{2\beta_1}^{\alpha_1+\cdots+\alpha_{i-1}+\alpha_{i+1}+\cdots+\alpha_n} \\ &= \|x\|_{2\beta_1}^{\alpha_1+\cdots+\alpha_n-1}. \end{aligned}$$

Thus $|\psi_i(x_1, \ldots, x_n)| \leq \|x\|_{2\beta_1}^{\alpha_1+\cdots+\alpha_n-1}$. Similarly we obtain

$$|\psi_i(x_1, \ldots, x_n)| \leq \|x\|_{2\beta_2}^{\alpha_1+\cdots+\alpha_n-1}, \ldots, |\psi_i(x_1, \ldots, x_n)| \leq \|x\|_{2\beta_m}^{\alpha_1+\cdots+\alpha_n-1},$$

which by multiplication gives us

$$|\psi_i(x_1, \ldots, x_n)| \leq \left(\|x\|_{2\beta_1} \cdots \|x\|_{2\beta_m}\right)^{\frac{\alpha_1+\cdots+\alpha_n-1}{m}}.$$

□

2. The main results

The next proposition is the main result of this paper. It is a natural addition to Proposition 2 in [3].

Proposition 3. *Let n, m, $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_m$, $n \geq 2$ be positive integers, $\gamma_1, \ldots, \gamma_m$ be positive real numbers and let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by*

$$f(x_1, \ldots, x_n) = \begin{cases} \dfrac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{\prod_{i=1}^{m} \left(x_1^{2\beta_i} + \cdots + x_n^{2\beta_i}\right)^{\gamma_i}} & \text{if } (x_1, \ldots, x_n) \neq (0, \ldots, 0) \\ 0 \text{ if } (x_1, \ldots, x_n) = (0, \ldots, 0) \end{cases}.$$

Then:

(i) *f is continuous at $(0, \ldots, 0)$ if and only if*

$$\alpha_1 + \cdots + \alpha_n > 2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m.$$

(ii) *f is Fréchet differentiable at $(0, \ldots, 0)$ if and only if*

$$\alpha_1 + \cdots + \alpha_n > 2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m + 1.$$

(iii) *f is twice Fréchet differentiable at $(0, \ldots, 0)$ if and only if*

$$\alpha_1 + \cdots + \alpha_n > 2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m + 2.$$

Proof. (i) and (ii) were proved in [3, Proposition 2]. Moreover, $f'(0) = 0$.
(iii) Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and define $S_k = x_1^{2\beta_k} + \cdots + x_n^{2\beta_k} = \|x\|_{2\beta_k}^{2\beta_k}$ for $1 \le k \le m$. Let $1 \le i \le n$. For all $(x_1, \ldots, x_n) \ne (0, \ldots, 0)$ we have

$$\frac{\partial f}{\partial x_i} = \frac{\psi_i(x_1, \ldots, x_n)(\alpha_i S_1 \cdots S_m - 2T_i)}{S_1^{1+\gamma_1} \cdots S_m^{1+\gamma_m}}, \tag{2.2}$$

where

$$\psi_i(x_1, \ldots, x_n) = x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} x_i^{\alpha_i - 1} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n},$$

$$T_i = \beta_1\gamma_1 x_i^{2\beta_1} S_2 \cdots S_m + \beta_2\gamma_2 x_i^{2\beta_2} S_1 S_3 \cdots S_m + \cdots + \beta_m\gamma_m x_i^{2\beta_m} S_1 \cdots S_{m-1}$$

if $m \ge 2$ and $T_i = \beta_1\gamma_1 x_i^{2\beta_1}$ if $m = 1$.

Let us suppose that f is twice Fréchet differentiable at $(0, \ldots, 0)$. We prove that $\frac{\partial^2 f}{\partial x_i \partial x_j}(0) = 0$, $1 \le i, j \le n$.

Case $n = 2$. In this case for $(x_1, x_2) \ne (0, 0)$ we have

$$\frac{\partial f}{\partial x_1} = \frac{x_1^{\alpha_1 - 1} x_2^{\alpha_2}(\alpha_1 S_1 \cdots S_m - 2T_1)}{S_1^{1+\gamma_1} \cdots S_m^{1+\gamma_m}}; \frac{\partial f}{\partial x_2} = \frac{x_1^{\alpha_1} x_2^{\alpha_2 - 1}(\alpha_2 S_1 \cdots S_m - 2T_2)}{S_1^{1+\gamma_1} \cdots S_m^{1+\gamma_m}}$$

and $\frac{\partial f}{\partial x_1}(0,0) = \frac{\partial f}{\partial x_2}(0,0) = 0$. Then $\frac{\partial f}{\partial x_1}(x_1, 0) = 0$, $\forall x_1 \in \mathbb{R}$; $\frac{\partial f}{\partial x_2}(0, x_2) = 0$, $\forall x_2 \in \mathbb{R}$ which gives us $\frac{\partial^2 f}{\partial x_1^2}(0,0) = \frac{\partial^2 f}{\partial x_2^2}(0,0) = 0$.

If $\alpha_1 - 1 \ne 0$ then $\frac{\partial f}{\partial x_1}(0, x_2) = 0$, $\forall x_2 \in \mathbb{R}$ from where $\frac{\partial^2 f}{\partial x_2 \partial x_1}(0,0) = 0$.

If $\alpha_1 - 1 = 0$ then $\frac{\partial f}{\partial x_1}(0, x_2) = x_2^{\alpha_2 - (2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m)}$, $\forall x_2 \in \mathbb{R}$.

If $\alpha_2 - 1 \ne 0$ then $\frac{\partial f}{\partial x_2}(x_1, 0) = 0$, $\forall x_1 \in \mathbb{R}$ from where $\frac{\partial^2 f}{\partial x_1 \partial x_2}(0,0) = 0$.

If $\alpha_2 - 1 = 0$ then $\frac{\partial f}{\partial x_2}(x_1, 0) = x_1^{\alpha_1 - (2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m)}$, $\forall x_1 \in \mathbb{R}$.

We have the following cases:

a) $\alpha_1 - 1 \ne 0$ and $\alpha_2 - 1 \ne 0$. In this case $\frac{\partial^2 f}{\partial x_1 \partial x_2}(0,0) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(0,0) = 0$.

b) $\alpha_1 - 1 \ne 0$ and $\alpha_2 - 1 = 0$ (respectively $\alpha_1 - 1 = 0$ and $\alpha_2 - 1 \ne 0$). In this case we have $\frac{\partial^2 f}{\partial x_2 \partial x_1}(0,0) = 0$ (respectively $\frac{\partial^2 f}{\partial x_1 \partial x_2}(0,0) = 0$). Since f is twice Fréchet differentiable at $(0,0)$ by Schwarz's symmetry theorem, $\frac{\partial^2 f}{\partial x_1 \partial x_2}(0,0) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(0,0) = 0$.

c) $\alpha_1 - 1 = 0$ and $\alpha_2 - 1 = 0$. In this case we have $\frac{\partial f}{\partial x_1}(0, x_2) = x_2^{1 - (2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m)}$, $\frac{\partial f}{\partial x_2}(x_1, 0) = x_1^{1 - (2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m)}$ and

$$\frac{\partial^2 f}{\partial x_2 \partial x_1}(0,0) = \lim_{x_2 \to 0} \frac{\frac{\partial f}{\partial x_1}(0, x_2) - \frac{\partial f}{\partial x_1}(0,0)}{x_2} = \lim_{x_2 \to 0} \frac{1}{x_2^{2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m}} = \infty$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(0,0) = \lim_{x_1 \to 0} \frac{\frac{\partial f}{\partial x_2}(x_1, 0) - \frac{\partial f}{\partial x_2}(0,0)}{x_1} = \lim_{x_2 \to 0} \frac{1}{x_1^{2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m}} = \infty$$

so this case is impossible.

Case $n \geq 3$. Let $1 \leq i \leq n$. For $t \in \mathbb{R}$ we have

$$\frac{\partial f}{\partial x_i}(0, \dots, 0, \underbrace{t}_{i}, 0, \dots, 0) = 0$$

which gives us $\frac{\partial^2 f}{\partial x_i^2}(0, \dots, 0) = 0$. Also

$$\frac{\partial f}{\partial x_i}(0, \dots, 0, \underbrace{t}_{j}, 0, \dots, 0) = 0$$

for $j \neq i$ (since $n \geq 3$ there exist $1 \leq k \leq n$ with $k \neq i$ and $k \neq j$ such that the kth position of the vector $(0, \dots, 0, \underbrace{t}_{j}, 0, \dots, 0)$ is 0, so $\psi_i(0, \dots, 0, \underbrace{t}_{j}, 0, \dots, 0) = 0$) which gives us that $\frac{\partial^2 f}{\partial x_j \partial x_i}(0, \dots, 0) = 0$.

Since f is twice Fréchet differentiable at $(0, \dots, 0)$,

$$\lim_{x \to 0} \frac{\|f'(x) - f'(0) - f''(0)(x)\|_2}{\|x\|_2} = 0,$$

where $f''(0) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right)$. Thus, $\lim\limits_{x \to 0} \frac{\|f'(x)\|_2}{\|x\|_2} = 0$.

Since $\|\cdot\|_\infty \leq \|\cdot\|_2 \leq \sqrt{n}\,\|\cdot\|_\infty$ we deduce $\lim\limits_{x \to 0} \frac{\|f'(x)\|_\infty}{\|x\|_\infty} = 0$ and this implies, in particular,

$$\lim_{t \to 0} \frac{\|f'(t, \dots, t)\|_\infty}{|t|} = 0.$$

Let $1 \leq i \leq n$. From (2.2) for each $t \in \mathbb{R}$, $t \neq 0$, we have

$$\frac{\partial f}{\partial x_i}(t, \dots, t) = \frac{n\alpha_i - (2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m)}{n^{1+\gamma_1+\cdots+\gamma_m}} \cdot t^{\alpha_1+\cdots+\alpha_n-2\beta_1\gamma_1-\cdots-2\beta_m\gamma_m-1}.$$

We show there exists $1 \leq i \leq n$ such that $n\alpha_i - (2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m) \neq 0$. Otherwise, if $n\alpha_i - (2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m) = 0$ for all $1 \leq i \leq n$ we get, by summing up, that $\alpha_1 + \cdots + \alpha_n = 2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m$ which contradicts (i). (We supposed f is twice Fréchet differentiable at 0, hence f is continuous at 0.)

Let $I = \{1 \leq i \leq n \mid n\alpha_i - (2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m) \neq 0\}$ which, as we proved is non-empty. Since

$$\frac{\partial f}{\partial x_i}(t, \dots, t) = 0, \forall i \in \{1, \dots, n\} \setminus I,$$

we deduce

$$\|f'(t, \dots, t)\|_\infty = \max_{1 \leq i \leq n} \left| \frac{\partial f}{\partial x_i}(t, \dots, t) \right| = M\,|t|^{\alpha_1+\cdots+\alpha_n-2\beta_1\gamma_1-\cdots-2\beta_m\gamma_m-1},$$

where $M = \frac{1}{n^{1+\gamma_1+\cdots+\gamma_m}} \max_{i\in I} |n\alpha_i - (2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m)| > 0$. Then,

$$\frac{\|f'(t,\ldots,t)\|_\infty}{|t|} = M\,|t|^{\alpha_1+\cdots+\alpha_n-2\beta_1\gamma_1-\cdots-2\beta_m\gamma_m-2}$$

so $\lim_{t\to 0} |t|^{\alpha_1+\cdots+\alpha_n-2\beta_1\gamma_1-\cdots-2\beta_m\gamma_m-2} = 0$, which implies

$$\alpha_1 + \cdots + \alpha_n > 2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m + 2.$$

Conversely, let us suppose that $\alpha_1+\cdots+\alpha_n > 2\beta_1\gamma_1+\cdots+2\beta_m\gamma_m+2$. Let $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, $x \neq 0$. For all $1 \leq i \leq n$ from (2.2) we deduce

$$\left|\frac{\partial f}{\partial x_i}\right| \leq |\psi_i(x_1,\ldots,x_n)| \, \frac{\alpha_i S_1\cdots S_m + 2T_i}{S_1^{1+\gamma_1}\cdots S_m^{1+\gamma_n}}.$$

Since $x_i^{2\beta_1} \leq \|x\|_{2\beta_1}^{2\beta_1}, \ldots, x_i^{2\beta_m} \leq \|x\|_{2\beta_m}^{2\beta_m}$ we get $T_i \leq L_m \|x\|_{2\beta_1}^{2\beta_1}\cdots\|x\|_{2\beta_m}^{2\beta_m}$, where $L_m = \beta_1\gamma_1 + \beta_2\gamma_2 + \cdots + \beta_m\gamma_m$. From Proposition 2 we obtain

$$\left|\frac{\partial f}{\partial x_i}\right| \leq \frac{(\alpha_i + 2L_m)\left(\|x\|_{2\beta_1}\cdots\|x\|_{2\beta_m}\right)^{\frac{\alpha_1+\cdots+\alpha_n-1}{m}}}{\|x\|_{2\beta_1}^{2\beta_1\gamma_1}\cdots\|x\|_{2\beta_m}^{2\beta_m\gamma_m}}.$$

Then

$$\frac{\|f'(x) - f'(0)\|_2}{\|x\|_2} \leq K_m \cdot \frac{\left(\|x\|_{2\beta_1}\cdots\|x\|_{2\beta_m}\right)^{\frac{\alpha_1+\cdots+\alpha_n-1}{m}}}{\|x\|_{2\beta_1}^{2\beta_1\gamma_1}\cdots\|x\|_{2\beta_m}^{2\beta_m\gamma_m}\,\|x\|_2},$$

where $K_m = \left(\sum_{i=1}^{n} (\alpha_i + 2L_m)\right)^{\frac{1}{2}}$.

Since $\alpha_1 + \cdots + \alpha_n - 1 > 2\beta_1\gamma_1 + \cdots + 2\beta_m\gamma_m + 1$ from Proposition 1 we have

$$\lim_{x\to 0} \frac{\left(\|x\|_{2\beta_1}\cdots\|x\|_{2\beta_m}\right)^{\frac{\alpha_1+\cdots+\alpha_n-1}{m}}}{\|x\|_{2\beta_1}^{2\beta_1\gamma_1}\cdots\|x\|_{2\beta_m}^{2\beta_m\gamma_m}\,\|x\|_2} = 0$$

and thus $\lim_{x\to 0} \frac{\|f'(x)-f'(0)\|_2}{\|x\|_2} = 0$, i.e. f is twice Fréchet differentiable at $(0,\ldots,0)$ and $f''(0) = 0$. □

From Proposition 3 we obtain the next corollary which is an additional result to [3, Corollary 4].

Corollary 4. *Let $n \geq 2$ be a positive integer and $f : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$f(x_1,\ldots,x_n) = \begin{cases} \dfrac{x_1x_2\cdots x_n}{\left(x_1^2+\cdots+x_n^2\right)\left(x_1^4+\cdots+x_n^4\right)} & \text{if } (x_1,\ldots,x_n) \neq (0,\ldots,0) \\ 0 & \text{if } (x_1,\ldots,x_n) = (0,\ldots,0) \end{cases}.$$

Then:

(i) *f is continuous at $(0,\ldots,0)$ if and only if $n \geq 7$.*

(ii) *f is Fréchet differentiable at $(0,\ldots,0)$ if and only if $n \geq 8$.*

(iii) *f is twice Fréchet differentiable at $(0,\dots,0)$ if and only if $n \geq 9$.*

Another particular case of Proposition 3 is the following

Corollary 5. *Let α, β be positive integers and let $f:\mathbb{R}^2\to\mathbb{R}$,*

$$f(x,y)=\begin{cases}\frac{x^\alpha y^\beta}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}.$$

Then:

(i) *f is continuous at $(0,0)$.*
(ii) *f is Fréchet differentiable at $(0,0)$ if and only if $\alpha+\beta\geq 3$.*
(iii) *f is twice Fréchet differentiable at $(0,0)$ if and only if $\alpha+\beta\geq 4$.*

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Traian Lalescu national mathematical contest for university students, Braşov 2015

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Abstract. This note presents the solutions to the problems proposed at the 2015 edition of the Traian Lalescu national mathematical contest for university students.

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The 2015 edition of the Traian Lalescu National Mathematics contest for university students has been hosted between May 21st and May 23rd by the Transilvania University of Braşov.

73 students participated at the contest, representing 12 universities from Braşov, Bucureşti, Constanţa, Craiova, Iaşi and Timişoara.

The contest was organized in four sections: Section A for Mathematics faculties students, Section B for first-year students of technical faculties, electric specialization, Section C for first-year students of technical faculties, mechanics and constructions specializations, and Section D for second-year students of technical faculties.

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We present in the sequel the statements and solutions of the problems pertaining to Sections A and B of the contest.

Section A

Problem 1. Let $n \geq 2$ be an integer. Prove the equivalence of the following statements:

i) Every element of $\mathbb{Z}_n$ can be written as a product of an idempotent element and an invertible one.

ii) n is squarefree.

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Solution. „$\Rightarrow$": Suppose n is not squarefree. Then there is a prime p such that $p^2 \mid n$. Suppose we have $\hat{p} = \hat{u} \cdot \hat{e}$ in $\mathbb{Z}_n$, $\hat{u}$ being invertible and $\hat{e}$ idempotent. Let $\hat{v} = \hat{u}^{-1}$. Then $(\widehat{vp})^2 = \widehat{vp}$, and thus $p^2 \mid n \mid vp(vp-1)$. Since $\hat{v}$ is invertible, $(n, v) = 1$, so $p \nmid v$. Thus, $p \mid vp - 1$, contradiction.

„$\Leftarrow$" If n is squarefree, it can be written as $p_1 \cdots p_s$, where $p_1, \ldots, p_s$ are pairwise distinct primes. We then have the ring isomorphism

$$\varphi : \mathbb{Z}_n \to \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}, \quad \varphi(a + n\mathbb{Z}) = (a + p_1\mathbb{Z}, \ldots, a + p_s\mathbb{Z}),$$

and for each $a + n\mathbb{Z} \in \mathbb{Z}_n$ we may write

$$a + n\mathbb{Z} = \varphi^{-1}(b_1 + p_1\mathbb{Z}, , \ldots, b_s + p_s\mathbb{Z})\varphi^{-1}(\varepsilon_1 + p_1\mathbb{Z}, \ldots, \varepsilon_s + p_s\mathbb{Z}),$$

where $b_i = \begin{cases} a & \text{if } p_i \nmid a \\ 1 & \text{if } p_i \mid a \end{cases}$ and $\varepsilon_i = \begin{cases} 1 & \text{if } p_i \nmid a \\ 0 & \text{if } p_i \mid a \end{cases}$. Since the first factor of the product is invertible and the second is idempotent, we are done.

Problem 2. a) Let $p \geq 3$ be a prime number and let $A \in \mathcal{M}_p(\mathbb{C})$ be such that $\operatorname{tr}(A) = 0$ and $\det(A - I_p) \neq 0$. Show that $A^p \neq I_p$.

b) Show that for every composite number $n \geq 4$ there is $A \in \mathcal{M}_n(\mathbb{C})$ such that $\operatorname{tr}(A) = 0$, $\det(A - I_n) \neq 0$, and $A^n = I_n$.

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Solution. a) Let us denote by μ_A the minimal polynomial of A and by P_A the characteristic polynomial of A. Suppose $A^p = I_p$. Then $\mu_A \mid X^p - 1$. Since $\det(A - I_p) \neq 0$, $\mu_A \mid X^{p-1} + X^{p-2} + \cdots + 1$. Since μ_A and P_A have the same roots, we derive that $P_A = \prod\limits_{k=1}^{p-1}(X - \varepsilon^k)^{\alpha_k}$, where $\varepsilon = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}$, $\alpha_k \in \mathbb{N}$, and $\sum\limits_{k=1}^{p-1}\alpha_k = p$. Then $\sum\limits_{k=1}^{p-1}\alpha_k\varepsilon^k = \operatorname{tr}(A) = 0$, and therefore ε is a root of $\sum_{k=1}^{p-1}\alpha_k X^{k-1}$. But $f = X^{p-1} + X^{p-2} + \cdots + 1$ is irreducible over $\mathbb{Q}$, so it is the minimal polynomial of ε over $\mathbb{Q}$. We derive that $f \mid \sum_{k=1}^{p-1} \alpha_k X^{k-1}$, whence $\alpha_1 = \alpha_2 = \cdots = \alpha_{p-1} = 0$, contradiction.

b) Let $n \geq 4$ be composite. Let p be a prime factor of n, and $q = \frac{n}{p}$. Since n is composite, $q > 1$. Let $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, $\delta = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}$, and

$$A = \begin{pmatrix} \varepsilon I_q & 0 & \dots & 0 \\ 0 & \delta\varepsilon I_q & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \delta\varepsilon^{p-1} I_q \end{pmatrix} \in \mathcal{M}_n(\mathbb{C}).$$

We then have:

$$\mathrm{tr}(A) = (1 + \delta + \cdots + \delta^{p-1})q\varepsilon = 0,$$

$$\begin{aligned} \det(I_n - A) &= [(1-\varepsilon)(1-\delta\varepsilon)\cdots(1-\delta^{p-1}\varepsilon)]^q \\ &= \delta^{\frac{pq(p-1)}{2}}(\varepsilon - 1)(\varepsilon - \delta^{-1})\cdots(\varepsilon - \delta^{1-p}) \\ &= \delta^{\frac{pq(p-1)}{2}}(\varepsilon^p - 1) \neq 0, \end{aligned}$$

and, obviously, $A^n = I_n$.

Remark. In the particular case when $n = 5$ the problem was given at the final stage of the National Olympiad, XI-th class, 2015, problem 2 (see onm2015.ssmr.ro/subiecte).

Problem 3. Show that the least positive constant C such that the inequality

$$\int_0^a \frac{x}{\int_0^x f(t)\,\mathrm{d}t}\,\mathrm{d}x < C \int_0^a \frac{1}{f(x)}\,\mathrm{d}x$$

holds for every $a > 0$ and every continuous function $f : [0, \infty) \to (0, \infty)$ is $C = 2$.

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Solution. Let $a > 0$. We first notice that

$$\lim_{x \searrow 0} \frac{x}{\int_0^x f(t)\mathrm{d}t} = \frac{1}{f(0)} \in \mathbb{R},$$

so the integral $\int_0^a \frac{x}{\int_0^x f(t)\mathrm{d}t}\mathrm{d}x$ is convergent.

For every $x > 0$ the Cauchy-Schwarz inequality yields

$$\frac{x^4}{4} = \left[\int_0^x \left(\sqrt{f(t)} \cdot \frac{t}{\sqrt{f(t)}}\right) \mathrm{d}t\right]^2 \leq \left(\int_0^x f(t)\mathrm{d}t\right) \cdot \left(\int_0^x \frac{t^2}{f(t)}\mathrm{d}t\right),$$

and thus

$$\frac{x}{\int_0^x f(t)\mathrm{d}t} \leq \frac{4}{x^3} \cdot \left(\int_0^x \frac{t^2}{f(t)}\mathrm{d}t\right).$$

Since $\lim\limits_{x\searrow 0}\frac{\int_0^x t^2/f(t)\mathrm{d}t}{x^3}=\frac{1}{3f(0)}\in\mathbb{R}$, $\int_0^a\frac{\int_0^x t^2/f(t)\mathrm{d}t}{x^3}\mathrm{d}x$ is convergent. We may thus write

$$\int_0^a\frac{x}{\int_0^x f(t)\mathrm{d}t}\mathrm{d}x\overset{(2)}{\leq}\int_0^a\frac{4}{x^3}\left(\int_0^x\frac{t^2}{f(t)}\mathrm{d}t\right)\mathrm{d}x=\int_0^a\frac{t^2}{f(t)}\left(\int_t^a 4x^{-3}\mathrm{d}x\right)\mathrm{d}t$$
$$=2\int_0^a\frac{t^2}{f(t)}\left(\frac{1}{t^2}-\frac{1}{a^2}\right)\mathrm{d}t<2\int_0^a\frac{1}{f(t)}\mathrm{d}t.$$

Therefore, the required constant is at most 2.

On the other hand, let $C>0$ be a constant such that the inequality in the statement holds. Using the given inequality for $f(x)=x+1$, we get

$$C\geq\lim_{a\to\infty}\frac{\int_0^a\frac{x}{\int_0^x(t+1)\mathrm{d}t}\mathrm{d}x}{\int_0^a\frac{1}{x+1}\mathrm{d}x}=2\lim_{a\to\infty}\frac{\log(a+2)-\log 2}{\log(a+1)}=2,$$

and this completes the solution.

Problem 4. Let m,n be positive integers and let K be a field. Prove that:
a) There are nonzero ring homomorphisms from $\mathcal{M}_m(K)$ to $\mathcal{M}_n(K)$ if and only if $m\leq n$.
b) There are unitary ring homomorphisms from $\mathcal{M}_m(K)$ to $\mathcal{M}_n(K)$ if and only if $m\mid n$.

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Solution. Let $f:\mathcal{M}_m(K)\to\mathcal{M}_n(K)$ be a ring homomorphism. We denote by E_{ij} the $m\times n$ matrix which has as its only nonzero entry a 1 on the i^{th} row and j^{th} column. Let us start with a few remarks:

1. For every $i,j\in\{1,2,\ldots,m\}$ we have $f(E_{ij})f(E_{jj})f(E_{ji})=f(E_{ii})$, so $\operatorname{rank} f(E_{ii})\leq\operatorname{rank} f(E_{jj})$.

Swapping roles for i and j, we get $\operatorname{rank} f(E_{ii})=\operatorname{rank} f(E_{jj})$.

2. $f(E_{ii})$, $i\in\{1,2,\ldots,m\}$, and $f(I_m)$ are idempotent, thus they are diagonalizable, and they only have eigenvalues equal to 0 or 1.

3. $f(E_{ii})$, $i\in\{1,2,\ldots,m\}$, and $f(I_m)$ commute pariwise, so there is a basis with respect to which all of them are diagonal.

Let us denote by λ_{ik}, $k\in\{1,2,\ldots,n\}$, the eigenvalues of $f(E_{ii})$ and with $\mu_1,\mu_2,\ldots,\mu_n$ the eigenvalues of $f(I_m)$.

a) "$\Rightarrow$" We have $\sum\limits_{i=1}^m f(E_{ii})=f(I_m)$. Switching to diagonal forms, we get

$$\sum_{i=1}^m \operatorname{diag}(\lambda_{i1},\ldots,\lambda_{in})=\operatorname{diag}(\mu_1,\ldots,\mu_n).\tag{0.1}$$

Since all the λ's and the μ's are in $\{0,1\}$, we derive that for every $k\in\{1,2,\ldots,n\}$ all the numbers $\lambda_{1k},\lambda_{2k},\ldots,\lambda_{mk}$ are zero, with one possible exception which is then equal to 1. Since $f\neq 0$, $f(I_m)\neq 0$, the (common) rank of the matrices E_{ii} cannot be zero. Thus, each term in the left hand side of relation (0.1) is at least 1. Therefore, $m \leq \operatorname{rank} f(I_m) \leq n$.

"$\Leftarrow$" Let $m \leq n$. Then $f : \mathcal{M}_m(K) \to \mathcal{M}_n(K)$, $f(A) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, is obviously a nonzero ring homomorphism.

b) "$\Rightarrow$" If f is unitary, then $\displaystyle\sum_{i=1}^{m} f(E_{ii}) = I_n$. Switching to diagonal forms, we get $\displaystyle\sum_{i=1}^{m} \operatorname{diag}(\lambda_{i1}, \ldots, \lambda_{in}) = I_n$. Reasoning like above, we get for each $k \in \{1, 2, \ldots, n\}$ the relation $\displaystyle\sum_{i=1}^{m} \lambda_{ik} = 1$, which yields $mr = \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ik} = n$, so $m \mid n$.

"$\Leftarrow$" Let $m \mid n$. Then $f : \mathcal{M}_m(K) \to \mathcal{M}_n(K)$ defined by

$$f(A) = \begin{pmatrix} A & 0 & \ldots & 0 \\ 0 & A & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & A \end{pmatrix},$$

is obviously a unitary ring homomorphism.

Section B

Problem 1. Determine the minimal value of the integral $I(f) = \int_0^1 (f(x))^2 \mathrm{d}x$ on the set of polynomials f of degree at most 2 for which $f(1) = 1$.

* * *

Although this was considered an easy problem, it turned out to be a suitable one since it contains ideas from three domains which are taught in the first year of university classes: mathematical analysis, analytic geometry (conics and quadratics), linear algebra (the distance from a vector to a linear variety in an euclidean space). We present four solutions to this problem. In the first three of them, we let $f(x) = ax^2 + bx + c, a, b, c \in \mathbb{R}$ with $f(1) = 1$, that is $a + b + c = 1$. Note that we have that

$$\int_0^1 f^2(x)\mathrm{d}x = \frac{1}{5}a^2 + \frac{1}{3}b^2 + c^2 + \frac{1}{2}ab + \frac{2}{3}ac + bc.$$

Solution 1. A first analysis of the problem leads to a problem of conditional extremum, the minimum of the function

$$\phi : \mathbb{R}^3 \to \mathbb{R}, \quad \phi(a, b, c) = \frac{1}{5}a^2 + \frac{1}{3}b^2 + c^2 + \frac{1}{2}ab + \frac{2}{3}ac + bc,$$

with the condition $a+b+c=1$. We apply Lagrange multipliers method by considering the function

$$\Phi:\mathbb{R}^4\to\mathbb{R},\quad \Phi(a,b,c,\lambda)=\phi(a,b,c)-\lambda(a+b+c-1),$$

for which we determine its stationary points from the relations

$$\frac{\partial\Phi}{\partial a}=\frac{\partial\Phi}{\partial b}=\frac{\partial\Phi}{\partial c}=\frac{\partial\Phi}{\partial\lambda}=0,$$

which leads to the system of equations

$$\begin{cases}\frac{2}{5}a+\frac{1}{2}b+\frac{2}{3}c-\lambda=0,\\ \frac{1}{2}a+\frac{2}{3}b+c-\lambda=0,\\ \frac{2}{3}a+b+2c-\lambda=0,\\ a+b+c-1=0.\end{cases}$$

This system has a unique solution

$$a=\frac{10}{3},b=-\frac{8}{3},c=\frac{1}{3},\lambda=\frac{2}{9}.$$

The calculation of the second differential of the function Φ shows that the polynomial $f_0(x)=\frac{10}{3}x^2-\frac{8}{3}x+\frac{1}{3}$, with the condition $\frac{10}{3}-\frac{8}{3}+\frac{1}{3}=1$, realizes the minimum of the function Φ and the minimal value is $\int_0^1(f_0(x))^2\mathrm{d}x=\frac{1}{9}$.

Remark. The geometrical interpretation of this method is the following: the quadratic form $\phi(x,y,z)=\frac{1}{5}x^2+\frac{1}{3}y^2+z^2+\frac{1}{2}xy+\frac{2}{3}xz+yz$ is positively defined. This can be verified using the Sylvester criterion for the matrix

$$A_\varphi=\begin{pmatrix}\frac{1}{5}&\frac{1}{4}&\frac{1}{3}\\ \frac{1}{4}&\frac{1}{3}&\frac{1}{2}\\ \frac{1}{3}&\frac{1}{2}&1\end{pmatrix}.$$

For any $m>0$, large enough, the surface $S:\varphi(x,y,z)=m$ is an ellipsoid with center $I(0,0,0)$ which intersects the plane $P:x+y+z=1$ at an ellipse. The problem asks for finding m such that the surface S is tangent to the plane P, that is, the ellipse reduces to a point. The condition of being tangent is $\operatorname{grad}\varphi\perp P$ or $\frac{\frac{\partial\varphi}{\partial x}}{1}=\frac{\frac{\partial\varphi}{\partial y}}{1}=\frac{\frac{\varphi}{\partial z}}{1}$ and $x+y+z=1$, that is the same system obtained by the method of conditioned extremum.

Solution 2. The condition $a+b+c=1$ allows passing to a problem of free extremum with two variables by replacing $c=1-a-b$. We have

$$I(f)=\int_0^1(ax^2+bx+1-a-b)^2\mathrm{d}x=\frac{8}{15}a^2+\frac{1}{3}b^2+\frac{5}{6}ab-\frac{4}{3}a-b+1,$$

so we have to determine the minimum of the function $g:\mathbb{R}^2\to\mathbb{R},g(a+b)=I(f)$. The critical points verify the system $\frac{\partial g}{\partial a}=\frac{\partial g}{\partial b}=0$, that is

$$\begin{cases} \dfrac{16}{15}a + \dfrac{5}{6}b = \dfrac{4}{3}, \\ \dfrac{5}{6}a + \dfrac{2}{3}b = 1, \end{cases}$$

with the unique solution $a = \frac{10}{3}, b = -\frac{8}{3}$. The second differential for function g is

$$d^2g = \frac{\partial^2 g}{\partial a^2}(da)^2 + 2\frac{\partial^2 g}{\partial a \partial b}dadb + \frac{\partial^2 g}{\partial b^2}(db)^2 = \frac{16}{15}(da)^2 + 2 \cdot \frac{5}{6}dadb + \frac{2}{3}(db)^2,$$

which is positively defined (because $\Delta = (\frac{5}{6})^2 - \frac{16}{15} \cdot \frac{2}{3} < 0$). It follows that the polynomial $f_0(x) = \frac{10}{3}x^2 - \frac{8}{3}x + \frac{1}{3}$ is a point of minimum for the function g and the minimal value is $I(f_0) = \frac{1}{9}$.

Remark. The geometrical interpretation of this method is the following: the quadratic form

$$\Psi(x, y) = \frac{8}{15}x^2 + \frac{1}{3}y^2 + \frac{5}{6}xy$$

is positively defined. The equation

$$\frac{8}{15}x^2 + \frac{1}{3}y^2 + \frac{5}{6}xy - \frac{4}{3}x - y + 1 = m$$

is an ellipse with center at $(x_0, y_0) = (\frac{10}{3}, -\frac{8}{3})$ for m large enough, a point if $m = m_0$, and the empty set if $m < m_0$. The problem asks to determine m_0.

Solution 3. Similarly to the previous solution we have to determine the minimum of the function

$$h : \mathbb{R}^2 \to \mathbb{R}, \quad h(x, y) = \frac{8}{15}x^2 + \frac{1}{3}y^2 + \frac{5}{6}xy - \frac{4}{3}x - y + 1.$$

We apply Gauss method for reducing to canonical form of a conic $h(x, y) = m$. We have

$$h(x, y) = \frac{8}{15}\left(x + \frac{25}{32}y - \frac{5}{4}\right)^2 + \frac{1}{128}\left(y + \frac{8}{3}\right)^2 + \frac{1}{9} \geq \frac{1}{9},$$

and the equality is obtained for $Y + \frac{8}{3} = 0$ and $x + \frac{25}{32}y - \frac{5}{4} = 0$, that is $x = \frac{10}{3}$ and $y = -\frac{8}{3}$.

Solution 4. We consider the euclidean space $\mathbb{R}[x]$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\mathrm{d}x,$$

in which the set

$$V_1 = \{f \in \mathbb{R}[x] \mid \mathrm{grad} f \leq 2, f(1) = 1\}$$

is a linear variety:

$$V_1 = \{a(x-1)^2 + b(x-1) + 1 \mid a, b \in \mathbb{R}\} = 1 + V_0,$$

where $V_0 = \mathrm{Span}\{x - 1, (x-1)^2\}$. The problem asks for finding the minimum of the norm

$$\| f \|^2 = \int_0^1 f^2(x)\mathrm{d}x$$

for $f \in V_1$, which represents the square of the distance from 0 to V_1. We have

$$d^2(0,V_1) = d^2(0,1+V_0) = d^2(-1,V_0) = \frac{G(x-1,(x-1)^2,-1)}{G(x-1,(x-1)^2)} = \frac{G_1}{G_2},$$

where by G we denoted the Gram determinants. A calculation shows that

$$G_1 = \begin{vmatrix} \frac{1}{3} & -\frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{5} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{3} & 1 \end{vmatrix} = \frac{1}{3^3 \cdot 5 \cdot 16}, \qquad G_2 = \begin{vmatrix} \frac{1}{3} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{5} \end{vmatrix} = \frac{1}{3 \cdot 5 \cdot 16},$$

and $\frac{G_1}{G_2} = \frac{1}{9}$.

Problem 2. Let $A \in \mathcal{M}_3(\mathbb{R})$ be a matrix with the property that $A^3 = A + I_3$.

(a) Show that the matrix A is invertible and determine its real eigenvalues with one exact decimal.

(b) Show that $\det A$ and $\det(A - I_3)$ have the same sign.

(c) Show that the function $f : \mathbb{R}^9 = \mathcal{M}_3(\mathbb{R}) \to \mathbb{R},\ f(X) = \det(AXA^{-1})$, is differentiable and determine its critical points.

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A version of this problem was given at International Mathematical Contest for University Students, Budapest, 1999, problem 1, day 1, and the problem was about proving the existence of a real matrix of order $n \geq 2$ which verifies the given relation and showing that such matrices have strictly positive determinant (see www.imc-math.org.uk).

Solution. (a) The given relation is equivalent to $A(A^2 - I_3) = I_3$, which shows that $\det A \neq 0$ and $A^{-1} = A^2 - I_3$. The eigenvalues of A verify the equation $\lambda^3 = \lambda + 1$. The polynomial $g(x) = x^3 - x - 1$ has one real root and two conjugate complex roots. We have $g(1) < 0$ and $g(2) > 0$, hence the real root is between 1 and 2. We deduce that it is between $\frac{13}{10}$ and $\frac{14}{10}$, so $x_1 \cong 1.3$. Since the characteristic polynomial has degree three, the eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = x_1$ or $\lambda_1 = x_1$ and λ_2, λ_3 are the other two nonreal roots of the polynomial g.

(b) Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of A. We have $\det A = \lambda_1 \cdot \lambda_2 \cdot \lambda_2$. If $\lambda_1 = \lambda_2 = \lambda_3 = x_1 > 0$ it follows that $\det A = x_1^3 > 0$. If $\lambda_1 = x_1$ and $\lambda_3 = \overline{\lambda_2}$ we have $\det A = x_1 \cdot \lambda_2 \cdot \overline{\lambda_2} > 0$. The matrix $A - I_3$ has the eigenvalues $\lambda_1 - 1, \lambda_2 - 1, \lambda_3 - 1$.

When $\lambda_1 = \lambda_2 = \lambda_3 = x_1$ we have $\det A = (x_1 - 1)^3 > 0$ and, when $\lambda_1 = x_1$, $\lambda_3 = \overline{\lambda_2}$ we have $\det(A - I_3) = (x_1 - 1)(\lambda_2 - 1)\overline{(\lambda_2 - 1)} > 0$.

A different method is writing the relation in the equivalent form $(A - I_3)(A^2 + A + I_3) = A$, whence

$$\det(A - I_3) \cdot \det(A^2 + A + I_3) = \det A.$$

Since

$$\det(A^2 + A + I_3) = \det(A - \varepsilon I_3)(A - \overline{\varepsilon} I_3) = |\det(A - \varepsilon I_3)|^2 > 0,$$

where $\varepsilon^2 + \varepsilon + 1 = 0$, we obtain the conclusion.

(c) We have $f(X) = \det A \cdot \det X \cdot \det(A^{-1}) = \det X$, hence

$$f(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}.$$

The critical points are defined by the relations $\frac{\partial f}{\partial x_{ij}} = 0, ij \in \{1, 2, 3\}$. By differentiation we get $\frac{\partial f}{\partial x_{ij}} = \Delta_{ij}$. The critical points are the matrices X for which $\Delta_{ij} = 0, i, j \in \{1, 2, 3\}$, hence the matrices which have rank zero or one.

Problem 3. Let $n \geq 0$ be an integer, let T_{2n} be the Taylor polynomial associated to the cosine function at 0

$$T_{2n}(x) = \sum_{k=1}^{n+1} (-1)^{k-1} \frac{x^{2k-2}}{(2k-2)!}$$

and let

$$I_n = \int_0^\infty \frac{T_{2n}(x) - \cos x}{x^{2n+2}} \mathrm{d}x.$$

(a) Prove that $I_n = -\dfrac{1}{(2n+1)(2n)} I_{n-1}$, $n \geq 1$.

(b) Calculate I_n using that $\displaystyle\int_0^\infty \frac{\sin x}{x} \mathrm{d}x = \frac{\pi}{2}$.

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Solution. (a) We integrate by parts twice and we get that

$$\begin{aligned} I_n &= -\frac{T_{2n}(x) - \cos x}{(2n+1)x^{2n+1}} \Big|_0^\infty + \frac{1}{2n+1} \int_0^\infty \frac{T'_{2n}(x) + \sin x}{x^{2n+1}} \mathrm{d}x \\ &= \frac{1}{2n+1} \int_0^\infty \frac{T'_{2n}(x) + \sin x}{x^{2n+1}} \mathrm{d}x \\ &= -\frac{T'_{2n}(x) + \sin x}{(2n+1)(2n)x^{2n}} \Big|_0^\infty + \frac{1}{(2n+1)(2n)} \int_0^\infty \frac{T''_{2n}(x) + \cos x}{x^{2n}} \mathrm{d}x \\ &= \frac{1}{(2n+1)(2n)} \int_0^\infty \frac{T''_{2n}(x) + \cos x}{x^{2n}} \mathrm{d}x. \end{aligned}$$

A calculation shows that

$$T'_{2n}(x) = \sum_{k=2}^{n+1} (-1)^{k-1} \frac{x^{2k-3}}{(2k-3)!}$$

and

$$T''_{2n}(x)=\sum_{k=2}^{n+1}(-1)^{k-1}\frac{x^{2k-4}}{(2k-4)!}=-\sum_{i=1}^{n}(-1)^{i-1}\frac{x^{2i-2}}{(2i-2)!}=-T_{2n-2}(x).$$

Thus

$$I_n=-\frac{1}{(2n+1)(2n)}\int_0^\infty\frac{T''_{2n-2}(x)-\cos x}{x^{2n}}\mathrm{d}x=-\frac{1}{(2n+1)(2n)}I_{n-1}.$$

(b) The integral equals $\dfrac{(-1)^n}{2(2n+1)!}\pi$.

We have, based on the recurrence formula, that

$$I_n=\frac{(-1)^n}{(2n+1)(2n)\cdots 3\cdot 2}I_0=\frac{(-1)^n}{(2n+1)!}\int_0^\infty\frac{1-\cos x}{x^2}\mathrm{d}x.$$

We calculate the preceding integral by parts and we have

$$\int_0^\infty\frac{1-\cos x}{x^2}\mathrm{d}x=-\frac{1-\cos x}{x}\Big|_0^\infty+\int_0^\infty\frac{\sin x}{x}\mathrm{d}x=\frac{\pi}{2},$$

from which it follows that

$$I_n=\frac{(-1)^n}{2(2n+1)!}\pi.$$

Remark. Similarly, one can prove that, if $n\in\mathbb{N}^*$ and

$$T_{2n-1}(x)=\sum_{k=1}^{n}(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!}$$

is the Taylor polynomial associated to the sine function at 0 and

$$I_n=\int_0^\infty\frac{T_{2n-1}(x)-\sin x}{x^{2n+1}}\mathrm{d}x,$$

then:

(a) $I_n=-\dfrac{1}{2n(2n-1)}I_{n-1}$, $n\geq 1$,

(b) $I_n=\dfrac{(-1)^{n-1}}{2(2n)!}\pi$.

Problem 4. Identical with Problem 2 from section A.

NOTE MATEMATICE

Asymptotic expansion of an integral considered in a problem of IMC 2015

ULRICH ABEL[1)]

Abstract. We derive a complete asymptotic expansion of the integral

$$\int_0^A A^{1/x}\mathrm{d}x$$

as the parameter A tends to infinity. This integral was considered in a Problem of IMC 2015, Blagoevgrad, Bulgaria.

Keywords: One-variable calculus, asymptotic expansions

MSC: Primary 26A06; Secondary 41A60.

1. INTRODUCTION

In this note we derive a complete asymptotic expansion of the integral

$$\int_0^A A^{1/x}\mathrm{d}x$$

as the parameter A tends to infinity. It originates from the Problem 7 of Day 2 of IMC 2015, Blagoevgrad, Bulgaria, proposed by Jan Šustek, University of Ostrava: Calculate the limit

$$\lim_{A\to+\infty}\frac{1}{A}\int_0^A A^{1/x}\mathrm{d}x=1. \tag{1.1}$$

Two official solutions of this problem can be found online on the site of the competition at http://www.imc-math.org.uk. Recently, Ovidiu Furdui [1] gave a further solution by application of l'Hôspital's rule. We are going to show that

$$\frac{1}{A}\int_0^A A^{1/x}\mathrm{d}x\sim\sum_{k=0}^{\infty}\frac{k!}{(\log A)^k}\qquad(A\to+\infty). \tag{1.2}$$

This means that, for every $q\in\mathbb{N}_0$,

$$\frac{1}{A}\int_0^A A^{1/x}\mathrm{d}x=\sum_{k=0}^{q}\frac{k!}{(\log A)^k}+o\left((\log A)^{-q}\right)\qquad(A\to+\infty).$$

A direct consequence of (1.2) is

$$\lim_{A\to+\infty}(\log A)^{q+1}\left(\frac{1}{A}\int_0^A A^{1/x}\mathrm{d}x-\sum_{k=0}^{q}\frac{k!}{(\log A)^k}\right)=(q+1)!\ \ (q=0,1,\ldots).$$

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The special case $q=0$, i.e.,

$$\lim_{A\to+\infty}(\log A)\left(\frac{1}{A}\int_0^A A^{1/x}\mathrm{d}x-1\right)=1$$

is a stronger form of the equation (1.1).

If we put $a=\log A$ the limit (1.1) is equivalent to

$$\lim_{a\to+\infty} g(a)=1,$$

where

$$g(a)=\mathrm{e}^{-a}\int_1^{\mathrm{e}^a}\mathrm{e}^{a/x}\mathrm{d}x \qquad (a>0).$$

In the following we study the function g. The change of variable $t=a/x$ yields

$$g(a)=a\mathrm{e}^{-a}\int_{a\mathrm{e}^{-a}}^{a}\mathrm{e}^t t^{-2}\mathrm{d}t \qquad (a>0).$$

We split the integral into three parts:

$$\begin{aligned}\int_{a\mathrm{e}^{-a}}^{a}\mathrm{e}^t t^{-2}\mathrm{d}t &= \int_{a\mathrm{e}^{-a}}^{a/2}\frac{\mathrm{e}^t-1-t}{t^2}\mathrm{d}t+\int_{a\mathrm{e}^{-a}}^{a/2}\left(\frac{1}{t}+\frac{1}{t^2}\right)\mathrm{d}t+\int_{a/2}^{a}\mathrm{e}^t t^{-2}\mathrm{d}t\\ &= I_1+I_2+I_3,\end{aligned}$$

say. We take advantage of the fact that $\left(\mathrm{e}^t-1-t\right)t^{-2}$ is bounded on each interval $[0,a]$. It is a matter of calculus to show that $\mathrm{e}^t-1-t\le t^2\mathrm{e}^t$, for $t\ge 0$. It can be directly seen by power series expansion:

$$\mathrm{e}^t-1-t=\sum_{\nu=2}^{\infty}\frac{t^\nu}{\nu!}=\sum_{\nu=0}^{\infty}\frac{t^{\nu+2}}{(\nu+2)!}\le t^2\mathrm{e}^t \qquad \text{for } t\ge 0.$$

Therefore, the first term I_1 can be estimated by

$$I_1\le\int_{a\mathrm{e}^{-a}}^{a/2}\mathrm{e}^t\mathrm{d}t\le \mathrm{e}^{a/2}-1.$$

The second term I_2 is easily evaluated to be

$$I_2=\left(\log t-\frac{1}{t}\right)\Bigg|_{a\mathrm{e}^{-a}}^{a/2}=a-\log 2-\frac{2}{a}+\frac{1}{a}\mathrm{e}^a.$$

As regards the third term I_3, the change of variable replacing t with $a(1-t)$ leads to

$$I_3=\frac{1}{a}\mathrm{e}^a\int_0^{1/2}\mathrm{e}^{-at}(1-t)^{-2}\,\mathrm{d}t.$$

Combining the latter relations we obtain

$$g(a)=1+O\left(a\mathrm{e}^{-a/2}\right)+\int_0^{1/2}\mathrm{e}^{-at}(1-t)^{-2}\,\mathrm{d}t \qquad (a\to+\infty).$$

The Laplace integral can be treated by application of Watson's lemma (see, e.g., [2, p. 106f]). This standard result states

$$\int_0^b \mathrm{e}^{-st} f(t)\,\mathrm{d}t \sim \sum_{k=0}^{\infty} \frac{k!c_k}{s^{k+1}} \qquad (s \to +\infty),$$

provided that f is bounded on $[0, b]$ and admits a power series expansion $f(t) = \sum\limits_{k=0}^{\infty} c_k t^k$ in a neighborhood of $t = 0$. Noting that

$$(1-t)^{-2} = \sum_{k=1}^{\infty} k t^{k-1} \qquad (|t| < 1)$$

yields the complete asymptotic expansion

$$g(a) \sim 1 + \sum_{k=1}^{\infty} \frac{k!}{a^k} \qquad (a \to +\infty).$$

Substituting $a = \log A$ we obtain the equation (1.2).

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A Putnam problem revisited and a formula in disguise

Ovidiu Furdui[1)], Alina Sîntămărian[2)]

Abstract. In this note we bring into light and discuss a Putnam problem involving the calculation of an improper integral over the interval $[0, \infty)$.

Keywords: Gamma function, improper integral, integral formula, Putnam problem

MSC: Primary 26A42; Secondary 33B15.

It is a Putnam problem (see [2, p. 2]) to calculate the improper integral

$$\int_0^{\infty} t^{-\frac{1}{2}} \mathrm{e}^{-1985\left(t+\frac{1}{t}\right)}\,\mathrm{d}t.$$

In this short note we give a new method for calculating this integral, which is based on splitting the interval $[0, \infty)$ into two intervals and then by

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using two special substitutions. This solution is then generalized to obtain an integral formula involving a generic function of a special form. For the sake of completeness we record below the solution of this problem as it appears in [2, pp. 62–63].

Solution 1. Let $a > 0$ and let $I(a) = \int_0^\infty t^{-\frac{1}{2}} \mathrm{e}^{-a\left(t+\frac{1}{t}\right)}\,\mathrm{d}t$. We have

$$\begin{aligned}
I(a) &= \int_0^1 t^{-\frac{1}{2}} \mathrm{e}^{-a\left(t+\frac{1}{t}\right)}\,\mathrm{d}t + \int_1^\infty t^{-\frac{1}{2}} \mathrm{e}^{-a\left(t+\frac{1}{t}\right)}\,\mathrm{d}t \\
&\overset{\frac{1}{t}=u}{=} \int_1^\infty \frac{1}{u\sqrt{u}} \mathrm{e}^{-a\left(u+\frac{1}{u}\right)}\,\mathrm{d}u + \int_1^\infty t^{-\frac{1}{2}} \mathrm{e}^{-a\left(t+\frac{1}{t}\right)}\,\mathrm{d}t \\
&= \int_1^\infty \left(\frac{1}{\sqrt{t}} + \frac{1}{t\sqrt{t}}\right) \mathrm{e}^{-a\left(t+\frac{1}{t}\right)}\,\mathrm{d}t \\
&\overset{y=\sqrt{t}-\frac{1}{\sqrt{t}}}{=} 2\int_0^\infty \mathrm{e}^{-a(y^2+2)}\,\mathrm{d}y = 2\mathrm{e}^{-2a} \int_0^\infty \mathrm{e}^{-ay^2}\,\mathrm{d}y \\
&\overset{ay^2=v}{=} \frac{\mathrm{e}^{-2a}}{\sqrt{a}} \int_0^\infty v^{-\frac{1}{2}} \mathrm{e}^{-v}\,\mathrm{d}v = \frac{\mathrm{e}^{-2a}}{\sqrt{a}} \Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{a}} \mathrm{e}^{-2a},
\end{aligned}$$

since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. When $a = 1985$ it follows that

$$\int_0^\infty t^{-\frac{1}{2}} \mathrm{e}^{-1985\left(t+\frac{1}{t}\right)}\,\mathrm{d}t = \sqrt{\frac{\pi}{1985}} \mathrm{e}^{-3970},$$

and the problem is solved. □

Now we give a new solution of this problem which is based on a completely different technique.

Solution 2. We have

$$I(a) = \int_0^1 t^{-\frac{1}{2}} \mathrm{e}^{-a\left(t+\frac{1}{t}\right)}\,\mathrm{d}t + \int_1^\infty t^{-\frac{1}{2}} \mathrm{e}^{-a\left(t+\frac{1}{t}\right)}\,\mathrm{d}t.$$

Using the substitution $t + \frac{1}{t} = y$ we put $t = \frac{y-\sqrt{y^2-4}}{2}$ in the first integral and $t = \frac{y+\sqrt{y^2-4}}{2}$ in the second integral, and we get that

$$\begin{aligned}
I(a) &= \int_2^\infty \mathrm{e}^{-ay} \frac{\sqrt{y-\sqrt{y^2-4}}}{\sqrt{2}\sqrt{y^2-4}}\,\mathrm{d}y + \int_2^\infty \mathrm{e}^{-ay} \frac{\sqrt{y+\sqrt{y^2-4}}}{\sqrt{2}\sqrt{y^2-4}}\,\mathrm{d}y = \\
&= \int_2^\infty \mathrm{e}^{-ay} \frac{\sqrt{y-\sqrt{y^2-4}} + \sqrt{y+\sqrt{y^2-4}}}{\sqrt{2}\sqrt{y^2-4}}\,\mathrm{d}y.
\end{aligned}$$

One can check that

$$\sqrt{y \pm \sqrt{y^2-4}} = \sqrt{\frac{y+2}{2}} \pm \sqrt{\frac{y-2}{2}}.$$

It follows that

$$I(a)=\int_2^{\infty}\mathrm{e}^{-ay}\frac{\sqrt{y+2}}{\sqrt{y^2-4}}\,\mathrm{d}y=\int_2^{\infty}\frac{\mathrm{e}^{-ay}}{\sqrt{y-2}}\,\mathrm{d}y$$
$$\overset{y-2=u}{=}\int_0^{\infty}\frac{\mathrm{e}^{-a(u+2)}}{\sqrt{u}}\,\mathrm{d}u=\mathrm{e}^{-2a}\int_0^{\infty}\frac{\mathrm{e}^{-au}}{\sqrt{u}}\,\mathrm{d}u$$
$$\overset{au=v}{=}\frac{\mathrm{e}^{-2a}}{\sqrt{a}}\Gamma\left(\frac{1}{2}\right)=\sqrt{\frac{\pi}{a}}\mathrm{e}^{-2a},$$

and the problem is solved. □

It is worth mentioning that one can prove, by using the same technique as in Solution 2, that the following integral formula holds

$$\int_0^{\infty}\frac{1}{\sqrt{t}}f\left(t+\frac{1}{t}\right)\,\mathrm{d}t=2\int_0^{\infty}f\left(t^2+2\right)\,\mathrm{d}t,$$

where f is a function for which both integrals exist.

Solution 3. The third solution of this problem is based on making the substitution $t=x^2$ and then by using the formula [1, Problem 2314, p. 171]

$$\int_0^{\infty}f\left(x+\frac{1}{x}\right)\,\mathrm{d}x=\int_0^{\infty}f\left(\sqrt{x^2+4}\right)\,\mathrm{d}x,$$

where f is a function for which both integrals exist. This integral formula, which is left as an exercise to the interested reader, can be proved by using a technique similar to that in Solution 2.

□

In closing this note we collect below two problems which can be solved by the method that was discussed above, and we leave them as exercises to the interested reader.

An exponential integral. Let $a,b>0$. Prove that

$$\int_0^{\infty}x^{-\frac{1}{2}}\mathrm{e}^{-\left(ax+\frac{b}{x}\right)}\,\mathrm{d}x=\sqrt{\frac{\pi}{a}}\mathrm{e}^{-2\sqrt{ab}}.$$

A Fresnel integral. Let a,b be real numbers such that $ab>0$. Prove that

$$\int_0^{\infty}\frac{1}{\sqrt{x}}\cos\left(ax+\frac{b}{x}\right)\,\mathrm{d}x=\sqrt{\frac{\pi}{a}}\cos\left(2\sqrt{ab}+\frac{\pi}{4}\right).$$

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PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of December 2016**.

PROPOSED PROBLEMS

439. For every $n \geq 1$ let

$$S_n = \int_0^1 \frac{\mathrm{d}x_1}{x_1+1} + \int_{[0,1]^2} \frac{\mathrm{d}x_1 \mathrm{d}x_2}{\sqrt{x_1 x_2}+2} + \cdots + \int_{[0,1]^n} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{\sqrt[n]{x_1 \cdots x_n}+n} - \ln n.$$

Prove that the sequence $(S_n)_{n\geq 1}$ is convergent and, if $S = \lim\limits_{n\to\infty} S_n \in \mathbb{R}$, find the value of the limit $\lim\limits_{n\to\infty} n\,(S_n - S)$.

Proposed by Dumitru Popa, Department of Mathematics, Ovidius University, Constanţa, Romania.

440. Let V be a linear space with finite dimension and let $T : V \to V$ be an endomorphism.

a) Prove that there exist an endomorphism $P : V \to V$ and a positive integer k such that

$$P^2 = P, \quad \operatorname{Ker} P = \operatorname{Ker} T^k, \quad \operatorname{Im} P = \operatorname{Im} T^k.$$

b) Is the assertion a) true when V does not have finite dimension?

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania.

441. Let F be a field of characeristic 2 and let $F^2 := \{a^2 \mid a \in A\}$. We denote by ε the image of X in the quotion ring $F[X]/(X^2)$. Prove that $R := F^2 + F\varepsilon$ is a subring of $F[\varepsilon]$ and define an operation $\boxplus : R \times R \to R$ such that $(R, \boxplus, \cdot)$ is a ring which is not isomorphic to $(R, +, \cdot)$.

Proposed by Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

442. Let $f : [0,1] \to \mathbb{R}$ be a function with the following properties:

a) f is convex.

b) $f'_r(0) > f(0)$ and $f'_l(1) < f(1)$.

c) For any $x \in (0,1) \cap \mathbb{Q}$ we have $(f'_l(x) - f(x))(f'_r(x) - f(x)) > 0$.

Prove that there is $c \in (0,1)$ such that f is differentiable at c and $f'(c) = f(c)$.

Here $f_l'(x)$ and $f_r'(x)$ denote the left and right derivatives of f at x.

Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.

443. Let $\mathcal{F}$ be the set of continuous functions on $[0, 2\pi]$ that satisfy the equalities

$$\int_0^{2\pi} f(x)\cos kx \,\mathrm{d}x = \int_0^{2\pi} f(x)\sin kx \,\mathrm{d}x = \pi, \quad \text{for all } k = 1, 2, \ldots, n.$$

Prove that there exists a function $f_0 \in \mathcal{F}$ such that

$$\int_0^{2\pi} f_0^2(x)\,\mathrm{d}x \le \int_0^{2\pi} f^2(x)\,\mathrm{d}x, \quad \text{for all } f \in \mathcal{F},$$

and find such a function.

Remark. The problem is true in a more general case, considering the equalities

$$\int_0^{2\pi} f(x)\cos kx \,\mathrm{d}x = a_k \text{ and } \int_0^{2\pi} f(x)\sin kx \,\mathrm{d}x = b_k, \text{ for all } k = 1, 2, \ldots, n,$$

with a_k, b_k arbitrary real numbers.

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania.

444. Let $A, B \in M_n(\mathbb{R})$ such that $A^2 + B^2 = e(AB - BA)$.

Prove that $AB - BA \notin GL_n(\mathbb{R})$.

Proposed by Luigi-Ionuţ Catană, University of Bucharest, Romania.

A question from the editor. For a given $n \ge 2$ try to determine all values of $\alpha \in \mathbb{R}$ such that there are $A, B \in M_n(\mathbb{R})$ with the property that $A^2 + B^2 = \alpha(AB - BA)$ and $AB - BA \in GL_n(\mathbb{R})$.

445. Prove that

$$\lim_{n\to\infty} n^3 \iint\limits_{[0,1]^2} \left(\frac{x+y}{2}\right)^n \left(\sqrt[n]{1 + x^n y^n} - 1\right) \mathrm{d}x\mathrm{d}y = 4\ln 2 - 16 + 2\pi + 8K,$$

where $K = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2}$ is the Catalan constant.

Proposed by Dumitru Popa, Department of Mathematics, Ovidius University, Constanţa, Romania.

446. Let $A \in \mathcal{M}_2(\mathbb{Z})$. Prove that $\mathrm{e}^A \in \mathcal{M}_2(\mathbb{Z})$ if and only if $A^2 = O_2$.

Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Romania.

SOLUTIONS

417. Calculate
$$\int_0^1\int_0^1 \frac{\log(1+x)-\log(1+y)}{x-y}\mathrm{d}x\mathrm{d}y.$$

Proposed by Ovidiu Furdui Technical University of Cluj-Napoca, Romania and Cornel Vălean, Teremia Mare, Timiş, Romania.

Solution by the authors. The integral equals $\zeta(2)-2\ln^2 2$.

First we note that
$$\log(1+x)-\log(1+y)=\int_0^1\left(\frac{x}{1+tx}-\frac{y}{1+ty}\right)\mathrm{d}t=\int_0^1\frac{x-y}{(1+tx)(1+ty)}\mathrm{d}t.$$

Thus,
$$I=\int_0^1\int_0^1\frac{\log(1+x)-\log(1+y)}{x-y}\mathrm{d}x\mathrm{d}y=\int_0^1\int_0^1\int_0^1\frac{\mathrm{d}x\mathrm{d}y\mathrm{d}t}{(1+tx)(1+ty)}=$$
$$=\int_0^1\left(\int_0^1\frac{1}{1+tx}\mathrm{d}x\right)\left(\int_0^1\frac{1}{1+ty}\mathrm{d}y\right)\mathrm{d}t=\int_0^1\left(\frac{\ln(1+t)}{t}\right)^2\mathrm{d}t.$$

We calculate the preceding integral by parts, with $f(t)=\ln^2(1+t)$ and $g'(t)=\frac{1}{t^2}$, and we get
$$\begin{aligned}
I&=-\frac{\ln^2(1+t)}{t}\Bigg|_{t=0}^{t=1}+2\int_0^1\frac{\ln(1+t)}{t(1+t)}\mathrm{d}t=\\
&=-\ln^2 2+2\int_0^1\left(\frac{\ln(1+t)}{t}-\frac{\ln(1+t)}{1+t}\right)\mathrm{d}t=\\
&=-2\ln^2 2+2\int_0^1\frac{\ln(1+t)}{t}\mathrm{d}t\\
&=-2\ln^2 2+2\int_0^1\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^{n-1}}{n}\right)\mathrm{d}t=\\
&=-2\ln^2 2+2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}\\
&=\zeta(2)-2\ln^2 2.
\end{aligned}$$

The problem is solved. □

418. (i) Let $R=k[X,Y]/(XY^2)$, k a field. Denote by x, respectively y, the residue classes of X, respectively Y, modulo the ideal (XY^2). Show that the elements x and $x(1+y)$ are associates, that is, $xR=x(1+y)R$, but there is no invertible element $u\in R$ such that $ux=x(1+y)$.

(ii) Show that we can't find such elements in $\mathbb{Z}/n\mathbb{Z}$.

Proposed by Cornel Băeţica, Faculty of Mathematics and Informatics, University of Bucharest, Romania.

Solution by the author. (i) The elements x and $x(1+y)$ are associates since $x = x(1+y)(1-y)$.

Suppose there is an invertible element $u \in R$ such that $ux = x(1+y)$. This means that in $k[X,Y]$ we have $U(X,Y)X - X(1+Y) \in (XY^2)$, that is, $U(X,Y) - (1+Y) \in (Y^2)$ and therefore we can write $U(X,Y) = 1 + Y + Y^2V(X,Y)$. Since $xy^2 = 0$ in R, we may assume that V is a polynomial only in Y, so $u = 1 + y + y^2v(y)$. But u can't be invertible in R because it lies in a maximal ideal: consider an irreducible polynomial $f \in K[Y]$ such that $f(Y) \mid 1 + Y + Y^2V(Y)$, and let $\mathfrak{m} = (x, f(y))$.

(ii) By Chinese Remainder Theorem it suffices to consider the case $n = p^k$ is a prime power. Every ideal of $\mathbb{Z}/p^k\mathbb{Z}$ is of the form $p^n\mathbb{Z}/p^k\mathbb{Z}$ for some $n = 0, \ldots, k$. Then one can easily check that $p^n + p^k\mathbb{Z}$ and $p^m + p^k\mathbb{Z}$ are associates if and only if $n = m$. □

419. Suppose that $n \geq 1$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a function such that the image under f of the interior of any sphere of codimension 1 is the interior of a sphere of codimension 1 of the same radius. Prove that f is an isometry.

Proposed by Marius Cavachi, Ovidius University of Constanţa, Romania.

Solution by the author. We denote by $S(x,r)$ the sphere of center $x \in \mathbb{R}^n$ and radius r and by $B(x,r)$ its interior.

For any $x, y \in \mathbb{R}$ and any $\varepsilon > 0$ we have $x, y \in B(O, |x-y|/2+\varepsilon)$, where O is the midpoint of the segment xy, $O = (x+y)/2$. Then $f(x), f(y)$ belong to $f(B(O, |x-y|/2+\varepsilon))$, which, by hypothesis, is the interior of a sphere of radius $|x-y|/2+\varepsilon$. Then $|f(x)-f(y)|$ is smaller than the diameter of this sphere, which is $|x-y|+2\varepsilon$. Since $|f(x)-f(y)| < |x-y|+2\varepsilon$ for any $\varepsilon > 0$, we have $|f(x)-f(y)| \leq |x-y|$. In particular, if $a \in \mathbb{R}^n$, $r > 0$ then for any $x \in B(a,r)$ we have $|f(x)-f(a)| \leq |x-a| < r$, so $f(x) \in B(f(a), r)$. Thus $f(B(a,r)) \subseteq B(f(a),r)$. Hence $f(B(a,r))$ is the interior of a sphere of radius r which is contained in $B(f(a),r)$. It follows that $f(B(a,r)) = B(f(a),r)$.

For any $a \in \mathbb{R}$ and any neighborhood $B(f(a),r)$ of $f(a)$ the neighborhood $B(a,r)$ of a is contained in $f^{-1}(B(f(a),r))$. Thus f is continuous.

Since f is continuous, for any $a \in \mathbb{R}$, $r > 0$ we have

$$f(B(a,r)) \subseteq f(\overline{B(a,r)}) \subseteq \overline{f(B(a,r))},$$

i.e.,

$$\overline{B(f(a),r)} \subseteq f(\overline{B(a,r)}) \subseteq \overline{B(f(a),r)}.$$

But $\overline{B(a,r)}$ is compact, so $f(\overline{B(a,r)})$ is compact and therefore closed. It follows that $f(\overline{B(a,r)}) = \overline{B(f(a),r)}$.

In particular, for any $y \in S(f(a), r)$ there is some $x_y \in \overline{B(a,r)}$ such that $f(x_y) = y$. We cannot have $x_y \in B(a,r)$ since this would imply $y = f(x_y) \in f(B(a,r)) = B(f(a), r)$. Hence $x_y \in S(a,r)$. We obtained a function $g : S(f(a), r) \to S(a,r)$, given by $y \mapsto x_y$. It has the property that $f(g(y)) = f(x_y) = y$, i.e., $f \circ g = 1_{S(f(a),r)}$. Note that for any $y, y' \in S(f(a), r)$ we have $|y' - y| = |f(g(y')) - f(g(y))| \leq |g(y') - g(y)|$.

Let $y_1, y_2 \in S(f(a), r)$ and let $y_1' \in S(f(a), r)$ be the antipodal point of y_1. Then $|g(y_1') - g(y_1)| \geq |y_1' - y_1| = 2r$. Since $g(y_1), g(y_1') \in S(a,r)$, we must have $|g(y_1') - g(y_1)| = 2r$, so $g(y_1'), g(y_1)$ are antipodal on $S(a,r)$. By Pythagora's theorem we have $|y_1 - y_2|^2 + |y_1' - y_2|^2 = |y_1' - y_1|^2 = 4r^2$ and $|g(y_1) - g(y_2)|^2 + |g(y_1') - g(y_2)|^2 = |g(y_1') - g(y_1)|^2 = 4r^2$. It follows that both inequalities $|g(y_1) - g(y_2)| \geq |y_1 - y_2|$ and $|g(y_1') - g(y_2)| \geq |y_1' - y_2|$ must be in fact equalities. Since $|g(y_1) - g(y_2)| = |y_1 - y_2|$ $\forall y_1, y_2 \in S(f(a), r)$, the map $g : S(f(a), r) \to S(a,r)$ is an isometry. In particular, it is a bijection. Since $f(g(y)) = y$, the inverse of g is the restriction of f to $S(a,r)$. Hence, $f(S(a,r)) = S(f(a), r)$.

If $a, b \in \mathbb{R}^n$, $a \neq b$, then $b \in S(a, |b-a|)$, so $f(b) \in f(S(a, |b-a|)) = S(f(a), |b-a|)$. It follows that $|f(b) - f(a)| = |b-a|$. Thus f is an isometry.

420. Let $a, b, c, d \in \mathbb{R}$, $c \neq 0$, $d \neq 0$, such that $\frac{a}{c} < \frac{b}{d}$. We consider the Maclaurin expansion $\mathrm{e}^{\frac{az+b}{cz+d}} = \sum\limits_{n \geq 0} a_n z^n$.

(i) Find an exact formula as a finite sum for a_n.

(ii) Determine the asymptotic behaviour of a_n as $n \to \infty$.

Try to solve (ii) without using the result from (i).

Proposed by Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

Solution by the author. (i) Let $f = f_{a,b,c,d}$ be given by $z \mapsto \mathrm{e}^{\frac{az+b}{cz+d}}$. Since $\frac{az+b}{cz+d} = \frac{b}{d} + \frac{(ad-bc)z}{d(cz+d)}$, we have

$$f(z) = \mathrm{e}^{\frac{b}{d}} \mathrm{e}^{\frac{(ad-bc)z}{d(cz+d)}} = \mathrm{e}^{\frac{b}{d}} \sum_{k \geq 0} \frac{1}{k!} \left(\frac{(ad-bc)z}{d(cz+d)} \right)^k .$$

From

$$\binom{-k}{l} = \frac{(-k)(-k-1)\cdots(-k-l+1)}{l!} =$$
$$= (-1)^k \frac{(k+l-1)\cdots k}{l!} = (-1)^l \binom{k+l-1}{l}$$

we obtain

$$\left(\frac{(ad-bc)z}{d(cz+d)} \right)^k = \left(\frac{ad-bc}{d^2} \right)^k z^k \left(1 + \frac{c}{d} z \right)^{-k} =$$

$$= \left(\frac{ad-bc}{d^2}\right)^k z^k \sum_{l\geq 0} \binom{-k}{l} \left(\frac{c}{d}z\right)^l =$$

$$= \sum_{l\geq 0} \binom{k+l-1}{l} \left(\frac{ad-bc}{d^2}\right)^k \left(-\frac{c}{d}\right)^l z^{k+l}.$$

It follows that, for any $n \geq 0$, z^n appears in $\left(\frac{(ad-bc)z}{d(cz+d)}\right)^k$ with the coefficient

$$\binom{n-1}{n-k} \left(\frac{ad-bc}{d^2}\right)^k \left(-\frac{c}{d}\right)^{n-k} = \binom{n-1}{n-k} \left(-\frac{ad-bc}{cd}\right)^k \left(-\frac{c}{d}\right)^n$$

if $k \leq n$ and with the coefficient 0 otherwise. It follows that the coefficient of z^n in f is

$$a_n = \mathrm{e}^{\frac{b}{d}} \left(-\frac{c}{d}\right)^n \sum_{k=0}^{n} \frac{1}{k!} \binom{n-1}{n-k} \left(-\frac{ad-bc}{cd}\right)^k.$$

(ii) One may find an asymptotic formula for a_n by using the exact formula from (i) and the Stirling formula $n! \sim \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n}$. We must evaluate $\sum\limits_{k=0}^{n} \frac{1}{k!}\binom{n-1}{n-k}\alpha^k$, where $\alpha = -\frac{ad-bc}{cd} = \frac{b}{d} - \frac{a}{c} > 0$, for $n \gg 0$.

Let $b_k = \frac{1}{k!}\binom{n-1}{n-k}\alpha^k = \frac{(n-1)!}{k!(k-1)!(n-k)!}\alpha^k$. We want to determine the monotony of the sequence $b_0, \ldots, b_n$. We have $b_{k+1}/b_k = \frac{\alpha(n-k)}{k(k+1)}$, so $b_{k+1} \geq b_k$ iff $\alpha(n-k) \geq k(k+1)$, i.e., iff $k^2 + (\alpha+1)k - n\alpha \leq 0$. Since $T = \frac{-\alpha-1+\sqrt{(\alpha+1)^2+4n\alpha}}{2}$ is the only non-negative root of $X^2 + (\alpha+1)X - n\alpha$, this is equivalent to $k \leq C$. Thus b_k increases on $[0, [C]+1]$ and decreases on $[[C]+1, n]$. Thus b_k reaches its maximum at $k = [C]+1$.

Since $C \cong \sqrt{n\alpha}$, we want to estimate b_k for k in the vicinity of $\sqrt{n\alpha}$. If both k and $n-k$ are large then

$$\begin{aligned}
b_k &= \frac{k}{n} \cdot \frac{n!\alpha^k}{k!^2(n-k)!} \sim \frac{k}{n} \cdot \frac{\sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n}}{2\pi k^{2k+1} \mathrm{e}^{-2k} \sqrt{2\pi}(n-k)^{n-k+1/2} \mathrm{e}^{-n+k}} \cdot \alpha^k \\
&= \frac{1}{2\pi} \cdot n^{n-1/2} \mathrm{e}^k k^{-2k} (n-k)^{-n+k-1/2} \alpha^k \\
&= \frac{1}{2\pi} \cdot n^{n-1/2} \mathrm{e}^k (n\alpha)^{-k} \left(\frac{k}{\sqrt{n\alpha}}\right)^{-2k} n^{-n+k-1/2} \left(\frac{n-k}{n}\right)^{-n+k-1/2} \alpha^k \\
&= \frac{1}{2\pi} \cdot n^{-1} \mathrm{e}^k \left(\frac{k}{\sqrt{n\alpha}}\right)^{-2k} \left(\frac{n-k}{n}\right)^{-n+k-1/2}.
\end{aligned}$$

Suppose now that $k=\sqrt{n\alpha}+x$ with $|x|\ll\sqrt{n}$. Then $n-k=m-x$, where $m=n-\sqrt{n\alpha}$. Hence,

$$\begin{aligned}\log\left(\frac{k}{\sqrt{n\alpha}}\right)^{-2k}&=-2(\sqrt{n\alpha}+x)\log\left(1+\frac{x}{\sqrt{n\alpha}}\right)\\&=-2(\sqrt{n\alpha}+x)\left(\frac{x}{\sqrt{n\alpha}}-\frac{x^2}{2n\alpha}+O\left(\frac{x^3}{n^{3/2}}\right)\right)\\&=-2x-\frac{x^2}{\sqrt{n\alpha}}+O\left(\frac{x^3}{n}\right).\end{aligned}$$

Next we evaluate

$$\log\left(\frac{n-k}{n}\right)^{-n+k-1/2}=\log\left(\frac{m}{n}\right)^{-m+x-1/2}+\log\left(\frac{m-x}{m}\right)^{-m+x-1/2}.$$

We have

$$\begin{aligned}\log\left(\frac{m}{n}\right)^{-m+x-1/2}&=-(m-x+1/2)\log\left(1-\frac{\sqrt{n\alpha}}{n}\right)\\&=(n-\sqrt{n\alpha}-x+1/2)\left(\sqrt{\frac{\alpha}{n}}+\frac{\alpha}{2n}+O\left(\frac{1}{n^{3/2}}\right)\right)\\&=\sqrt{n\alpha}-\frac{\alpha}{2}+o(1).\end{aligned}$$

(Recall that $x\ll\sqrt{n}$, so $\frac{x}{\sqrt{n}}=o(1)$.) Also

$$\begin{aligned}\log\left(\frac{m-x}{m}\right)^{-m+x-1/2}&=-(m-x+1/2)\log\left(1-\frac{x}{m}\right)\\&=(m-x+1/2)\left(\frac{x}{m}+O\left(\frac{x^2}{m^2}\right)\right)\\&=x+O\left(\frac{x^2}{m}\right)=x+o(1).\end{aligned}$$

(We have $m=n-\sqrt{n\alpha}\sim n$, so $x^2\ll n\sim m$.)

From the three relations above, together with $\log \mathrm{e}^k=k=\sqrt{n\alpha}+x$, we get

$$\log \mathrm{e}^k\left(\frac{k}{\sqrt{n\alpha}}\right)^{-2k}\left(\frac{n-k}{n}\right)^{-n+k-1/2}=2\sqrt{n\alpha}-\frac{\alpha}{2}-\frac{x^2}{\sqrt{n\alpha}}+O\left(\frac{x^3}{n}\right)+o(1),$$

so

$$b_k=\frac{1}{2\pi n}\mathrm{e}^{2\sqrt{n\alpha}-\frac{\alpha}{2}-\frac{x^2}{\sqrt{n\alpha}}+O\left(\frac{x^3}{n}\right)+o(1)}=Ac_k,$$

where $A=\frac{1}{2\pi n}\mathrm{e}^{2\sqrt{n\alpha}-\frac{\alpha}{2}}$ and $c_k=\mathrm{e}^{-\frac{x^2}{\sqrt{n\alpha}}+O\left(\frac{x^3}{n}\right)+o(1)}$ when $|x|\ll\sqrt{n}$.

For every $i \in \mathbb{Z}$ let $k_i = i + [\sqrt{n\alpha}]$. Then $\sum_{k=0}^{n} c_k = \sum_{i=-[\sqrt{n\alpha}]}^{n-[\sqrt{n\alpha}]} c_{k_i}$. For every i we have $k_i = \sqrt{n\alpha} + x_i$, where $x_i = i + [\sqrt{n\alpha}] - \sqrt{n\alpha} \in (i-1, i]$. Hence $c_{k_i} = e^{-\frac{x_i^2}{\sqrt{n\alpha}} + O\left(\frac{x_i^3}{n}\right) + o(1)}$ when $x_i \ll \sqrt{n}$, i.e., when $|i| \ll \sqrt{n}$.

Let now $j = -[B] - 1$, $l = [B] + 2$, where $B = (n\alpha)^{1/4}\sqrt{2\log n}$. For $j \leq i \leq l$ we have $|k| \ll \sqrt{n}$, so the above estimate for c_{k_i} holds. Also for such i we have $-[B]-2 < x_i \leq [B]+2$ (recall, $x_i \in (i-1, i]$), so $x_i = O(n^{1/4}\sqrt{\log n})$, which implies that $O\left(\frac{x_i^3}{n}\right) = o(1)$. Hence $c_{k_i} = e^{-\frac{x_i^2}{\sqrt{n\alpha}} + o(1)}$, i.e., $c_{k_i} \sim \mathrm{e}^{-\frac{x_i^2}{\sqrt{n\alpha}}}$.

Note that $x_l > l - 1 = [B] + 1 > B$ and $x_j \leq j = -[B] - 1 < -B$. It follows that $\frac{x_j^2}{\sqrt{n\alpha}}, \frac{x_l^2}{\sqrt{n\alpha}} > \frac{B^2}{\sqrt{n\alpha}} = 2\log n$. Then $c_{k_j} \sim e^{-\frac{x_j^2}{\sqrt{n\alpha}}} < e^{-2\log n} = \frac{1}{n^2}$, so $c_{k_j} = O\left(\frac{1}{n^2}\right)$. Similarly $c_{k_l} = O\left(\frac{1}{n^2}\right)$. Recall that b_k increases on $[0, [C]+1]$ and decreases on $[[C]+1, n]$ and so does $c_k = \frac{1}{A} b_k$. But the difference between $[C]+1 = [\frac{-\alpha-1+\sqrt{(\alpha+1)^2+4n\alpha}}{2}]$ and $\sqrt{n\alpha}$ is bounded, while $B \to \infty$ as $n \to \infty$. So for n large we have

$$k_j = \sqrt{n\alpha} + x_j < \sqrt{n\alpha} - B < [C] + 1 < \sqrt{n\alpha} + B < \sqrt{n\alpha} + x_l = k_l.$$

It follows that for $i \leq j$ we have $k_i \leq k_J$, so $c_{k_i} \leq c_{k_j}$, and for $i \geq l$ we have $k_i \geq k_l$, so $c_{k_i} \leq c_{k_l}$. In both cases $c_{k_i} = O\left(\frac{1}{n^2}\right)$. Since the number of these terms is $< n$, their contribution to $\sum_{k=0}^{n} c_k$ is $O\left(\frac{1}{n}\right)$.

We now evaluate the rest of the sum, that is, $\sum_{i=-[B]}^{[B]+1} c_{k_i}$, which is $\sim \sum_{i=-[B]}^{[B]+1} \mathrm{e}^{-\frac{x_i^2}{\sqrt{n\alpha}}}$. Let $t_i = \frac{x_i}{(n\alpha)^{1/4}}$. Then $\sum_{i=-[B]}^{[B]+1} \mathrm{e}^{-\frac{x_i^2}{\sqrt{n\alpha}}} = \sum_{i=-[B]}^{[B]+1} \mathrm{e}^{-t_i^2}$.

But $t_i \in (\frac{i-1}{(n\alpha)^{1/4}}, \frac{i}{(n\alpha)^{1/4}}]$. It follows that $\frac{1}{(n\alpha)^{1/4}} \sum_{i=-[B]}^{[B]+1} \mathrm{e}^{-t_i^2}$ is a Riemann sum with intervals of lenght $\frac{1}{(n\alpha)^{1/4}}$ for the interval $\left[\frac{-[B]-1}{(n\alpha)^{1/4}}, \frac{[B]+1}{(n\alpha)^{1/4}}\right]$.

But $\frac{[B]+1}{(n\alpha)^{1/4}} > \frac{B}{(n\alpha)^{1/4}} = \sqrt{2\log n}$, which $\to \infty$ when $n \to \infty$. So the limit of the Riemann sum when $n \to \infty$ is $\int_{-\infty}^{\infty} \mathrm{e}^{-t^2}\mathrm{d}t = \sqrt{\pi}$. It follows that $\sum_{i=-[B]}^{[B]+1} c_{k_i} \sim (n\alpha)^{1/4}\sqrt{\pi}$. Together with $\left(\sum_{i\leq-[B]-1} + \sum_{i\geq[B]+2}\right) c_{k_i} = O\left(\frac{1}{n}\right)$, this implies

$$\sum_{k=0}^{n} c_k = \sum_{i=-[\sqrt{n\alpha}]}^{n-[\sqrt{n\alpha}]} c_{k_i} \sim (n\alpha)^{1/4}\sqrt{\pi}.$$

Since $a_n = \mathrm{e}^{\frac{b}{d}} \left(-\frac{c}{d}\right)^n \sum\limits_{k=0}^{n} b_k$ and $b_k = Ac_k = \frac{1}{2\pi n}\mathrm{e}^{2\sqrt{n\alpha}-\frac{\alpha}{2}}$, we get

$$a_n \sim \mathrm{e}^{\frac{b}{d}} \left(-\frac{c}{d}\right)^n \cdot \frac{1}{2\pi n}\mathrm{e}^{2\sqrt{n\alpha}-\frac{\alpha}{2}} \cdot (n\alpha)^{\frac{1}{4}}\sqrt{\pi} = \frac{1}{2\sqrt{\pi}}\alpha^{\frac{1}{4}} n^{-\frac{3}{4}}\mathrm{e}^{2\sqrt{n\alpha}+\frac{b}{d}-\frac{\alpha}{2}} \left(-\frac{c}{d}\right)^n,$$

i.e.

$$a_n \sim \frac{1}{2\sqrt{\pi}}\left(\frac{b}{d}-\frac{a}{c}\right)^{\frac{1}{4}} n^{-\frac{3}{4}} \mathrm{e}^{2\sqrt{n\left(\frac{b}{d}-\frac{a}{c}\right)}+\frac{1}{2}\left(\frac{a}{c}+\frac{b}{d}\right)} \left(-\frac{c}{d}\right)^n.$$

421. (i) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. Find the value of the limit

$$\lim_{n\to\infty} n^2 \iiint\limits_{x^2+y^2\leq 1,\ 0\leq z\leq 1} \left(\frac{\sqrt{x^2+y^2}+z}{2}\right)^n f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

(ii) Let $f : \mathbb{R}^4 \to \mathbb{R}$ be a continuous function. Find the value of the limit

$$\lim_{n\to\infty} n^2 \iiiint\limits_{x^2+y^2\leq 1,\ z^2+t^2\leq 1} \left(\frac{\sqrt{x^2+y^2}+\sqrt{z^2+t^2}}{2}\right)^n f(x,y,z,t)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t.$$

`Proposed by Dumitru Popa, Universitatea Ovidius, Constanţa, Constanţa, Romania.`

Solution by the author. (i) By Fubini's theorem

$$\iiint\limits_{x^2+y^2\leq 1,\ 0\leq z\leq 1} \left(\frac{\sqrt{x^2+y^2}+z}{2}\right)^n f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z =$$

$$= \int_0^1 \left(\iint\limits_{x^2+y^2\leq 1} \left(\frac{\sqrt{x^2+y^2}+z}{2}\right)^n f(x,y,z)\,\mathrm{d}x\mathrm{d}y\right)\mathrm{d}z.$$

Let $z \in [0,1]$. By the change of variable in the polar coordinates $x = \rho\cos\theta$, $y = \rho\sin\theta$ we obtain

$$\iint\limits_{x^2+y^2\leq 1} \left(\frac{\sqrt{x^2+y^2}+z}{2}\right)^n f(x,y,z)\,\mathrm{d}x\mathrm{d}y =$$

$$= \iint\limits_{[0,1]\times[0,2\pi]} \left(\frac{\rho+z}{2}\right)^n \rho f(\rho\cos\theta,\rho\sin\theta,z)\,\mathrm{d}\rho\,\mathrm{d}\theta.$$

Then again by Fubini's theorem we get

$$\iiint\limits_{x^2+y^2\le 1,\ 0\le z\le 1}\left(\frac{\sqrt{x^2+y^2}+z}{2}\right)^n f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z =$$

$$= \int_0^1\left(\iint\limits_{[0,1]\times[0,2\pi]}\left(\frac{\rho+z}{2}\right)^n \rho f(\rho\cos\theta,\rho\sin\theta,z)\,\mathrm{d}\rho\,\mathrm{d}\theta\right)\mathrm{d}z =$$

$$= \iint\limits_{[0,1]\times[0,1]}\left(\frac{\rho+z}{2}\right)^n\left(\int_0^{2\pi}\rho f(\rho\cos\theta,\rho\sin\theta,z)\,\mathrm{d}\theta\right)\mathrm{d}\rho\,\mathrm{d}z =$$

$$= \iint\limits_{[0,1]\times[0,1]}\left(\frac{\rho+z}{2}\right)^n g(\rho,z)\,\mathrm{d}\rho\,\mathrm{d}z,$$

where $g:[0,1]^2\to\mathbb{R}$, $g(\rho,z)=\int_0^{2\pi}\rho f(\rho\cos\theta,\rho\sin\theta,z)\,\mathrm{d}\theta$.

By Corollary 7 (ii) in [1] we deduce

$$\lim_{n\to\infty} n^2 \iiint\limits_{x^2+y^2\le 1,\ 0\le z\le 1}\left(\frac{\sqrt{x^2+y^2}+z}{2}\right)^n f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \lim_{n\to\infty} n^2 \iint\limits_{[0,1]\times[0,1]}\left(\frac{\rho+z}{2}\right)^n g(\rho,z)\,\mathrm{d}\rho\,\mathrm{d}z = 4g(1,1)$$

$$= 4\int_0^{2\pi} f(\cos\theta,\sin\theta,1)\,\mathrm{d}\theta.$$

(ii) By Fubini's theorem

$$\iiiint\limits_{x^2+y^2\le 1,\ z^2+t^2\le 1}\left(\frac{\sqrt{x^2+y^2}+\sqrt{z^2+t^2}}{2}\right)^n f(x,y,z,t)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t$$

$$= \iint\limits_{x^2+y^2\le 1}\left(\iint\limits_{z^2+t^2\le 1}\left(\frac{\sqrt{x^2+y^2}+\sqrt{z^2+t^2}}{2}\right)^n f(x,y,z,t)\,\mathrm{d}z\mathrm{d}t\right)\mathrm{d}x\mathrm{d}y.$$

Let $(x,y)\in\mathbb{R}^2$ be such that $x^2+y^2\le 1$. By the change of variable in the polar coordinates $z=\rho_2\cos\theta_2$, $t=\rho_2\sin\theta_2$ we obtain

$$\iint\limits_{z^2+t^2\le 1}\left(\frac{\sqrt{x^2+y^2}+\sqrt{z^2+t^2}}{2}\right)^n f(x,y,z,t)\,\mathrm{d}z\mathrm{d}t$$

$$= \iint\limits_{[0,1]\times[0,2\pi]}\left(\frac{\sqrt{x^2+y^2}+\rho_2}{2}\right)^n \rho_2 f(x,y,\rho_2\cos\theta_2,\rho_2\sin\theta_2)\,\mathrm{d}\rho_2\mathrm{d}\theta_2.$$

Again by Fubini's theorem

$$\iiiint\limits_{x^2+y^2\le 1,\ z^2+t^2\le 1} \left(\frac{\sqrt{x^2+y^2}+\sqrt{z^2+t^2}}{2}\right)^n f(x,y,z,t)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t =$$

$$=\iint\limits_{x^2+y^2\le 1}\left(\iint\limits_{[0,1]\times[0,2\pi]}\left(\frac{\sqrt{x^2+y^2}+\rho_2}{2}\right)^n \rho_2 f(x,y,\rho_2\cos\theta_2,\rho_2\sin\theta_2)\,\mathrm{d}\rho_2\mathrm{d}\theta_2\right)\mathrm{d}x\mathrm{d}y=$$

$$=\iint\limits_{[0,1]\times[0,2\pi]}\rho_2\left(\iint\limits_{x^2+y^2\le 1}\left(\frac{\sqrt{x^2+y^2}+\rho_2}{2}\right)^n f(x,y,\rho_2\cos\theta_2,\rho_2\sin\theta_2)\,\mathrm{d}x\mathrm{d}y\right)\mathrm{d}\rho_2\mathrm{d}\theta_2.$$

Let $(\rho_2,\theta_2)\in[0,1]\times[0,2\theta]$. By the change of variable in the polar coordinates $x=\rho_1\cos\theta_1$, $t=\rho_1\sin\theta_1$ we obtain

$$\iint\limits_{x^2+y^2\le 1}\left(\frac{\sqrt{x^2+y^2}+\rho_2}{2}\right)^n f(x,y,\rho_2\cos\theta_2,\rho_2\sin\theta_2)\,\mathrm{d}x\mathrm{d}y=$$

$$=\iint\limits_{[0,1]\times[0,2\pi]}\left(\frac{\rho_1+\rho_2}{2}\right)^n \rho_1 f(\rho_1\cos\theta_1,\rho_1\sin\theta_1,\rho_2\cos\theta_2,\rho_2\sin\theta_2)\,\mathrm{d}\rho_1\mathrm{d}\theta_1.$$

Thus

$$\iiiint\limits_{x^2+y^2\le 1,\ z^2+t^2\le 1} \left(\frac{\sqrt{x^2+y^2}+\sqrt{z^2+t^2}}{2}\right)^n f(x,y,z,t)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t =$$

$$=\iint\limits_{[0,1]^2}\left(\frac{\rho_1+\rho_2}{2}\right)^n\left(\rho_1\rho_2\iint\limits_{[0,2\pi]^2} f(\rho_1\cos\theta_1,\rho_1\sin\theta_1,\rho_2\cos\theta_2,\rho_2\sin\theta_2)\,\mathrm{d}\theta_1\mathrm{d}\theta_2\right)\mathrm{d}\rho_1\mathrm{d}\rho_2=$$

$$=\iint\limits_{[0,1]^2}\left(\frac{\rho_1+\rho_2}{2}\right)^n g(\rho_1,\rho_2)\,\mathrm{d}\rho_1\mathrm{d}\rho_2,$$

where $g:[0,1]^2\to\mathbb{R}$,

$$g(\rho_1,\rho_2)=\rho_1\rho_2\iint\limits_{[0,2\pi]^2} f(\rho_1\cos\theta_1,\rho_1\sin\theta_1,\rho_2\cos\theta_2,\rho_2\sin\theta_2)\,\mathrm{d}\theta_1\mathrm{d}\theta_2.$$

By Corollary 7 (ii) in [1] we deduce

$$\lim_{n\to\infty} n^2 \iiiint\limits_{x^2+y^2\le 1,\ z^2+t^2\le 1} \left(\frac{\sqrt{x^2+y^2}+\sqrt{z^2+t^2}}{2}\right)^n f(x,y,z,t)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t =$$

$$=\lim_{n\to\infty} n^2\iint\limits_{[0,1]^2}\left(\frac{\rho_1+\rho_2}{2}\right)^n g(\rho_1,\rho_2)\,\mathrm{d}\rho_1\mathrm{d}\rho_2=4g(1,1)=$$

$$= 4 \iint\limits_{[0,2\pi]^2} f(\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2)\, \mathrm{d}\theta_1 \mathrm{d}\theta_2.$$

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[1] D. Popa, The limit of some sequences of double integrals on the unit square, G.M.A., **32** (2014), Nr. 1–2, 19–26.

422. Find a sequence $(x_n)_{n\geq 1}$ with the following properties: $x_n \searrow 0$, $\sqrt{n}/(x_1 + \cdots + x_n) \to 0$ and $x_{[\sqrt{n}]}/x_n \to 1$, where $[\sqrt{n}]$ denotes the integer part of $\sqrt{n}$.

Can you find a sequence with the above properties, in which $\sqrt{n}$ is replaced by $\ln n$?

Proposed by George Stoica, University of New Brunswick, Saint John, Canada.

Solution by the author. If it weren't for the last property (which means that $(x_n)_{n\geq 1}$ is varying very slowly), any simple sequence such as $x'_n = c\big[(n+1)^{3/4} - n^{3/4}\big]$ with $c > 0$, fulfills the requirements. Hence it remains to modify $(x'_n)_{n\geq 1}$ to satisfy the last property. For this purpose choose $c > 0$ such that $x'_n < 1$ for $n \geq 1$, and define $y'_n > 0$ through $y'_1 + \cdots + y'_n = -\ln x'_n$ for $n \geq 1$. Put $z'_n = y'_{2^{2^{n-1}}-1} + \cdots + y'_{2^{2^n}-2}$. Then $\sum\limits_{n=1}^{\infty} z'_n = \lim(-\ln x'_{2^{2^n}-2}) = +\infty$. Therefore, one can choose a sequence $(z_n)_{n\geq 1}$ with the following properties:

(a) $0 < z_n \leq z'_n$,

(b) $z_n \to 0$,

(c) $\sum\limits_{n=1}^{\infty} z_n = +\infty$.

Moreover, by property (a), for each $n \geq 1$, one can choose nonnegative real numbers $y_{2^{2^{n-1}}-1} \leq y'_{2^{2^{n-1}}-1}, \ldots, y_{2^{2^n}-2} \leq y'_{2^{2^n}-2}$ such that $y_{2^{2^{n-1}}-1} + \cdots + y_{2^{2^n}-2} = z_n$. Finally, put $x_n = \mathrm{e}^{-(y_1+\cdots+y_n)}$. It follows that $(x_n)_{n\geq 1}$ is decreasing, $0 < x'_n \leq x_n < 1$ (we have $x'_n = \mathrm{e}^{-(y'_1+\cdots+y'_n)}$) and, by property (c), we have $x_{2^{2^n}-2} = \mathrm{e}^{-(z_1+\cdots+z_n)} \to 0$; hence $x_n \searrow 0$. As $x_1 + \cdots + x_n \geq x'_1 + \cdots + x'_n$, we have $\sqrt{n}/(x_1 + \cdots + x_n) \to 0$ as $\sqrt{n}/(x'_1 + \cdots + x'_n) \to 0$. Finally, by property (b) we have $x_{2^{2^{n-1}}-2}/x_{2^{2^n}-2} = \mathrm{e}^{z_n} \to 1$. But this implies that $x_{[\sqrt{n}]}/x_n \to 1$. Indeed, for $n \geq 0$ let $t_n = x_{2^{2^n}-2}$. So $t_{n-1}/t_n \to 1$. For any $n \geq 1$ there is a unique $m \geq 1$ such that $2^{2^{m-1}} - 1 \leq n \leq 2^{2^m} - 2$. Moreover $m \to \infty$ as $n \to \infty$. Then $[\sqrt{n}] \geq [\sqrt{2^{2^{m-1}} - 1}] = 2^{2^{m-2}} - 1$. (For any positive integer a we have $[\sqrt{a^2 - 1}] = a - 1$.) Since $x_n \searrow 0$, we have $x_n \geq x_{2^{2^m}-2} = t_n$ and $x_{[\sqrt{n}]} \leq x_{2^{2^{m-2}}-2} = t_{m-2}$. It follows that $1 \leq x_{[\sqrt{n}]}/x_n \leq t_{m-2}/t_m$. But $t_{m-2}/t_m = t_{m-2}/t_{m-1} \cdot t_{m-1}/t_m \to 1$, so $x_{[\sqrt{n}]}/x_n \to 1$.

The problem is now solved. We invite the readers to answer the question of replacing $[\sqrt{n}]$ by $\ln n$. □

423. Determine all differentiable functions $f : [0, \infty) \to \mathbb{R}$ with $f(0) = 0$ such that

a) f' is strictly positive and increasing.

b) $\int_0^x (f'(t))^2 \mathrm{d}t \geq f(x + f(x)) - f(x)$ $\forall x \in [0, \infty)$

Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.

Solution by the author. From f' increasing we get that f' is continuous. (By Darboux's theorem f' has the intermediate value property.) Also $f'(x) \geq f'(0) > 0$ and $f(x) > 0$ when $x > 0$.

By Lagrange's theorem, for any $x > 0$ there is some $c_x \in (x, x + f(x))$ such that $f(x+f(x)) - f(x) = f(x)f'(c_x)$. Since $f(x) > 0$ and f' is increasing we get $f(x + f(x)) - f(x) \geq f(x)f'(x)$, so $\int_0^x (f'(t))^2 \mathrm{d}t \geq f(x)f'(x)$ and

$$\frac{f'(x)}{f(x)} - \frac{(f'(x))^2}{\int_0^x (f'(t))^2 \mathrm{d}t} \geq 0,$$

i.e., $\phi'(x) \geq 0$, where $\phi : (0, \infty) \to \mathbb{R}$, $\phi(x) = \log f(x) - \log(\int_0^x (f'(t))^2 \mathrm{d}t)$. Since ϕ is increasing, so is $\psi = \mathrm{e}^{\phi}$, i.e., $\psi(x) = \frac{f(x)}{\int_0^x (f'(t))^2 \mathrm{d}t}$. Then by l'Hospital rule we have $\lim\limits_{x \to 0} \psi(x) = \lim\limits_{x \to 0} \frac{f'(x)}{(f'(x))^2} = \frac{1}{f'(0)}$. Since ψ is increasing, this implies that for any $x > 0$ we have $\frac{f(x)}{\int_0^x (f'(t))^2 \mathrm{d}t} = \psi(x) \geq \frac{1}{f'(0)}$, whence

$$f(x)f'(0) \geq \int_0^x (f'(t))^2 \mathrm{d}t \geq f(x)f'(x),$$

so $f'(0) \geq f'(x)$. As f' is increasing, we must have $f'(x) = f'(0)$ $\forall x$. If $f'(0) = a > 0$ then $f(x) = ax + b$ for some $b \in \mathbb{R}$. But $f(0) = 0$, so $f(x) = ax$ with $a > 0$. Obviously the functions of this type satisfy all the required conditions. □

424. Let $f, g \in \mathbb{C}[X]$ be monic polynomials of the same degree with the property that $|f(z)| = |g(z)| = 1$ for an infinity of values of $z \in \mathbb{C}$. Prove that $f = g$.

Proposed by Marius Cavachi, Universitatea Ovidius, Constanţa, Romania.

Solution by the author. Let $\overline{f}, \overline{g}$ be the polynomials obtained from f, g by conjugating the coefficients. Let M be the infinite set of all $z \in \mathbb{C}$ satisfying $|f(z)| = |g(z)| = 1$. Then for any $z \in M$ we have $f(z)\overline{f}(\overline{z}) - 1 = g(z)\overline{g}(\overline{z}) - 1 = 0$. Then the polynomials $f(z)\overline{f}(Y) - 1, g(z)\overline{g}(Y) - 1 \in \mathbb{C}[Y]$ have $\overline{z}$ as a common root, so their resultant is zero. But this resultant is a polynomial in the variable z. Since it has an infinity of roots (all $z \in M$), it must be zero for any $z \in \mathbb{C}$. Therefore for any $z \in \mathbb{C}$ with $f(z)g(z) \neq 0$ there is some $y_z \in \mathbb{C}$ such that $f(z)\overline{f}(y_z) - 1 = g(z)\overline{g}(y_z) - 1 = 0$.

Let a be a root of f. Then $\lim_{z\to a} f(y_z) = \lim_{z\to a} \frac{1}{f(z)} = \frac{1}{0} = \infty$. It follows that $y_z \to \infty$. It follows that $g(a) = \lim_{z\to a} g(z) = \lim_{z\to a} \frac{1}{\overline{g}(y_z)} = \frac{1}{\infty} = 0$. So a is a root of g. So every root of f is also a root of g. Similarly every root of g is also a root of f.

Let a be a root of f and g and let m and n be the multiplicity of a for f and g, respectively. We prove that $m = n$. Assume that, say, $m > n$. Write $f(X) = (X-a)^m f_1(X)$, $g(X) = (X-a)^n g_1(X)$ with $f_1(a), g_1(a) \neq 0$. Then taking limit for $z \to a$ in the equality $\frac{(z-a)^m f_1(z)}{(z-a)^n g_1(z)} = \frac{f(z)}{g(z)} = \frac{\overline{g}(y_z)}{\overline{f}(y_z)}$ one gets $0 = 1$. ($\overline{f}$ and $\overline{g}$ are monic of the same degree and $y_z \to \infty$ as $z \to a$.) Contradiction.

Since f, g are unitary and have the same roots, with the same multiplicities, we have $f = g$. □

Note from the editor. If we remove the condition that f, g are monic and we denote by α the leading coefficient of g divided by the leading coefficient of f then by the same proof we have that f and g have the same roots with the same multiplicities. (The only difference is that when we take limit in $\frac{(z-a)^m f_1(z)}{(z-a)^n g_1(z)} = \frac{f(z)}{g(z)} = \frac{\overline{g}(y_z)}{\overline{f}(y_z)}$ one gets $0 = \overline{\alpha} \neq 0$ instead of $0 = 1$.) It follows that $g = \alpha f$, from which we conclude that $|\alpha| = 1$. So f, g must differ by a factor which has the absolute value 1.

If we drop the condition that f, g have the same degree then let $m = \deg f$, $n = \deg g$. We have $m = m'd$, $n = n'd$, where $d = (m,n)$ and $(m', n') = 1$. We consider the polynomials $F = f^{n'}$, $G = g^{m'}$. Then $\deg F = \deg G = m'n'd = [m,n]$ and there are an infinity of $z \in \mathbb{C}$ with $|F(z)| = |G(z)| = 1$. It follows that F and G have the same roots, with the same multiplicities. If m_a is the multiplicity of a in F and G, since $F = f^{n'}$, $G = g^{m'}$, we have $m', n' \mid m_a$, so $m'n' \mid m_a$. As seen above, we have $F = \alpha \prod_a (X-a)^{m_a}$ and $G = \beta \prod_a (X-a)^{m_a}$, where $\alpha, \beta \in \mathbb{C}^*$ with $|\alpha| = |\beta|$. If $h = \prod_a (X-a)^{\frac{m_a}{m'n'}}$ then h is monic, of degree $\frac{[m,n]}{m'n'} = d$, and we have $f^{n'} = F = \alpha h^{m'n'}$ and $g^{m'} = G = \beta h^{m'n'}$. It follows that $f = \gamma h^{m'}$, $g = \delta h^{n'}$ with $\gamma^{n'} = \alpha$, $\delta^{m'} = \beta$. In particular, $|\gamma|^{n'} = |\delta|^{m'}$.

So in the general case the condition is that $f = \gamma h^{m'}$, $g = \delta h^{n'}$ for some monic polynomial h of degree d and some $\gamma, \delta \in \mathbb{C}^*$ with $|\gamma|^{n'} = |\delta|^{m'}$. □

425. Let $n \geq 2$ and $a_1, \ldots, a_n \geq 0$ be integers and let $b_1, \ldots, b_n$ and λ be positive real numbers. Find the necessary and sufficient condition for the function

$f:\mathbb{R}^n\backslash\{(0,\ldots,0)\}\to\mathbb{R}$ defined by $f(x_1,\ldots,x_n)=\frac{x_1^{a_1}\cdots x_n^{a_n}}{\left(|x_1|^{b_1}+\cdots+|x_n|^{b_n}\right)^\lambda}$ to have a finite limit at $(0,\ldots,0)$.

Proposed by Dumitru Popa, Universitatea Ovidius, Constanţa, Romania.

Solution by the author. We use in the sequel, without supplementary explanations, the following well-known result: Let $\alpha\in\mathbb{R}$. Then

$$\lim_{k\to\infty}k^\alpha=\begin{cases}0 \text{ if } \alpha<0\\ 1 \text{ if } \alpha=0\\ \infty \text{ if } \alpha>0\end{cases}.$$

We use the well-known characterization for the existence of the limit of a function defined on $\mathbb{R}^n$: A function $f:\mathbb{R}^n\backslash\{(0,\ldots,0)\}\to\mathbb{R}$ has the limit $l\in\mathbb{R}$ at the point $(0,\ldots,0)$ if and only if for all sequences $\left(x_k^1,\ldots,x_k^n\right)_{k\in\mathbb{N}}\subset\mathbb{R}^n\backslash\{(0,\ldots,0)\}$ with $\lim\limits_{k\to\infty}\left(x_k^1,\ldots,x_k^n\right)=(0,\ldots,0)$ we have $\lim\limits_{k\to\infty}f(x_k^1,\ldots,x_k^n)=l$.

Let us suppose that there exists $\lim\limits_{(x_1,\ldots,x_n)\to(0,\ldots,0)}f(x_1,\ldots,x_n)=l\in\mathbb{R}$. Since $b_1>0$, …, $b_n>0$, $\lim\limits_{k\to\infty}\left(k^{-\frac{1}{b_1}},\ldots,k^{-\frac{1}{b_n}}\right)=(0,\ldots,0)$ (the limit in $\mathbb{R}^n$ is taken on components), it follows that $\lim\limits_{k\to\infty}f\left(k^{-\frac{1}{b_1}},\ldots,k^{-\frac{1}{b_n}}\right)=l$. From

$$f\left(k^{-\frac{1}{b_1}},\ldots,k^{-\frac{1}{b_n}}\right)=\frac{k^{-\left(\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}\right)}}{\left(\frac{n}{k}\right)^\lambda}=\frac{1}{n^\lambda}\cdot k^{\lambda-\left(\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}\right)}$$

we obtain $\lim\limits_{k\to\infty}\frac{1}{n^\lambda}\cdot k^{\lambda-\left(\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}\right)}=l\in\mathbb{R}$.

Then we deduce: $l=\frac{1}{n^\lambda}$ if $\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}-\lambda=0$, or $l=0$ if $\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}-\lambda>0$.

We show that the situation $\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}-\lambda=0$ is not possible.

Since $n\geq 2$, $\lim\limits_{k\to\infty}\left(k^{-\frac{2}{b_1}},k^{-\frac{1}{b_2}},\ldots,k^{-\frac{1}{b_n}}\right)=(0,\ldots,0)$, and thus

$$\lim_{k\to\infty}f\left(k^{-\frac{2}{b_1}},k^{-\frac{1}{b_2}},\ldots,k^{-\frac{1}{b_n}}\right)=l=\frac{1}{n^\lambda}.$$

By simple calculation

$$f\left(k^{-\frac{2}{b_1}},k^{-\frac{1}{b_2}},\ldots,k^{-\frac{1}{b_n}}\right)=\frac{k^{-\left(\frac{2a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_n}{b_n}\right)}}{\left(\frac{1}{k^2}+\frac{n-1}{k}\right)^\lambda}=$$

$$=k^{\lambda-\left(\frac{2a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_n}{b_n}\right)}\cdot\frac{1}{\left(\frac{1}{k}+n-1\right)^\lambda},$$

from where we get $\lim\limits_{k\to\infty} \frac{k^{\lambda-\left(\frac{2a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_n}{b_n}\right)}}{\left(\frac{1}{k}+n-1\right)^{\lambda}} = \frac{1}{n^{\lambda}}$.

From $\frac{k^{\lambda-\left(\frac{2a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_n}{b_n}\right)}}{(n-1)^{\lambda}} = \frac{k^{\lambda-\left(\frac{2a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_n}{b_n}\right)}}{\left(\frac{1}{k}+n-1\right)^{\lambda}} \cdot \frac{\left(\frac{1}{k}+n-1\right)^{\lambda}}{(n-1)^{\lambda}}$ it follows that $\lim\limits_{k\to\infty} \frac{k^{\lambda-\left(\frac{2a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_n}{b_n}\right)}}{(n-1)^{\lambda}} = \frac{1}{n^{\lambda}}$, so $\frac{2a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_n}{b_n}-\lambda=0$ and $\frac{1}{(n-1)^{\lambda}} = \frac{1}{n^{\lambda}}$, which is impossible.

Thus, if f has finite limit at $(0,\ldots,0)$, then $\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n} > \lambda$.

Conversely, let us suppose that $\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n} > \lambda$.

Let $(x_1,\ldots,x_n) \in \mathbb{R}^n$. For any i we have $0 \le |x_i|^{b_i} \le |x_1|^{b_1}+\cdots+|x_n|^{b_n}$ and since $\frac{a_1}{b_1} \ge 0$, we deduce $0 \le |x_i|^{a_i} = \left(|x_i|^{b_i}\right)^{\frac{a_i}{b_i}} \le \left(|x_1|^{b_1}+\cdots+|x_n|^{b_n}\right)^{\frac{a_i}{b_i}}$. Multiplying these inequalities for $1 \le i \le n$ we get

$$|x_1|^{a_1}\cdots|x_n|^{a_n} \le \left(|x_1|^{b_1}+\cdots+|x_n|^{b_n}\right)^{\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}}.$$

Hence, for all $(x_1,\ldots,x_n) \in \mathbb{R}^n \setminus \{(0,\ldots,0)\}$ we obtain

$$|f(x_1,\ldots,x_n)| = \frac{|x_1|^{a_1}\cdots|x_n|^{a_n}}{\left(|x_1|^{b_1}+\cdots+|x_n|^{b_n}\right)^{\lambda}} \le \left(|x_1|^{b_1}+\cdots+|x_n|^{b_n}\right)^{\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}-\lambda}.$$

Since $\lim\limits_{(x_1,\ldots,x_n)\to(0,\ldots,0)} \left(|x_1|^{b_1}+\cdots+|x_n|^{b_n}\right) = 0$ (recall that $b_1 > 0$, …, $b_n > 0$) and $\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}-\lambda > 0$, it follows that

$$\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} \left(|x_1|^{b_1}+\cdots+|x_n|^{b_n}\right)^{\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}-\lambda} = 0$$

and therefore $\lim\limits_{(x_1,\ldots,x_n)\to(0,\ldots,0)} f(x_1,\ldots,x_n) = 0$.

In conclusion, the function f has finite limit at $(0,\ldots,0)$ if and only if one has $\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n} > \lambda$. □

426. Find all functions $f:\mathbb{R}\to\mathbb{R}$, continuous in at least one point, and that satisfy the following inequalities: $f(x-1) \le f(x)-1$, $f(x+\sqrt{2}) \le f(x)+\sqrt{2}$ for $x \in \mathbb{R}$.

`Proposed by George Stoica, Department of Mathematical Sciences, University of New Brunswick, Canada`

Solution by Viktor Makanin, Sankt Petersburg, Russia. The functions f of the form $f(x) = x + c$ (with c some real number) satisfy (obviously) the conditions of the enounce, and we prove below that these are the only solutions to our problem.

Inductively we get $f(x-n) \leq f(x)-n$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}^*$, and, also, $f(x+m\sqrt{2}) \leq f(x)+m\sqrt{2}$, again for all $x \in \mathbb{R}$, and all $m \in \mathbb{N}^*$. Consequently,

$$f(x+m\sqrt{2}-n) \leq f(x+m\sqrt{2})-n \leq f(x)+m\sqrt{2}-n$$

for all $x \in \mathbb{R}$, and all positive integers m and n.

Now, let f be continuous at x_0. Because the set $\{m\sqrt{2}-n \mid m,n \in \mathbb{N}^*\}$ is dense in $\mathbb{R}$, there is a sequence $(m_k-n_k\sqrt{2})_{k\geq 1}$ such that $\lim\limits_{k\to\infty}(m_k-n_k\sqrt{2}) = x_0-x$. If we pass to the limit for k tending to ∞ in

$$f(x+m_k\sqrt{2}-n_k) \leq f(x)+m_k\sqrt{2}-n_k$$

(and use the continuity of f at x_0) we get

$$f(x_0) \leq f(x)+x_0-x \Leftrightarrow f(x_0)-x_0 \leq f(x)-x$$

for all $x \in \mathbb{R}$.

On the other hand, if we replace x by $x+1$ in the first of the given inequalities satisfied by f, we get $f(x+1) \geq f(x)+1$, for all x; and, similarly, from the second inequality, by plugging $x-\sqrt{2}$ in place of x, we get $f(x-\sqrt{2}) \geq f(x)$ for all x. Consequently, $f(x+m-n\sqrt{2}) \geq f(x)+m-n\sqrt{2}$ for all $x \in \mathbb{R}$, and all $m,n \in \mathbb{N}^*$. The same reasoning as above leads to

$$f(x_0) \geq f(x)+x_0-x \Leftrightarrow f(x_0)-x_0 \geq f(x)-x$$

for every real x. Combining the above two inequalities, we get $f(x)-x = f(x_0)-x_0$ for every x, thus the function $x \mapsto f(x)-x$ is constant, providing the desired result. □

427. Let F be a field of characteristic $\neq 2$ and let E/F be a finite multiquadratic extension so that $G := \mathrm{Gal}(E/F) \cong \mathbb{Z}_2^n$. Let $a \in E^*$ with the property that $a^{s-1} \in E^{*2}$ $\forall s \in G$. Prove that there are $b_s \in E^*$ with $s \in G$ such that $a^{s-1} = b_s^2$ $\forall s \in G$ and $b_{stu}b_{st}^{-1}b_{su}^{-1}b_{tu}^{-1}b_sb_tb_u = b_u^{(s-1)(t-1)}$ $\forall s,t,u \in G$.

How many $(b_s)_{s\in G} \in (E^*)^G$ with the properties above exist?

Here we use the exponential notation, if $c \in E^*$ and $x = \sum\limits_{s\in G} n_s s \in \mathbb{Z}G$ then $c^x := \prod\limits_{s\in G} s(c)^{n_s}$.

Proposed by Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

Solution by the author. If $c \in E^*$ and $s,t \in G = \mathrm{Gal}(E/F)$ then $(c^s)^t = t(s(c)) = c^{ts}$. More generally, $(c^x)^y = c^{yx}$ for any $x,y \in \mathbb{Z}G$. Since G is abelian, we have $(c^x)^y = c^{xy}$.

If $E = F(\sqrt{x_1},\ldots,\sqrt{x_n})$ then a basis of G as a $\mathbb{Z}/2\mathbb{Z}$-vector space is $s_1,\ldots,s_n$, where s_i is given by $\sqrt{x_i} = -\sqrt{x_i}$ and $\sqrt{x_j} = \sqrt{x_j}$ for $j \neq i$. For convenience, for every subset $I = \{i_1,\ldots,i_k\}$ of $\{1,\ldots,n\}$ we denote $s_I = s_{i_1}\cdots s_{i_k}$. ($s_\emptyset := 1$.) Then every $s \in G$ writes uniquely as $s = s_I$

for some $I \subseteq \{1, \dots, n\}$. Obviously$s_i$ is given by $\sqrt{x_i} = -\sqrt{x_i}$ if $i \in I$ and $\sqrt{x_i} = \sqrt{x_i}$ otherwise.

Note that if b_s are chosen arbitrarily with the property that $b_s^2 = a^{s-1}$ for every $s \in G$ then

$$(b_{stu}b_{st}^{-1}b_{su}^{-1}b_{tu}^{-1}b_sb_tb_u)^2 = a^{stu-1}a^{-(st-1)}a^{-(su-1)}a^{-(tu-1)}a^{s-1}a^{t-1}a^{u-1}$$

$$= a^{stu-st-su-tu+s+t+u-1} = a^{(s-1)(t-1)(u-1)} = (a^{u-1})^{(s-1)(t-1)} = b_u^{2(s-1)(t-1)}.$$

It follows that $b_{stu}b_{st}^{-1}b_{su}^{-1}b_{tu}^{-1}b_sb_tb_u = \pm b_u^{(s-1)(t-1)}$. The problem is to find $(b_s)_{s\in G}$ such that the $\pm$ sign is $+$ $\forall s,t,u \in G$.

Note that if (b_s) satisfies the required conditions then $b_1 = 1$. Indeed, if we put $s = t = u = 1$ in the relation $b_{stu}b_{st}^{-1}b_{su}^{-1}b_{tu}^{-1}b_sb_tb_u = b_u^{(s-1)(t-1)}$ we get $b_1 = b_1^0 = 1$. Assuming that $b_1 = 1$, if one of s,t,u is 1 then one checks that $b_{stu}b_{st}^{-1}b_{su}^{-1}b_{tu}^{-1}b_sb_tb_u = b_1 = 1$, while $b_u^{(s-1)(t-1)} = b_1^{(s-1)(t-1)} = 1$ if $u = 1$ and $= b_u^0 = 1$ if s or $t = 1$. Therefore b_1 can be ignored. Thus we have to find b_s for $s \in G \setminus \{1\}$ such that $b_s^2 = a^{s-1}$ and $b_{stu}b_{st}^{-1}b_{su}^{-1}b_{tu}^{-1}b_sb_tb_u = b_u^{(s-1)(t-1)}$ $\forall s,t,u \in G \setminus \{1\}$.

Note that the condition $b_{stu}b_{st}^{-1}b_{su}^{-1}b_{tu}^{-1}b_sb_tb_u = b_u^{(s-1)(t-1)}$ implies that b_{stu} is uniquely determined by b_x with $x = s,t,u,st,su,tu$. Moreover, assuming that $b_x^2 = a^{x-1}$ for $x = s,t,u,st,su,tu$, we also have $b_{stu}^2 = a^{stu-1}$. This can be obtained immediately by simply squaring the relation $b_{stu}b_{st}^{-1}b_{su}^{-1}b_{tu}^{-1}b_sb_tb_u = b_u^{(s-1)(t-1)}$. As a consequence, given that $b_{s_i} = b_i$ for $1 \le i \le n$ and $b_{s_is_j} = b_{i,j}$ for $1 \le i < j \le n$, where $b_i, b_{i,j} \in E^*$ satisfy $b_i^2 = a^{s_i-1}$ and $b_{i,j}^2 = a^{s_is_j-1}$, then $(b_s)_{s\in G\setminus\{1}$ satisfying the required conditions are uniquely determined (if they exist). Indeed we know b_{s_I} for $|I| = 1$ or 2. The remaining elements of $G \setminus \{1\}$ can be written as s_I with $|I| \ge 3$.

To show the unicity of b_{s_I}, we use induction on I. Assume that the statement is true for $|I| < k$, where $k \ge 3$. Let I with $|I| = k$. Let $i,j \in I$, $i < j$. Then $s_I = s_is_js_J$ where $J = I \setminus \{i,j\}$. Then by taking $s = s_i$, $t = s_j$ $u = s_J$ in the relation $b_{stu}b_{st}^{-1}b_{su}^{-1}b_{tu}^{-1}b_sb_tb_u = b_u^{(s-1)(t-1)}$ we get b_{s_I} in terms of b_x with $x = s_i, s_j, s_J, s_is_j, s_{J\cup\{i\}}, s_{J\cup\{j\}}$. But $b_{s_i} = b_i$, $b_{s_j} = b_j$, $b_{s_is_j} = b_{i,j}$ and $b_{s_J}, b_{s_{J\cup\{i\}}}, b_{s_{J\cup\{j\}}}$ are uniquely determined. (We have $|J| = k-2$ and $|J \cup \{i\}| = |J \cup \{j\}| = k-1$.) Therefore b_{s_I} is uniquely determined as well. We also prove by the same induction on $|I|$ that b_{s_I} constructed this way satisfy the condition $b_{s_I}^2 = a^{s_I-1}$. Since $b_s^2 = a^{s-1}$, one also has $b_{stu}b_{st}^{-1}b_{su}^{-1}b_{tu}^{-1}b_sb_tb_u = \pm b_u^{(s-1)(t-1)}$.

To prove the existence one must show that the $\pm$ sign is always $+$. However, this is technically very difficult. Therefore we will restate the required conditions in a more convenient way.

Let $\mathcal{I}$ be the subgroup of $\mathbb{Z}G$ generated by $s-1$, $s \in G$. In fact $\{s-1 \mid s \in G, s \neq 1\}$ is a basis for $\mathcal{I}$. Now $(s-1)t = (st-1) - (t-1) \in \mathcal{I}$

$\forall s, t \in G$. It follows that $\mathcal{I}$ is an ideal in $\mathbb{Z}G$. ($s-1$ generates $\mathcal{I}$ and t generates $\mathbb{Z}G$.)

Note that for any $s \in G$, $c \in F^*$ we have $c^{s-1} = 1$. More generally, by the linearity of $x \mapsto c^x$, we have $c^x = 1\ \forall x \in \mathcal{I}$. We are only interested in the fact that $(\pm 1)^x = 1\ \forall x \in \mathcal{I}$.

Given $b_s \in E^*$ for $s \in G \setminus \{1\}$ we define the function $f : \mathcal{I} \to E^*$ by $f(\sum_{s \in G\setminus\{1\}} n_s(s-1)) = \prod_{s \in G\setminus\{1\}} b_s^{n_s}$, i.e., f is the linear function satisfying $f(s-1) = b_s$. Obviously this gives a bijection between the linear functions $f : \mathcal{I} \to E^*$ and the elements $(b_s)_{s \in G\setminus\{1\}}$ of $(E^*)^{G\setminus\{1\}}$. (We have $b_s = f(s-1)$.)

Now the first required condition, $b_s^2 = a^{s-1}$, writes as $f(s-1)^2 = a^{s-1}$. The mappings from $\mathcal{I}$ to E^* defined by $x \mapsto f(x)^2$ and $x \mapsto a^x$, respectively, are both linear and coincide on elements of the form $x = s-1$. Since $s-1$ covers a base of $\mathcal{I}$, the two maps are equal, i.e., $f(x)^2 = a^x$. So the first condition is equivalent to $f(x)^2 = a^x\ \forall x \in \mathcal{I}$. Since $f(x)^2 = f(2x)$ (f is linear), this can be also written as $f(2x) = a^x$.

For the second condition note that $b_{stu}b_{st}^{-1}b_{su}^{-1}b_{tu}^{-1}b_s b_t b_u = f(stu-1)f(st-1)^{-1}f(su-1)^{-1}f(tu-1)^{-1}f(s-1)f(t-1)f(u-1) = f((stu-1)-(st-1)-(su-1)-(tu-1)+(s-1)+(t-1)+u-1) = f((s-1)(t-1)(u-1))$. We also have $b_u^{(s-1)(t-1)} = f(u-1)^{(s-1)(t-1)}$. So the second condition means that the functions from $\mathcal{I}^2 \times \mathcal{I}$ to E^* defined by $(x,y) \mapsto f(xy)$ and $(xy) \mapsto f(y)^x$, respectively, coincide on elements $(x,y) \in \mathcal{I}^2 \times \mathcal{I}$ of the form $((s-1)(t-1), u-1)$. But both functions are bilinear, $(s-1)(t-1)$ covers a system of generators of $\mathcal{I}^2$, and $u-1$ covers a basis of $\mathcal{I}$. It follows that the two functions must coincide. So the second condition is equivalent to $f(xy) = f(y)^x\ \forall x \in \mathcal{I}^2$, $y \in \mathcal{I}$.

We now prove that for any b_i with $1 \le i \le n$ and $b_{i,j}$ with $1 \le i < j \le n$ satisfying $b_i^2 = a^{s_i - 1}$ and $b_{i,j}^2 = a^{s_i s_j - 1}$ there is a unique linear function $f : \mathcal{I} \to E^*$ satisfying the conditions $f(x)^2 = a^x\ \forall x \in \mathcal{I}$, $f(xy) = f(y)^x$ $\forall x \in \mathcal{I}^2$, $y \in \mathcal{I}$ such that $f(s_i - 1) = b_i$ and $f(s_i s_j - 1) = b_{i,j}$. To do this we need some convenient bases for $\mathbb{Z}G$, $\mathcal{I}$ and $\mathcal{I}^2$.

Let $t_i = s_i - 1$. Similarly for s_I, for any $I \subseteq \{1, \dots, n\}$ we denote $t_I = \prod_{i \in I} t_i$ ($t_\emptyset := 1$.). Now for any I we have $s_I = \prod_{i \in I} s_i = \prod_{i \in I}(t_i + 1) = \sum_{J \subseteq I} t_J$. It follows that the set $\{t_I \mid I \subseteq \{1, \dots, n\}\}$ is a system of generators for $\mathbb{Z}G$. Since this set has 2^n elements and the rank of $\mathbb{Z}G$ is 2^n, it is in fact a basis.

Now $t_i = s_i - 1 \in \mathcal{I}$. It follow that $\mathcal{I}$ contains also the products t_I with $I \neq \emptyset$. But $\mathcal{I}$ is generated by $s_I - 1 = \sum_{J \subseteq I} t_J - 1 = \sum_{\emptyset \neq J \subseteq I} t_J$. It follows that t_I with $|I| \ge 1$ form a basis for $\mathcal{I}$

The relation $s_i^2 = 1$ writes as $(t_i + 1)^2 = 1$, i.e., $t_i^2 = -2t_i$. More generally, $t_I^2 = (-2)^{|I|} t_I$. As a consequence, if $I, J \subseteq \{1, \dots, n\}$ then $t_I =$

$t_{I\cap J}t_{I\setminus J}$ and $t_J = t_{I\cap J}t_{J\setminus I}$. It follows that

$$t_I t_J = t_{I\cap J}^2 t_{I\setminus J}t_{J\setminus I} = (-2)^{|I\cap J|}t_{I\cap J}t_{I\setminus J}t_{J\setminus I} = (-2)^{|I\cap J|}t_{I\cup J}.$$

In particular, if $i \in I$ then $t_i t_I = -2t_I$.

Since $\mathcal{I}$ contains t_i for all i, $\mathcal{I}^2$ will contain t_I for all I with $|I| \geq 2$. It also contains $2t_i = -t_i^2$. Thus $\mathcal{I}^2$ contains the $\mathbb{Z}$-module M generated by $2t_i$ and t_I with $|I| \geq 2$. Since $\mathcal{I}$ is generated by t_I with $|I| \geq 1$, $\mathcal{I}^2$ is generated by $t_I t_J$ with $|I|, |J| \geq 1$. If $|I \cup J| \geq 2$ then $t_I t_J = (-2)^{|I\cap J|}t_{I\cup J} \in M$. If $|I \cup J| = 1$ then $I = J = \{i\}$ for some i, so $t_I t_J = t_i^2 = -2t_i \in M$. Hence $\mathcal{I}^2 = M$, i.e., $2t_i$ and t_I with $|I| \geq 2$ form a basis for $\mathcal{I}^2$.

We now prove the existence and unicity of the linear function $f : \mathcal{I} \to E^*$ with the required properties. Since t_I with $|I| \geq 1$ are a basis for $\mathcal{I}$, it suffices to define f on this basis. We have $f(t_i) = f(s_i - 1) = b_i$. Also $f(s_i s_j - 1) = b_{i,j}$ writes as $f(t_i t_j + t_i + t_j) = b_{i,j}$. Together with $f(t_i) = b_i$, $f(t_j) = b_j$, this implies by linearity that $f(t_i t_j) = b_{i,j}b_i^{-1}b_j^{-1}$. For $|I| \geq 3$ we take $i \in I$ arbitrary and we have $t_{I\setminus\{i\}} \in \mathcal{I}^2$ ($|I \setminus \{i\}| \geq 2$) and $t_i \in \mathcal{I}$. It follows that $f(t_I) = f(t_{I\setminus\{i\}}t_i) = f(t_i)^{t_{I\setminus\{i\}}} = b_i^{t_{I\setminus\{i\}}}$. So we have proven the unicity.

Note that the formula for $f(t_I)$ for $|I| \geq 3$ is independent of the choice of $i \in I$. Indeed, if $i, j \in I$ then $b_i^2 = a^{s_i-1} = a^{t_i}$, so $(b_i^{t_j})^2 = a^{t_i t_j}$, and, similarly, $(b_j^{t_i})^2 = a^{t_i t_j}$. It follows that $b_i^{t_j} = \pm b_j^{t_i}$. But $t_{I\setminus\{i,j\}} \in \mathcal{I}$ ($|I \setminus \{i,j\}| \geq 1$), so $(\pm 1)^{t_{I\setminus\{i,j\}}} = 1$. It follows that $(b_i^{t_j})^{t_{I\setminus\{i,j\}}} = (b_j^{t_i})^{t_{I\setminus\{i,j\}}}$, i.e., $b_i^{t_{I\setminus\{i\}}} = b_j^{t_{I\setminus\{j\}}}$.

We now prove that f defined this way satisfies all the required conditions. We have $f(s_i - 1) = f(t_i) = b_i$ and $f(s_i s_j - 1) = f(t_i t_j + t_i + t_j) = f(t_i t_j)f(t_i)f(t_j) = (b_{i,j}b_i^{-1}b_j^{-1})b_i b_j = b_{i,j}$.

For the first condition, since both mappings $x \mapsto f(x)^2, x \mapsto a^x$ from $\mathcal{I}$ to E^* are linear, we only need to prove that the two functions are equal for x in the basis of $\mathcal{I}$, i.e., that $f(t_I)^2 = a^{t_I}$ when $|I| \geq 1$. If $I = \{i\}$ then $f(t_i)^2 = b_i^2 = a^{s_i-1} = a^{t_i}$. If $I = \{i,j\}$ then $f(t_i t_j)^2 = (b_{i,j}b_i^{-1}b_j^{-1})^2 = a^{s_i s_j - 1}a^{-(s_i-1)}a^{-(s_j-1)} = a^{s_i s_j - s_i - s_j + 1} = a^{t_i t_j}$.

For the second condition, since both mappings $(x,y) \mapsto f(xy), (x,y) \mapsto f(y)^x$ from $\mathcal{I}^2 \times \mathcal{I}$ to E^* are bilinear, it is enough to prove that $f(xy) = f(y)^x$ when x and y belong to the basis of $\mathcal{I}^2$ and I, respectively, i.e., when $x = 2t_i$ or t_I with $|I| \geq 2$ and $y = t_J$ with $|J| \geq 1$. Since we have already proved the first condition, $f(2x) = f(x)^2 = a^x$, we can use it in the proof of the second one.

If $x = 2t_i$ then $f(2t_i t_J) = a^{t_i t_J}$ and $f(t_J)^{2t_i} = (f(t_j)^2)^{t_i} = (a^{t_J})^{t_i} = f(2t_i t_J)$. So we may assume that $x = t_I$ with $|I| \geq 2$. We have two cases.

If $I \cap J \neq \emptyset$ then let $i \in I \cap J$. We have $t_i t_J = -2t_J$, so that

$$f(t_I t_J) = f(t_{I\setminus\{i\}}t_i t_J) = f(-2t_{I\setminus\{i\}}t_J) = a^{-t_{I\setminus\{i\}}t_J}.$$

Note also that $f(t_J)^{2t_i} = (f(t_J)^2)^{t_i} = a^{t_J t_i} = a^{-2t_J}$, hence $f(t_J)^{t_i} = \pm a^{-t_J}$. But $t_{I\setminus\{i\}} \in \mathcal{I}$ (we have $|I \setminus \{i\}| \geq 1$), so $((\pm 1)^{t_{I\setminus\{i\}}} = 1$. It follows that $(f(t_J)^{t_i})^{t_{I\setminus\{i\}}} = (a^{-t_J})^{t_{I\setminus\{i\}}}$, i.e., $f(t_J)^{t_I} = a^{-t_{I\setminus\{i\}}t_J} = f(t_I t_J)$.

Suppose now that $I \cap J = \emptyset$. If $J = \{j\}$ then $f(t_I t_j) = f(t_{I\cup\{j\}})$. But $|I \cup \{j\}| \geq 3$ and $j \in I \cup \{j\}$, so by definition $f(t_I t_j) = b_j^{t_I} = f(t_j)^{t_I}$. If $J = \{j_1, j_2\}$ then by definition $f(t_{j_1} t_{j_2} t_I) = f(t_{I\cup\{j_1,j_2\}}) = b_{j_1}^{I\cup\{j_2\}}$. On the other hand $f(t_{j_1}t_{j_2})^2 = a^{t_{j_1}t_{j_2}}$ and $(f(t_{j_1})^{t_{j_2}})^2 = (f(t_{j_1})^2)^{t_{j_2}} = a^{t_{j_1}t_{j_2}}$. It follows that $f(t_{j_1}t_{j_2}) = \pm f(t_{j_1})^{t_{j_2}} = \pm b_{j_1}^{t_{j_2}}$. But $t_I \in \mathcal{I}$ and therefore $(\pm 1)^{t_I} = 1$. It follows that $f(t_{j_1}t_{j_2}) = (b_{j_1}^{t_{j_2}})^{t_I} = b_{j_1}^{I\cup\{t_{j_2}\}}$. Finally, if $|J| \geq 3$ then let $j \in J$. We also have $j \in I \cup J$, so by definition $f(t_I t_J) = f(t_{I\cup J}) = b_j^{t_{I\cup J\setminus\{j\}}}$ and $f(t_J)^{t_I} = (b_j^{t_{J\setminus\{j\}}})^{t_I} = f(t_I t_J)$.

Now for any i there are two values of b_i such that $b_i^2 = a^{s_i}$ and if $i < j$ there are two values of $b_{i,j}$ such that $b_{i,j}^2 = a^{s_i s_j}$. Hence there are $2^{n+n(n-1)/2} = 2^{n(n+1)/2}$ ways of choosing the b_i's and the $b_{i,j}$'s. So there are $2^{n(n+1)/2}$ sequences $(b_s)_{s\in G}$ satisfying the required conditions.

Note. One may obtain an explicit formula for b_{s_I} in terms of b_i and $b_{i,j}$. We have

$$b_{s_I} = f(s_I - 1) = f(\sum_{\emptyset\neq J\subseteq I} t_J) = \prod_{i\in I} b_i \prod_{i,j\in I, i<j} b_{i,j} b_i^{-1} b_j^{-1} \prod_{j\subseteq I, |I|\geq 3} b_{\min J}^{J\setminus\{\min J\}}.$$

(For any J with $|J| \geq 3$ we use the formula $b_J = b_j^{J\setminus\{j\}}$ with $j = \min J$.) But, for every i, b_i^{-1} appears in the product $\prod_{i,j\in I,i<j} b_{i,j}b_i^{-1}b_j^{-1}$ exactly $|I|-1$ times. Thus

$$\prod_{i\in I} b_i \prod_{i,j\in I,i<j} b_{i,j} b_i^{-1} b_j^{-1} = \prod_{i\in I} b_i^{2-|I|} \prod_{i,j\in I,i<j} b_{i,j}.$$

Also, for every $i \in I$, b_i appears in the product $\prod_{j\subseteq I,|I|\geq 3} b_{\min J}^{J\setminus\{\min J\}}$ with the exponent $\sum_{J\subseteq I_i, |J|\geq 2} t_J$, where $I_i = \{j \in I \mid j > i\}$. But $\sum_{J\subseteq I_i} t_J = s_{I_i}$, so

$$\sum_{J\subseteq I_i,|J|\geq 2} t_J = s_{I_i} - \sum_{j\in I_i} t_j - 1 = \prod_{j\in I, j>i} s_i - \sum_{j\in I,j>i} (s_j - 1) - 1.$$

Therefore,

$$b_{s_I} = \prod_{i,j\in I,i<j} b_{i,j} \prod_{i\in I} b_i^{\prod_{j\in I,j>i} s_i - \sum_{j\in I,j>i}(s_j-1) - |I| + 1}.$$

This formula can also be obtained by induction on $|I|$ if we use the formula $b_u^{(s-1)(t-1)} = b_{stu}b_{st}^{-1}b_{su}^{-1}b_{tu}^{-1}b_sb_tb_u$ with $s=s_J, t=s_j, u=s_i$, where we choose $i=\min I$, $j=\min I\setminus\{i\}$ and $J=I\setminus\{i,j\}$. As seen before, this gives b_{s_I} in terms of b_x with $x=s_i,s_j,s_J,s_is_j,s_{J\cup\{i\}},s_{J\cup\{j\}}$.

One may try to obtain the formula $b_{stu}b_{st}^{-1}b_{su}^{-1}b_{tu}^{-1}b_sb_tb_u = b_u^{(s-1)(t-1)}$ for arbitrary s,t,u by writing $s=s_I$, $t=s_J$ $u=s_K$ for some $I,J,K\subseteq\{1,\dots,n\}$ and using the above formula for b_{s_I}. However this would imply some messy calculations.

428. Show that $f(x)=x^{4n}-3x^{3n}+4x^{2n}-2x^n+1$ is irreducible in $\mathbb{Z}[x]$ for all integers $n\geq 1$.

`Proposed by Cornel Băeţica, University of Bucharest, Romania.`

Solution by the author. We have $f(x)=g(x^n)$, where $g(x)=x^4-3x^3+4x^2-2x+1$. Note that $g(x+1)=x^4+x^3+x^2+x+1=\Phi_5(x)$. Since Φ_5 is irreducible, so is g. Since the roots of Φ_5 are ζ^k with $1\leq k\leq 4$, where $\zeta=\zeta_5=\exp(2\pi i/5)$, the roots of g are $\alpha_k=\zeta^k+1$, with $1\leq k\leq 4$.

We have $\alpha_k\in K:=\mathbb{Q}(\zeta)$. We denote by $\mathcal{O}$ the ring of integers in K, i.e., $\mathcal{O}=\mathbb{Z}[\zeta]$. The torsion of K^* is $\mu(K)=\mu_{10}=\{\pm\zeta^s\mid s\in\mathbb{Z}\}$. Since K is a totally imaginary quartic field, the unit group $\mathcal{O}^\times$ has rank 1, so it admits a fundamental unit.

Now α_k are algebraic integers of norm 1, so $\alpha_k\in\mathcal{O}^\times$. We claim that α_k are fundamental units. Note that $\alpha_k=\zeta^{3k}(\zeta^{-2k}+\zeta^{-3k})=\zeta^{3k}(\zeta^{2k}+\zeta^{-2k})=\zeta^{3k}\cdot 2\cos(4k\pi/5)$. Hence $\alpha_k=2\cos(4k\pi/5)$ in $\mathcal{O}^\times/\mu(K)$, so it is enough to prove that $\alpha_k=2\cos(4k\pi/5)$ is a fundamental unit of K. But $2\cos(2\pi/5)=2\cos(8\pi/5)=\frac{-1+\sqrt{5}}{2}=\phi^{-1}$ and $2\cos(4\pi/5)=2\cos(6\pi/5)=\frac{-1-\sqrt{5}}{2}=-\phi$, where $\phi=\frac{1+\sqrt{5}}{2}$. (Note that ζ^k is a solution of $0=x^2+x+1+x^{-1}+x^{-2}=(x+x^{-1})^2+(x+x^{-1})-1$, so $2\cos(2k\pi/5)=\zeta^k+\zeta^{-k}$ is a solution of $x^2+x-1=0$.) Hence it is enough to prove that ϕ is a fundamental unit of K. Suppose the contrary and let β be a fundamental unit of K. Then $\phi=\varepsilon\beta^k$ for some integer k with $|k|\geq 2$ and some $\varepsilon\in\mu(K)$.

We have $\overline{\phi}=\phi$ and $|\varepsilon|=1$, so that $\overline{\varepsilon}=\varepsilon^{-1}$. It follows that

$$1=\frac{\overline{\phi}}{\phi}=\frac{\overline{\varepsilon\beta^k}}{\varepsilon\beta^k}=\varepsilon^{-2}\left(\frac{\overline{\beta}}{\beta}\right)^k.$$

Hence $(\overline{\beta}/\beta)^k=\varepsilon^2\in\mu(K)$, so $\overline{\beta}/\beta\in\mu(K)$, i.e., $\overline{\beta}/\beta=\pm\zeta^l$ for some $l\in\mathbb{Z}$. If $\gamma=\zeta^{3l}\beta$ then γ too is a fundamental unit and we have $\overline{\gamma}/\gamma=\zeta^{-6l}\overline{\beta}/\beta=\pm\zeta^{-5l}=\pm 1$. Suppose that the $\pm$ sign is $-$, that is we have $\overline{\gamma}=-\gamma$. We write γ in the basis $\zeta,\zeta^2,\zeta^3,\zeta^4$ of $\mathbb{Z}[\zeta]$, $\gamma=a\zeta+b\zeta^2+c\zeta^3+d\zeta^4$, with $a,b,c,d\in\mathbb{Z}$. Then $\overline{\gamma}=a\zeta^{-1}+b\zeta^{-2}+c\zeta^{-3}+d\zeta^{-4}=d\zeta+c\zeta^2+b\zeta^3+a\zeta^4$, so the relation $\overline{\gamma}=-\gamma$ is equivalent to $d=-a$, $c=-b$. It follows that $\gamma=a\zeta+b\zeta^2-b\zeta^3-a\zeta^4$, so γ is divisible in $\mathcal{O}$ by $\zeta-1$. But $\zeta-1\notin\mathcal{O}^\times$.

(In fact $\zeta - 1$ is a prime element of $\mathcal{O}$ lying over the prime 5 of $\mathbb{Z}$.) It follows that $\gamma \notin \mathcal{O}^\times$, contradiction. So the $\pm$ sign is $+$ and we have $\overline{\gamma} = \gamma$, that is $\gamma \in \mathbb{R}$.

Let $\gamma' = \pm\gamma$ be such that $\gamma' > 0$. We have $\beta = \pm\zeta^{-3l}\gamma'$, so $\phi = \varepsilon(\pm\zeta^{-3l})^k\gamma'^k$. As β, γ' are both real and positive, so is $\varepsilon(\pm\zeta^{-3l})^k$. But $\varepsilon(\pm\zeta^{-3l})^k \in \mu(K)$, so we must have $\varepsilon(\pm\zeta^{-3l})^k = 1$. It follows that $\beta = \gamma'^k$. But $\gamma' \in K \cap \mathbb{R} = \mathbb{Q}(\cos(2\pi/5)) = \mathbb{Q}(\sqrt{5})$. Since γ' is an algebraic integer, we actually have $\gamma' \in \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$. But then the equality $\phi = \gamma'^k$ is impossible, since $\phi = \frac{1+\sqrt{5}}{2}$ is the fundamental unit of $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$. This contradiction shows that ϕ is a fundamental unit of $\mathcal{O}$.

It follows that α_k with $1 \leq k \leq 4$ are fundamental units of $\mathcal{O}$. It follows that α_k and $-\alpha_k$ cannot be powers in K. From this it follows that $x^n - \alpha_k$ is irreducible in $K[x]$. (Otherwise, by Vahlen-Capelli theorem we have either $p \mid n$ and $\alpha_k = t^p$ for some prime p or $4 \mid n$ and $-\alpha_k = 4t^4 = (2t^2)^2$.)

Suppose now that f is reducible, $f(x) = f_1(x)f_2(x)$ for some monic, nonconstant $f_1(x), f_2(x) \in \mathbb{Q}[x]$. In $K[x]$ we have $f_1(x)f_2(x) = f(x) = g(x^n) = (x^n - \alpha_1)\cdots(x^n - \alpha_4)$. But $x^n - \alpha_k$ are irreducible in $K[x]$, so we must have $f_1(x) = \prod_{k\in I}(x^n - \alpha_k)$, $f_2(x) = \prod_{k\in J}(x^n - \alpha_k)$, where I, J form a partition of $\{1,2,3,4\}$ with $I, J \neq \emptyset$. If $g_1(x) = \prod_{k\in I}(x - \alpha_k)$, $g_2(x) = \prod_{k\in J}(x - \alpha_k)$ then $g_1(x)g_2(x) = (x - \alpha_1)\cdots(x - \alpha_4) = g(x)$. But $g(x)$ is irreducible, so we cannot have $g_1(x), g_2(x) \in \mathbb{Q}[x]$. Since $f_i(x) = g_i(x^n)$, we cannot have $f_1(x), f_2(x) \in \mathbb{Q}[x]$ either. Contradiction. Hence f is irreducible. □