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On some stability concepts for real functions

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Abstract. In this paper we investigate some asymptotic properties for real functions defined on $\mathbb{R}_+$ as exponential stability, uniform exponential stability, polynomial stability and uniform polynomial stability. Our main objectives are to give characterizations for these concepts and to establish connections between them.

Keywords: Exponential stability, polynomial stability.

MSC : 34D05

1. Exponential stability

The exponential stability property plays a central role in the theory of asymptotic behaviors for differential equations. In this section we consider two concepts of exponential stability for the particular case of real functions defined on $\mathbb{R}_+$.

Let Δ_0 be the set defined by

$$\Delta_0 = \{(y, x) \in \mathbb{R}_+^2 \text{ with } y \geq x\}.$$

Definition 1. *A real function $F : \mathbb{R}_+ \to \mathbb{R}$ is called*

(i) *uniformly exponentially stable (and denote u.e.s), if there are $N \geq 1$ and $\alpha > 0$ such that*

$$|F(y)| \leq Ne^{-\alpha(y-x)}|F(x)|, \text{ for all } (y, x) \in \Delta_0;$$

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(ii) *(nonuniformly) exponentially stable (and denote e.s), if there exist* $\alpha > 0$ *and a nondecreasing function* $N : \mathbb{R}_+ \to [1, \infty)$ *such that*

$$|F(y)| \leq N(x)e^{-\alpha(y-x)}|F(x)|, \text{ for all } (y,x) \in \Delta_0.$$

Remark 1. *It is obvious that u.e.s ⇒ e.s. The converse implication is not true, as shown in*

Example 1. Let $F : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by

$$F(x) = e^{x\cos x - 3x}.$$

We observe that for all $(y,x) \in \Delta_0$, we have

$$|F(y)| = F(y) = e^{y\cos y - 3y - x\cos x + 3x}|F(x)| \leq$$
$$\leq e^{-2y+4x}|F(x)| \leq e^{2x}e^{-2(y-x)}|F(x)|$$

and hence F is e.s.

If we suppose that if F is u.e.s, then there exist $N \geq 1$ and $\alpha > 0$ such that

$$e^{y\cos y - 3y} \leq Ne^{-\alpha(y-x)}e^{x\cos x - 3x}, \text{ for all } (y,x) \in \Delta_0.$$

In particular, for $y = 2n\pi$ and $x = 2n\pi - \pi/2$, we obtain

$$e^{2n\pi} \leq Ne^{(3-\alpha)\pi/2},$$

which for $n \to \infty$, leads to a contradiction.

A necessary and sufficient condition for uniform exponential stability is given by

Proposition 1. *A function* $F : \mathbb{R}_+ \to \mathbb{R}$ *is uniformly exponentially stable if and only if there exists a decreasing function* $f : \mathbb{R}_+ \to \mathbb{R}_+^* = (0, \infty)$ *with* $\lim\limits_{x\to\infty} f(x) = 0$ *and*

$$|F(y)| \leq f(y-x)|F(x)|, \text{ for all } (y,x) \in \Delta_0.$$

Proof. *Necessity.* It is an immediate verification.

Sufficiency. Let $\delta > 1$ with $f(\delta) < 1$. Then for all $(y,x) \in \Delta_0$ there exist $n \in \mathbb{N}$ and $r \in [0, \delta)$ such that $y = x + n\delta + r$. Then

$$\begin{aligned}
|F(y)| &\leq f(r)|F(x+n\delta)| \leq f(0)|F(x+n\delta)| \leq \\
&\leq f(0)f(\delta)|F(x+(n-1)\delta)| \leq \cdots \leq f(0)f^n(\delta)|F(x)| = \\
&= f(0)e^{n\ln f(\delta)}|F(x)| = f(0)e^{\alpha r}e^{-\alpha(y-x)}|F(x)| \leq \\
&\leq f(0)e^{\alpha\delta}e^{-\alpha(y-x)}|F(x)| \leq Ne^{-\alpha(y-x)}|F(x)|,
\end{aligned}$$

where $\alpha = -\dfrac{\ln \delta}{f(\delta)}$ and $N = 1 + f(0)e^{\alpha\delta}$. □

Definition 2. *A function $F : \mathbb{R}_+ \to \mathbb{R}$ with the property that there are $M \geq 1$ and $\omega > 0$ such that*

$$|F(y)| \leq Me^{\omega(y-x)}|F(x)|, \text{ for all } (y,x) \in \Delta_0,$$

is called with exponential growth *(and we denote e.g).*

Remark 2. *It is obvious that u.e.s $\Rightarrow$ e.g and the converse implication is not true.*

Proposition 2. *A function $F : \mathbb{R}_+ \to \mathbb{R}$ is with exponential growth if and only if there exists a nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+^*$ with $\lim\limits_{x\to\infty} \varphi(x) = = \infty$ and*

$$|F(y)| \leq \varphi(y-x)|F(x)|, \text{ for all } (y,x) \in \Delta_0.$$

Proof. It is similar with the proof of Proposition 1. □

Another characterization of the u.e.s property is given by

Proposition 3. *Let $F : \mathbb{R}_+ \to \mathbb{R}$ be an integrable function on each compact interval $[a,b] \subset \mathbb{R}_+$ (i.e., F is locally integrable on $\mathbb{R}_+$) with exponential growth. Then F is uniformly exponentially stable if and only if there exists $D \geq 1$ with*

$$\int_x^\infty |F(y)|\mathrm{d}y \leq D|F(x)|, \text{ for all } x \in \mathbb{R}_+.$$

Proof. *Necessity.* It is an immediate verification.

Sufficiency. Because F is with e.g, it follows that there is a nondecreasing function $\varphi : \mathbb{R}_+ \to [1,\infty)$ with

$$|F(y)| \leq \varphi(y-x)|F(x)|, \text{ for all } (y,x) \in \Delta_0.$$

Then for all $(y,x) \in \Delta_0$ with $y \geq x+1$ we have

$$|F(y)| = \int_{y-1}^{y} |F(y)|\mathrm{d}z \leq \int_{y-1}^{y} \varphi(y-z)|F(z)|\mathrm{d}z \leq$$
$$\leq \varphi(1)\int_x^\infty |F(z)|\mathrm{d}z \leq D\varphi(1)|F(x)|.$$

If $(y,x) \in \Delta_0$ with $y \in [x, x+1)$, then

$$|F(y)| \leq \varphi(y-x)|F(x)| \leq D\varphi(1)|F(x)|$$

and hence

$$|F(y)| \leq D\varphi(1)|F(x)|, \text{ for all } (y,x) \in \Delta_0.$$

Thus we obtain

$$(y-x)|F(y)| = \int_x^y |F(y)|\mathrm{d}z \le D\varphi(1)\int_x^y |F(z)|\mathrm{d}z \le D^2\varphi(1)|F(x)|,$$

for all $(y,x)\in\Delta_0$ and

$$|F(y)| \le f(y-x)|F(x)|, \text{ for all } (y,x)\in\Delta_0,$$

where

$$f(z) = \frac{D^2\varphi(1)}{z+1}.$$

Using the preceding proposition we conclude that F is u.e.s. □

A generalization of the preceding result for the nonuniform case is given by

Proposition 4. *Let $F:\mathbb{R}_+\to\mathbb{R}$ be a locally integrable function on $\mathbb{R}_+$ with exponential growth. Then F is exponentially stable if and only if there are $\beta>0$ and $D:\mathbb{R}_+\to[1,\infty)$ such that*

$$\int_x^\infty e^{\beta y}|F(y)|\mathrm{d}y \le D(x)e^{\beta x}|F(x)|, \textit{ for all } x\ge 0.$$

Proof. *Necessity.* It is a simple verification for $\beta\in(0,\alpha)$, where α is given by Definition 1 (ii).

Sufficiency. If $(y,x)\in\Delta_0$ with $y\ge x+1$ then

$$e^{\beta y}|F(y)| = \int_{y-1}^y e^{\beta y}|F(y)|\mathrm{d}z \le \int_{y-1}^y \varphi(y-z)e^{\beta y}|F(z)| \le$$

$$\le \varphi(1)\int_{y-1}^y e^{\beta(y-z)}e^{\beta z}|F(z)|\mathrm{d}z \le \varphi(1)e^{\beta}\int_x^\infty e^{\beta z}|F(z)|\mathrm{d}z \le$$

$$\le D(x)e^{\beta}\varphi(1)e^{\beta x}|F(x)|.$$

If $(y,x)\in\Delta_0$ with $y\in[x,x+1)$ then

$$e^{\beta y}|F(y)| \le e^{\beta(y-x)}e^{\beta x}\varphi(y-x)|F(x)| \le$$

$$\le e^{\beta}\varphi(1)e^{\beta x}|F(x)| \le D(x)e^{\beta}\varphi(1)e^{\beta x}|F(x)|.$$

Finally, we obtain

$$|F(y)| \le N(x)e^{-\beta(y-x)}|F(x)|, \text{ for all } (y,x)\in\Delta_0,$$

where

$$N(x) = \varphi(1)e^{\beta}D(x).$$

□

In the particular case of u.e.s property we obtain:

Corollary 1. *Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a locally integrable function on $\mathbb{R}_+$ with exponential growth. Then F is uniformly exponentially stable if and only if there exist $D \geq 1$ and $\beta > 0$ such that*

$$\int_x^\infty e^{\beta y}|F(y)|\mathrm{d}y \leq De^{\beta x}|F(x)|, \text{ for all } x \geq 0.$$

2. **Polynomial stability**

In this section we consider two concepts of polynomial stability for real functions. Our approach is based on the extension of techniques of exponential stability to the case of polynomial stability. Two illustrating examples clarify the relations between the stability concepts considered in this paper.

Let Δ_1 be the set defined by

$$\Delta_1 = \{(y, x) \in \mathbb{R}^2 \text{ with } y \geq x \geq 1\}.$$

Definition 3. *A real function $F : \mathbb{R}_+ \to \mathbb{R}$ is called*

(i) *uniformly polynomially stable (and denote u.p.s) if there exist $N \geq 1$ and $\alpha > 0$ such that*

$$y^\alpha|F(y)| \leq Nx^\alpha|F(x)|, \text{ for all } (y, x) \in \Delta_1;$$

(ii) *(nonuniformly) polynomially stable (and denote p.s) if there are a nondecreasing function $N : \mathbb{R}_+ \to [1, \infty)$ and $\alpha > 0$ such that*

$$y^\alpha|F(y)| \leq N(x)x^\alpha|F(x)|, \text{ for all } (y, x) \in \Delta_1.$$

Remark 3. *If the function $F : \mathbb{R}_+ \to \mathbb{R}$ is u.e.s then it is u.p.s. Indeed, if F is u.e.s then using the monotony of the function*

$$f : [1, \infty) \to [e, \infty), \ f(t) = \frac{e^t}{t}$$

we obtain

$$y^\alpha|F(y)| \leq x^\alpha e^{\alpha(y-x)}|F(x)| \leq Nx^\alpha|F(x)|,$$

for all $(y, x) \in \Delta_1$ and hence F is u.p.s. The converse is not true, phenomenon illustrated by

Example 2. *The function $F : \mathbb{R}_+ \to \mathbb{R}$, $F(x) = \dfrac{1}{x^3 + 1}$, satisfies the inequality*

$$y^3|F(y)| = \frac{y^3}{y^3 + 1} \leq \frac{2x^3}{x^3 + 1} = 2x^3|F(x)|,$$

for all $(y, x) \in \Delta_1$. This shows that F is u.p.s. We show that F is not e.s and hence it is not u.e.s. Indeed, if we suppose that F is e.s then there are

$\alpha > 0$ and $N : \mathbb{R}_+ \to [1, \infty)$ such that

$$\frac{e^{\alpha y}}{y^3+1} \le N(x)\frac{e^{\alpha x}}{x^3+1}, \text{ for all } (y,x) \in \Delta_0.$$

For $x = 0$ and $y \to \infty$ we obtain a contradiction which proves that F is not e.s.

Remark 4. *Similarly, as in Remark 3 we can prove that if $F : \mathbb{R}_+ \to \mathbb{R}$ is e.s then it is p.s.*

The function considered in Example 2 shows that the converse is not true.

Remark 5. *It is obvious that if the function $F : \mathbb{R}_+ \to \mathbb{R}$ is u.p.s then it is p.s. The function F considered in Example 1 shows that the converse implication is not true. Indeed, F is e.s. and by Remark 4 it is p.s.*

If we suppose that F is u.p.s then there are $\alpha > 0$ and $N \ge 1$ such that

$$y^\alpha e^{y\cos y - 3y} \le N x^\alpha e^{x\cos x - 3x}, \text{ for all } (y,x) \in \Delta_1.$$

Then for $x = 2n\pi - \pi/2$, $y = 2n\pi$ and $n \to \infty$ we obtain a contradiction.

Remark 6. *The preceding considerations prove that between the asymptotic behaviors defined in this paper we have the following implications:*

$$\begin{array}{ccccc} e.g & \Leftarrow & u.e.s & \Rightarrow & e.s \\ \Downarrow & & \Downarrow & & \Downarrow \\ p.g & \Leftarrow & u.p.s & \Rightarrow & p.s \end{array}$$

Definition 4. *A function $F : \mathbb{R}_+ \to \mathbb{R}$ with the property that there are $M \ge 1$ and $p > 0$ such that*

$$x^p|F(y)| \le M y^p |F(x)|, \text{ for all } (y,x) \in \Delta_1$$

is called with polynomial growth *(and denote p.g).*

A necessary and sufficient condition for polynomial stability is given by

Proposition 5. *Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a locally integrable function on $\mathbb{R}_+$ with polynomial growth. Then F is polynomially stable with $\alpha > 1$ if and only if there exists $\beta > 0$ and $D : [1, \infty) \to [1, \infty)$ such that*

$$\int_x^\infty y^\beta |F(y)| \mathrm{d}y \le D(x) x^{\beta+1} |F(x)|, \text{ for all } x \ge 1.$$

Proof. *Necessity.* If F is p.s with $\alpha > 1$ then for $\beta \in (0, \alpha - 1)$ we have

$$\int_x^\infty y^\beta |F(y)| \mathrm{d}y \le N(x)|F(x)| x^\alpha \int_x^\infty y^{\beta-\alpha} \mathrm{d}y \le$$

$$\le \frac{N(x)}{\alpha - 1 - \beta} x^{\beta+1} |F(x)| \le D(x) x^{\beta+1} |F(x)|, \text{ for all } x \ge 1,$$

where

$$D(x) = 1 + \frac{N(x)}{\alpha - 1 - \beta}.$$

Sufficiency. If $(y, x) \in \Delta_1$ and $y \in [x, 2x)$ then

$$y^{\beta+1}|F(y)| \leq My^{\beta+1}\left(\frac{y}{x}\right)^{p}|F(x)| \leq Mx^{\beta+1}\left(\frac{y}{x}\right)^{p+\beta+1}|F(x)| \leq$$
$$\leq M2^{p+\beta+1}x^{\beta+1}|F(x)|,$$

where M and p are given by Definition 4.

If $(y, x) \in \Delta_1$ with $y \geq 2x$ and

$$C = \int_{1}^{2} \frac{\mathrm{d}z}{z^{p+\beta+2}} = \frac{1}{y^{p+\beta+1}} \int_{\frac{y}{2}}^{y} t^{p+\beta}\mathrm{d}t,$$

then

$$Cy^{\beta+1}|F(y)| = \frac{1}{y^p} \int_{\frac{y}{2}}^{y} t^{p+\beta}|F(y)|\mathrm{d}t \leq$$

$$\leq M \int_{x}^{\infty} t^{\beta}|F(t)|\mathrm{d}t \leq MD(x)x^{\beta+1}|F(x)|.$$

Finally, we obtain

$$y^{\alpha}|F(y)| \leq N(x)x^{\alpha}|F(x)|, \text{ for all } (y, x) \in \Delta_1,$$

where $\alpha = \beta + 1 > 1$ and

$$N(x) = M(2^{p+\beta+1} + D(x)) \geq 1,$$

which shows that F is p.s. with $\alpha > 1$. □

From the proof of the preceding result we obtain

Corollary 2. *Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a locally integrable function on $\mathbb{R}_+$ with polynomial growth. Then F is uniformly polynomially stable with $\alpha > 1$ if and only if there are $D \geq 1$ and $\beta > 0$ such that*

$$\int_{x}^{\infty} y^{\beta}|F(y)|\mathrm{d}y \leq Dx^{\beta+1}|F(x)|, \textit{ for all } x \geq 1.$$

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References

[1] M. Megan, T. Ceauşu, A. Minda, *On Barreira Valls stabilities of evolution operators in Banach spaces*, Seminar of Mathematical Analysis and Applications in Control Theory, West University of Timişoara, nr. 18 (2009), 1 – 15.

[2] M. Megan, C. Stoica, *Concepts of dichotomy for skew-evolution semiflows in Banach spaces*, Annals of the Academy of Romanian Scientists, Series on Mathematics and its Applications, vol. 2, nr. 2 (2010), 125 – 140.

[3] M. Megan, T. Ceauşu, M.L. Rămneanţu, *Polynomial stability of evolution operators in Banach spaces*, Opuscula Mathematica vol. 31, nr. 2 (2011), 283 – 292.

[4] A.A. Minda, M. Megan, *On $(h,k)-$stability of evolution operators on Banach spaces*, Applied Mathematics Letters 72 (2010), 1305 – 1313.

[5] C. Stoica, M. Megan, *On nonuniform exponential stability for skew-evolution operators in Banach spaces*, to appear in Carpathian J. Math.

Some inequalities about certain arithmetic functions which use the e-divisors and the e-unitary divisors

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Abstract. The purpose of this paper is to present several inequalities about the arithmetic functions $\sigma^{(e)}$, $\tau^{(e)}$, $\sigma^{(e)*}$, $\tau^{(e)*}$ and other well-known arithmetic functions. Among these, we have the following: $\frac{\tau(n)}{\tau^*(n)} \geq \frac{\tau^{(e)}(n)}{\tau^{(e)*}(n)}$, $\frac{\sigma(n)}{\sigma^*(n)} \geq \frac{\sigma^{(e)}(n)}{\sigma^{(e)*}(n)}$, $\tau(n)+1 \geq \tau^{(e)}(n)+\tau^*(n)$ and $\sigma(n)+n \geq \sigma^{(e)}(n)+\sigma^*(n)$, for any $n \geq 1$, where $\tau(n)$ is the number of natural divisors of n, $\tau^*(n)$ is the number of natural divisors of n, $\sigma(n)$ is the sum of the divisors of n, $\tau^*(n)$ is the number of unitary divisors of n, $\sigma^*(n)$ is the sum of the unitary divisors of n and γ is the „core" of n.

Keywords: arithmetic function, exponential divisor, exponential unitary divisor.

MSC : 11A25

1. Introduction

First we have to mention that the notion of „exponential divisor" was introduced by *M. V. Subbarao* in [9], in the following way: if $n>1$ is an integer of canonical form $n=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$, then the integer $d=\prod_{i=1}^{r}p_i^{b_i}$ is called an *exponential divisor* (or *e-divisors*) of $n=\prod_{i=1}^{r}p_i^{a_i}>1$, if $b_i|a_i$ for every $i=\overline{1,r}$. We note $d|_{(e)}n$. Let $\sigma^{(e)}(n)$ denote the sum of the exponential divisors of n and $\tau^{(e)}(n)$ denote the number of exponential divisors of n.

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For example, if $n = 2^4 3^2$, then the exponential divisors of n are the following:

$$2 \cdot 3,\ 2 \cdot 3^2,\ 2^2 \cdot 3,\ 2^2 \cdot 3^2,\ 2^4 \cdot 3 \ \text{ and } \ 2^4 \cdot 3^2.$$

For various properties of the arithmetic functions which use the e-divisors see the monograph of *J. Sándor* and *B. Crstici* [5].

J. Fabrykowski and *M. V. Subbarao* in [1] study the maximal order and the average order of the multiplicative function $\sigma^{(e)}(n)$. *E.G. Straus* and *M. V. Subbarao* in [8] obtained several results concerning e-perfect numbers (n is an e-perfect number if $\sigma^{(e)}(n) = 2n$). They conjecture that there is only a finite number of e-perfect numbers not divisible by any given prime p.

In [4], *J. Sándor* showed that, if n is a perfect square, then

$$2^{\omega(n)} \leq \tau^{(e)}(n) \leq 2^{\Omega(n)}, \tag{1}$$

where $\omega(n)$ and $\Omega(n)$ denote the number of distinct prime factors of n, and the total number of prime factors of n, respectively. It is easy to see that, for $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} > 1$, we have $\omega(n) = r$ and $\Omega(n) = a_1 + a_2 + ... + a_r$.

In [6], *J. Sándor* and *L. Tóth* proved the inequality

$$\frac{n^k + 1}{2} \geq \frac{\sigma_k^*(n)}{\tau^*(n)} \geq \sqrt{n^k}, \tag{2}$$

for all $n \geq 1$ and $k \geq 0$, where $\tau^*(n)$ is the number of the unitary divisors of n, $\sigma_k^*(n)$ is the sum of kth powers of the unitary divisors of n.

In [11] *L. Tóth* and *N. Minculete* presented the notion of „exponential unitary divisors“ or „e-unitary divisors“. The integer $d = \prod_{i=1}^{r} p_i^{b_i}$ is called an *e-unitary divisor* of $n = \prod_{i=1}^{r} p_i^{a_i} > 1$ if b_i is a unitary divisor of a_i, so $\left(b_i, \frac{a_i}{b_i}\right) = 1$, for every $i = \overline{1, r}$. Let $\sigma^{(e)*}(n)$ denote the sum of the e-unitary divisors of n, and $\tau^{(e)*}(n)$ denote the number of the e-unitary divisors of n. For example, if $n = 2^4 3^2$, then the exponential unitary divisors of n are the following:

$$2 \cdot 3,\ 2 \cdot 3^2,\ 2^4 \cdot 3 \text{ and } 2^4 \cdot 3^2.$$

By convention, 1 is an exponential divisor of itself, so that

$$\sigma^{(e)*}(1) = \tau^{(e)*}(1) = 1.$$

We notice that 1 is not an e-unitary divisor of $n > 1$, the smallest e-unitary divisor of $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} > 1$ is $p_1 p_2 \cdots p_r = \gamma(n)$ is called the „core“ of n.

In [3], it is show that

$$\sigma^{(e)}(n) \leq \psi(n) \leq \sigma(n), \tag{3}$$

where $\psi(n)$ is the function of Dedekind, and

$$\tau(n) \leq \frac{\sigma^{(e)}(n)}{\tau^{(e)}(n)}, \tag{4}$$

for all integers $n \geq 1$.

Other properties of the sum of the exponential divisors of n and of the number of the exponential divisors of n can be found in the papers [2, 6 and 10].

2. Inequalities for the functions τ^{e}, σ^{e}, $\tau^{\mathrm{e}*}$ and $\sigma^{\mathrm{e}*}$

Lemma 1. *For every $n \geq 1$, there is the following inequality*

$$n\tau^*(n) \geq \sigma(n), \tag{5}$$

and for all $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} > 1$, with $a_i \geq 2$, $(\forall)$ $i = \overline{1,r}$, there is the following inequality

$$n\tau^{\mathrm{e}}(n) \geq \sigma(n). \tag{6}$$

Proof. For $a \geq 1$ we show that

$$2p^a \geq p^a + p^{a-1} + \ldots + p + 1.$$

This inequality is rewritten as

$$p^a \geq p^{a-1} + \ldots + p + 1,$$

which by multiplication with $p-1$ becomes

$$p^{a+1} + 1 \geq 2p^a,$$

which is true, because $p^{a+1} \geq 2p^a$, for every prime number p and for all $a \geq 1$.

Therefore, $p^a\tau^*(p^a) = 2p^a \geq p^a + p^{a-1} + \ldots + p + 1 = \sigma(p^a)$, so $p^a\tau^*(p^a) \geq$ $\geq \sigma(p^a)$. Because the arithmetic functions τ^* and σ are multiplicative, we deduce the inequality

$$n\tau^*(n) \geq \sigma(n).$$

Since $a \geq 2$, it follows that $\tau(a) \geq 2$, so

$$p^a\tau(a) \geq 2p^a \geq p^a + \ldots + p + 1 = \sigma(p^a),$$

which is equivalent to $p^a\tau^{\mathrm{e}}(p^a) = p^a\tau(a) \geq \sigma(p^a)$. As the arithmetic functions τ^{e} and σ are multiplicative, we get the inequality of the statement. □

Remark 1. *It may be noted that the lemma readily implies the inequality*

$$n^2\tau^{\mathrm{e}}(n^2) \geq \sigma(n^2), \tag{7}$$

for all $n \geq 1$.

Theorem 2. *For every $n \geq 1$, there is the inequality*

$$\frac{\tau(n)}{\tau^*(n)} \geq \frac{\tau^{\mathrm{e}}(n)}{\tau^{\mathrm{e}*}(n)}. \tag{8}$$

Proof. For $n=1$, we have $\dfrac{\tau(1)}{\tau^*(1)}=1=\dfrac{\tau^{\mathrm{e}}(1)}{\tau^{\mathrm{e}*}(1)}$.
Let's consider $n=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$, with $a_i\geq 1$, $(\forall)\ i=\overline{1,r}$. According to lemma 1, we deduce the relation

$$a_i\tau^*(a_i)\geq\sigma(a_i),\ (\forall)\ i=\overline{1,r}. \tag{9}$$

From the inequality of *S. Sivaramakrishnan* and *C. S. Venkataraman* [7], $\sigma(n)\geq\sqrt{n}\tau(n)$, $(\forall)\ n\geq 1$, we get $\sigma(a_i)\geq\sqrt{a_i}\tau(a_i)$, $(\forall)\ i=\overline{1,r}$. This last inequality combined with inequality (9) implies the inequality

$$\sqrt{a_i}\ \geq\frac{\tau(a_i)}{\tau^*(a_i)},\ (\forall)\ i=\overline{1,r}, \tag{10}$$

which means that

$$\sqrt{\prod_{i=1}^{r}a_i}\geq\frac{\tau^{\mathrm{e}}(n)}{\tau^{\mathrm{e}*}(n)},\ \text{for all } n=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}>1,\ \text{because} \tag{11}$$

$\tau^{\mathrm{e}}(n)=\tau(a_1)\cdot\ldots\cdot\tau(a_r)$ and $\tau^{\mathrm{e}*}(n)=\tau^*(a_1)\cdot\ldots\cdot\tau^*(a_r)$.

But $a_i+1\geq 2\sqrt{a_i}$, $(\forall)\ i=\overline{1,r}$, and by taking the product, we obtain the inequality $\displaystyle\prod_{i=1}^{r}(a_i+1)\geq 2^r\sqrt{\prod_{i=1}^{r}a_i}$, which is equivalent to the relation $\tau(n)\geq\tau^*(n)\sqrt{\displaystyle\prod_{i=1}^{r}a_i}$, so

$$\frac{\tau(n)}{\tau^*(n)}\geq\sqrt{\prod_{i=1}^{r}a_i}\ \text{for all } n=p_1^{a_1}p_2^{a_2}...p_r^{a_r}>1. \tag{12}$$

Combining relations (11) and (12), we deduce inequality (8). Finally, the proof is completed. □

Theorem 3. *For every $n\geq 1$, there is the equality*

$$\frac{\sigma(n)}{\sigma^*(n)}\geq\frac{\sigma^{\mathrm{e}}(n)}{\sigma^{\mathrm{e}*}(n)}. \tag{13}$$

Proof. We distinguish the following cases:
Case I. For $n=1$, we have $\dfrac{\sigma(1)}{\sigma^*(1)}=1=\dfrac{\sigma^{\mathrm{e}}(1)}{\sigma^{\mathrm{e}*}(1)}$.
Case II. If n is squarefree, then $\sigma(n)=\sigma^*(n)$, and $\sigma^{\mathrm{e}}(n)=n=\sigma^{\mathrm{e}*}(n)$. Therefore, we obtain the relation of statement.
Case III. Let's consider $n=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}>1$, with $a_i\geq 2$, $(\forall)\ i=\overline{1,r}$, then

$$\frac{\sigma(p^a)}{\sigma^*(p^a)}=\frac{1+p+p^2+...+p^a}{1+p^a}=1+\frac{p+p^2+...+p^{a-1}}{1+p^a} \tag{14}$$

and, because the exponential unitary divisors $p^{d_1}, ..., p^{d_q}$ of p^a are among the exponential divisors $p^{d_1}, ..., p^{d_s}$ of p^a, we have the inequality

$$\frac{\sigma^{\mathrm{e}}(p^a)}{\sigma^{\mathrm{e}*}(p^a)} = \frac{p^{d_1} + p^{d_2} + ... + p^{d_s}}{p^{d_1} + ... + p^{d_q}} = 1 + \frac{p^{d_2} + ... + p^{d_{s-1}}}{p + ... + p^a} \leq$$

$$\leq 1 + \frac{p^2 + p^3 + ... + p^{a-1}}{p + p^a} \leq 1 + \frac{p + ... + p^{a-1}}{1 + p^a}.$$

Therefore, using the inequality (14), we get the relation

$$\frac{\sigma^{\mathrm{e}}(p^a)}{\sigma^{\mathrm{e}*}(p^a)} \leq \frac{\sigma(p^a)}{\sigma^*(p^a)}.$$

Because the arithmetic functions σ^{e}, $\sigma^{\mathrm{e}*}$, σ and σ^* are multiplicative, we deduce the inequality

$$\frac{\sigma(n)}{\sigma^*(n)} \geq \frac{\sigma^{\mathrm{e}}(n)}{\sigma^{\mathrm{e}*}(n)}, \text{ for all } \quad n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} > 1, \text{ with } a_i \geq 2, \ (\forall)\ i = \overline{1, r}.$$

Case IV. Let's consider $n = n_1 n_2$, where n_1 is squarefree, and $n_2 = \prod_{\substack{p \mid n_2 \\ a \geq 2}} p^a$.

Since $(n_1, n_2) = 1$, we deduce the following relation

$$\frac{\sigma(n)}{\sigma^*(n)} = \frac{\sigma(n_1)\sigma(n_2)}{\sigma^*(n_1)\sigma^*(n_2)} \geq \frac{\sigma^{\mathrm{e}}(n_1)\sigma^{\mathrm{e}}(n_2)}{\sigma^{\mathrm{e}*}(n_1)\sigma^{\mathrm{e}*}(n_2)} = \frac{\sigma^{\mathrm{e}}(n)}{\sigma^{\mathrm{e}*}(n)}.$$

Thus, the demonstration is complete. □

Remark 2. *In Theorem* 2 *and Theorem* 3 *the equality in relations* (8) *and* (13) *holds when* $n = 1$ *or* n *is squarefree.*

Theorem 4. *For any* $n \geq 1$ *the following inequality*

$$\tau(n) + 1 \geq \tau^{\mathrm{e}}(n) + \tau^*(n) \tag{15}$$

holds.

Proof. If $n = 1$, then we obtain $\tau(1) + 1 = 2 = \tau^{\mathrm{e}}(1) + \tau^*(1)$.
We consider $n > 1$. To prove the above inequality, will have to study several cases, namely:
Case I. If $n = p_1^2 p_2^2 ... p_r^2$, then $\tau(n) = 3^r$ and

$$\tau^{\mathrm{e}}(n) = \tau(a_1) \cdot \tau(a_2) \cdot ... \cdot \tau(a_r) = \tau^r(2) = 2^r = \tau^*(n),$$

which means that inequality (15) is equivalent to the inequality $3^r + 1 \geq$ $\geq 2 \cdot 2^r$, which is true because, by using *Jensen*'s inequality, we have

$$\frac{3^r + 1}{2} \geq \left(\frac{3 + 1}{2}\right)^r = 2^r.$$

Case II. If $a_k \neq 2$, $(\forall)\ k = \overline{1, r}$, then the numbers

$$\frac{n}{p_1}, \frac{n}{p_2}, ..., \frac{n}{p_r}, \frac{n}{p_1 p_2}, ..., \frac{n}{p_i p_j}, ..., \frac{n}{p_i p_j p_k}, ..., \frac{n}{p_1 p_2 \cdots p_r}$$

are not exponential divisors of n, so they are in a total number of $2^r - 1$, such that we have the inequality

$$\tau(n) = \sum_{d|_e n} 1 + \sum_{d\nmid_e n} 1 = \tau^e(n) + \sum_{d\nmid_e n} 1 \geq \tau^e(n) + 2^r - 1.$$

Therefore, we have

$$\tau(n) \geq \tau^e(n) + 2^r - 1, \text{ so } \tau(n) + 1 \geq \tau^e(n) + \tau^*(n).$$

Case III. If there is at least one $a_j = 2$ and at least one $a_k \neq 2$, where $j, k \in \{1, 2, ..., r\}$, then without loss of the generality we renumber the prime factors from the factorization of n and we obtain

$$n = p_1^2 p_2^2 \cdots p_s^2 p_{s+1}^{a_{s+1}} \cdots p_r^{a_r}, \text{ with } a_{s+1},\ a_{s+2}, ..., a_r \neq 2.$$

Therefore, we write $n = n_1 \cdot n_2$, where $n_1 = p_1^2 p_2^2 \cdots p_s^2$ and $n_2 = p_{s+1}^{a_{s+1}} \cdots p_r^{a_r}$, which means that $(n_1, n_2) = 1$, and

$$\tau(n) = \tau(n_1 \cdot n_2) = \tau(n_1) \cdot \tau(n_2) \geq (\tau^e(n_1) + \tau^*(n_1) - 1)(\tau^e(n_2) + \tau^*(n_2) - 1) =$$
$$= (\tau^e(n_1) + 2^s - 1)(\tau^e(n_2) + 2^{r-s} - 1) =$$
$$= \tau^e(n_1)\tau^e(n_2) + \tau^e(n_1)(2^{r-s} - 1) + \tau^e(n_2)(2^s - 1) + (2^s - 1)(2^{r-s} - 1) \geq$$
$$\geq \tau^e(n) + 2^{r-s} - 1 + 2^s - 1 + (2^s - 1)(2^{r-s} - 1) \geq \tau^e(n) + 2^r - 1 = \tau^e(n) + \tau^*(n) - 1.$$

Thus, the inequality of the statement is true. □

Lemma 5. *For any $x_i > 0$ with $i \in \{1, 2, ..., n\}$, there is the following inequality:*

$$\prod_{i=1}^{n}(1 + x_i + x_i^2) + \prod_{i=1}^{n} x_i^2 \geq \prod_{i=1}^{n}(x_i + x_i^2) + \prod_{i=1}^{n}(1 + x_i^2). \tag{16}$$

Proof. We apply the principle of mathematical induction. □

Theorem 6. *For any $n \geq 1$, the following inequality:*

$$\sigma(n) + n \geq \sigma^e(n) + \sigma^*(n) \tag{17}$$

holds.

Proof. If $n = 1$, then we obtain $\sigma(1) + 1 = 2 = \sigma^e(1) + \sigma^*(1)$. Let's consider $n > 1$. To prove the above inequality, we will have to study several cases namely:

Case I. If $n = p_1^2 p_2^2 \cdots p_r^2$, then we deduce the equalities $\sigma(n) = \prod_{i=1}^{r}(1 + p_i + p_i^2)$, $\sigma^e(n) = \prod_{i=1}^{r}(p_i + p_i^2)$ and $\sigma^*(n) = \prod_{i=1}^{r}(1 + p_i^2)$, which means that inequality (17) implies the inequality

$$\prod_{i=1}^{r}(1 + p_i + p_i^2) + \prod_{i=1}^{r} p_i^2 \geq \prod_{i=1}^{r}(p_i + p_i^2) + \prod_{i=1}^{r}(1 + p_i^2),$$

which is true, because we use inequality (16), for $n = r$ and $x_i = p_i$.
Case II. If $a_k \neq 2,\ (\forall)\ k = \overline{1, r}$, then the numbers

$$\frac{n}{p_1}, \frac{n}{p_2}, ..., \frac{n}{p_r}, \frac{n}{p_1p_2}, ..., \frac{n}{p_ip_j}, ..., \frac{n}{p_ip_jp_k}, ..., \frac{n}{p_1p_2\cdots p_r}$$

are not exponential divisors of n, so they are in a total number of $2^r - 1$, and their sum is $\psi(n) - n$, so that as we have the inequality

$$\sigma(n) = \sum_{d|_e n} d + \sum_{d\nmid_e n} d = \sigma^e(n) + \sum_{d\nmid_e n} d \geq \sigma^e(n) + \psi(n) - n.$$

Since we have the inequality

$$\psi(n) = n\prod_{i=1}^{r}\left(1 + \frac{1}{p_i}\right) \geq n\prod_{i=1}^{r}\left(1 + \frac{1}{p_i^{a_i}}\right) = \sigma^*(n),$$

it follows that

$$\sigma(n) \geq \sigma^e(n) + \sigma^*(n) - n, \text{ so } \sigma(n) + n \geq \sigma^e(n) + \sigma^*(n).$$

Case III. If there is at least one $a_k \neq 2$, and at last one $a_j = 2$, where $j, k \in \{1, 2, ..., r\}$, then without loss of generality, we renumber the prime factors from the factorization of n and we obtain

$$n = p_1^2p_2^2\cdots p_s^2p_{s+1}^2\cdots p_r^{a_r}, \text{ with } a_{s+1}, a_{s+2}, ..., a_r \neq 2.$$

Hence, we will write $n = n_1 \cdot n_2$, where $n_1 = p_1^2p_2^2\cdots p_s^2$ and $n_2 = p_{s+1}^{a_{s+1}}\cdots p_r^{a_r}$, which means that $(n_1, n_2) = 1$, and by simple calculations, it is easy to see that

$$\begin{aligned}
\sigma(n) &= \sigma(n_1 \cdot n_2) = \sigma(n_1)\cdot\sigma(n_2) \geq (\sigma^e(n_1) + \sigma^*(n_1) - n_1)(\sigma^e(n_2) + \sigma^*(n_2) - n_2) = \\
&= \sigma^e(n_1)\sigma^e(n_2) + \sigma^e(n_1)(\sigma^*(n_2) - n_2) + \sigma^*(n_1)(\sigma^e(n_2) - n_2) + \sigma^*(n_1)\cdot\sigma^*(n_2) - \\
&- n_1(\sigma^e(n_2) + \sigma^*(n_2)) + n_1n_2 \geq \sigma^e(n) + n_1(\sigma^*(n_2) - n_2) + n_1(\sigma^e(n_2) - n_2) + \\
&+ \sigma^*(n) - n_1\sigma^e(n_2) - n_1\sigma^*(n_2) + n_1n_2 \geq \sigma^e(n) + \sigma^*(n) - n.
\end{aligned}$$

Thus, the demonstration is complete. □

Remark 3. *In Theorem* 4 *and Theorem* 6 *the equality in relations* (15) *and* (17) *hold, when* $n = 1$ *or* n *is squarefree.*

References

[1] J. Fabrykowski and M. V. Subbarao, *The maximal order and the average order of multiplicative function* $\sigma^e(n)$, Théorie des Nombres (Québec, PQ, 1987), 201 – 206, de Gruyter, Berlin-New York, 1989.

[2] N. Minculete, *Several inequalities about arithmetic functions which use the e-divisors*, Proceedings of The Fifth International Symposium on Mathematical Inequalities-MATINEQ 2008", Sibiu.

[3] N. Minculete, *On certain inequalities about arithmetic functions which use the exponential divisors* (to appear).

[4] J. Sándor, *On exponentially harmonic numbers*, Scientia Magna, Vol. **2** (2006), No. 3, 44 – 47.

[5] J. Sándor and B. Crstici, *Handbook of Number Theory II*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2004.
[6] J. Sándor and L. Tóth , *On certain number-theoretic inequalities*, Fibonacci Quart. 28 (1990), 255 – 258.
[7] R. Sivaramakrishnan and C. S. Venkataraman, *Problem 5326*, Amer. Math. Monthly, **72** (1965), 915.
[8] E. G. Straus and M. V. Subbarao, *On exponential divisors*, Duke Math. J. **41** (1974), 465 – 471.
[9] M. V. Subbarao, *On some arithmetic convolutions in The Theory of Arithmetic Functions, Lecture Notes in Mathematics*, New York, Springer-Verlag, 1972.
[10] L. Tóth, *On Certain Arithmetic Functions Involving Exponential Divisors*, Annales Univ. Sci. Budapest., Sect. Comp. **24** (2004), 285 – 294.
[11] L. Tóth and N. Minculete, *Exponential unitary divisors* (to appear in Annales Univ. Sci. Budapest., Sect. Comp.).

Harmonic quadrilaterals revisited

Cosmin Pohoaţă[1)]

Abstract. With a plethora of instruments ranging from harmonic divisions to various geometric transformations, synthetic projective geometry has become very popular in olympiad geometry problems nowadays. In this note we expand a bit the literature concerned with applications of these techniques by discussing a topic of great importance (or better said, spectacularity) which might not be that covered in usual Euclidean geometry textbooks: harmonic quadrilaterals.

Keywords: harmonic, harmonic division, harmonic quadrilateral, inversion, symmedian.

MSC : 51A05, 51A20, 51A45.

1. Introduction

The name of **harmonic quadrilateral** dates from the middle of the nineteenth century, belonging to the famous mathematician R. Tucker [1]. He gives a slightly different definition from the ones popular these days:

Let A, B, C, D, M be five points in plane such that A, B, C, D are concyclic and M is the midpoint of the segment BD. Denote $BD = 2m$, i.e. $MB = MD = m$. Call vectorial inversion and denote (M, m^2, BD) the transformation which firstly inverts with respect to the pole M with constant m^2, and secondly takes the image of the inversed object in BD. Then, the quadrilateral $ABCD$ is **harmonic** *if and only if each of A, C could be obtained of the other after a vectorial inversion (M, m^2, BD).*

Remark. In the above statement it isn't necessary to firstly point out the concyclity of A, B, C, D. In fact, if we perform a vectorial inversion

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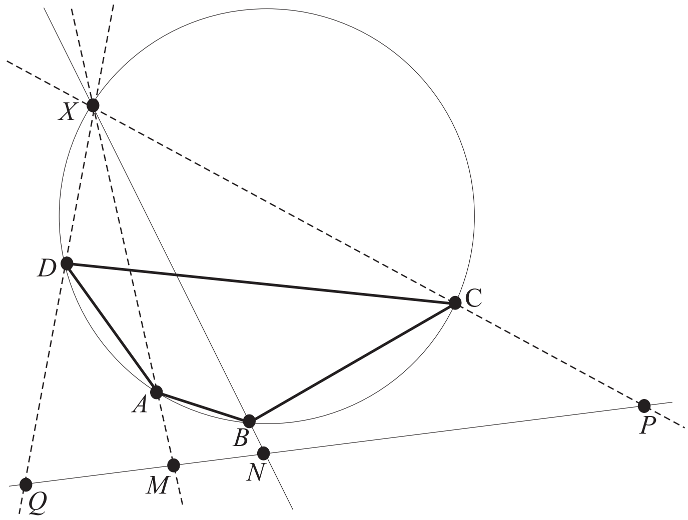
(M, m^2, BD) of a point T in plane, then its image under the transformation, T^*, lies on the circumcircle of $\triangle TBD$.

In the following, we will firstly present the most used characterization these days, a more synthetic view:

Consider $ABCD$ a cyclic quadrilateral inscribed in $\mathcal{C}(O)$ and X a point on $\mathcal{C}$. Let an arbitrary line d intersect the lines XA, XB, XC, XD at M, N, P and Q, respectively. If $\dfrac{NM}{NP} = \dfrac{QM}{QP}$, then the quadrilateral $ABCD$ is called **harmonic**. In addition, $ABCD$ is a harmonic quadrilateral if and only if $AB \cdot CD = BC \cdot DA$.



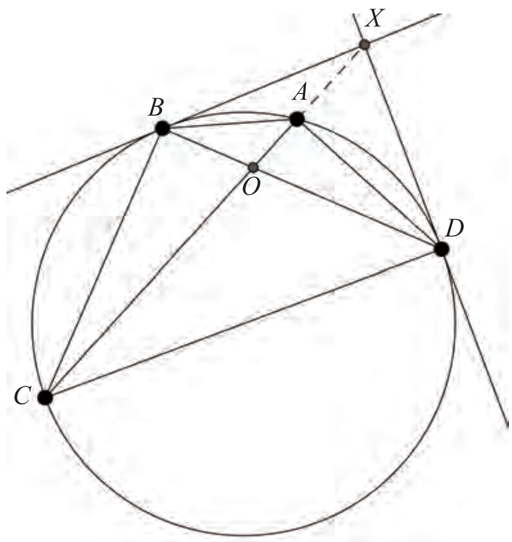
Proof. Denote by T the intersection of AC and BD. Since the pencil $X(A, B, C, D)$ is harmonic, it is clear that the pencil $A(A, B, C, D)$ is harmonic. Hence, if by S we note the intersection of BD with the tangent in A w.r.t. $\mathcal{C}$, then the quadruple $(SDTB)$ is harmonic.

Similarly, the division $(S'TDB)$ is harmonic, where by S' we denoted the intersection of BD with the tangent in C to $\mathcal{C}$. Therefore, $S \equiv S'$, i.e. BD, the tangent in A, respectively the tangent in C w.r.t. $\mathcal{C}$ are concurrent. But since $\triangle SAD = \triangle SBA$, if follows that $SA/SB = AD/BA$.

Likewise, $\triangle SCD = \triangle SBC$, i.e. $SD/SC = CD/BC$. Hence, $AB \cdot CD = BC \cdot DA$. $\square$

2. **Another classical property**

Consider $ABCD$ a cyclic quadrilateral inscribed in $\mathcal{C}$, having O as the intersection of its diagonals. Then, BO is the B-symmedian in $\triangle ABC$ **and** DO is the D-symmedian in $\triangle ADC$ if and only if $ABCD$ is **harmonic**.



Proof. As above, one can notice that a cyclic quadrilateral is harmonic if and only the tangents in two opposite points w.r.t. the circumscribed circle concur on the diagonal determined by the other two points, i.e. BB, DD and AC are concurrent in a point X (where by BB we denote the tangent in B to the circumcircle of ABC).

On the other hand, it is well-known that in a triangle MNP, having M_1 as the foot of the M-symmedian and M_2 as the intersection of the tangent in M to its circumcircle with the side BC, the quadruple (M_2, N, M_1, P) is a harmonic division. Hence, BO is the B-symmedian in $\triangle ABC$ and DO is the D-symmedian in $\triangle ADC$ if and only if BB, DD and AC are concurrent. □

Remark. In the initial statement it isn't really necessary to stress from the beginning that $ABCD$ is cyclic. One can give the following enuntiation:

Consider $ABCD$ a convex quadrilateral inscribed in $\mathcal{C}$, having O as the intersection of its diagonals. Then, BO is the B-symmedian in $\triangle ABC$ and DO is the D-symmedian in $\triangle ADC$ if and only if $ABCD$ is harmonic.

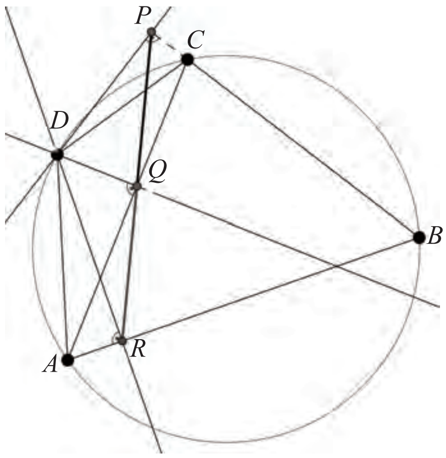
As probably yet noticed, the condition of BO, DO beeing the symmedians in $\triangle ABC$, respectively $\triangle ADC$, together with the harmonicity of (X, A, O, C), imply the coincidence of the polars of X w.r.t. $\triangle ABC$ and $\triangle ADC$ (the common polar beeing the line BD). Hence, the two circles coincide as well.

Corollary 1. Consider ABC a triangle and T the intersection of the tangents at B and C w.r.t. the circumcircle of ABC. Then AT is the A-symmedian.

Corollary 2. In the initially described configuration, AO, CO are the A, respectively C-symmedians in $\triangle BAD$, $\triangle BCD$.

3. An IMO-type alternative definition

Consider ABC a triangle and D a point on its circumcircle. Draw the Simson line of D w.r.t. $\triangle ABC$. This line cuts its sides BC, CA, AB in P, Q and R, respectively. Then, the quadrilateral $ABCD$ is **harmonic** if and only if $PQ = QR$.



The above statement is slightly modifying the original one from IMO 2003, problem 4:

Let $ABCD$ be a cyclic quadrilateral. Let P, Q and R be the feet of the perpendiculars from D to the lines BC, CA and AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\sphericalangle ABC$ and $\sphericalangle ADC$ meet on AC.

First proof. As we first stated, it is well-known that P, Q, R are collinear on the Simson line of D. Moreover, since $\sphericalangle DPC$ and $\sphericalangle DQC$ are right angles, the points D, P, Q, C are concyclic and so $\sphericalangle DCA = \sphericalangle DPQ = \sphericalangle DPR$. Similarly, since D, Q, R, A are concyclic, we have $\sphericalangle DAC = \sphericalangle DRP$. Therefore, $\triangle DCA \sim \triangle DPR$.

Likewise, $\triangle DAB \sim \triangle DQP$ and $\triangle DBC \sim \triangle DRQ$. Then

$$\frac{DA}{DC} = \frac{DR}{DP} = \frac{DB \cdot \dfrac{QR}{BC}}{DB \cdot \dfrac{PQ}{BA}} = \frac{QR}{PQ} \cdot \frac{BA}{BC}.$$

Thus $PQ = QR$ if and only if $DA/DC = BA/BC$, whence by the converse of the bisector theorem, we deduce that it is equivalent with the concurrence of the bisectors of $\sphericalangle ABC$ and $\sphericalangle ADC$ on AC. $\square$

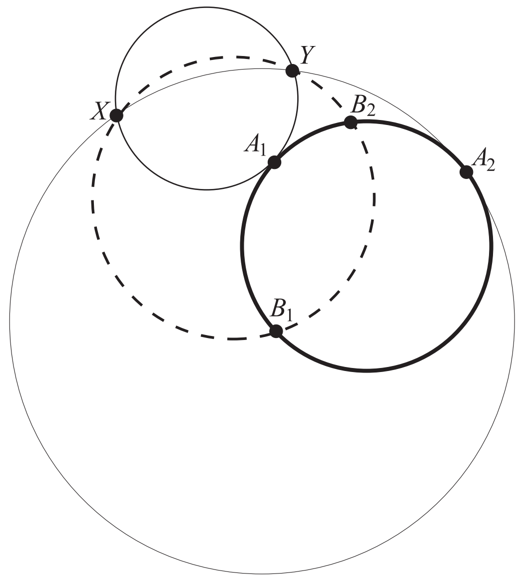
Second proof. Because $DP \perp BC$, $DQ \perp AC$, $DR \perp AB$, the circles with diameters DC and DA contain the pairs of points P, Q and Q, R, respectively. It follows that $\sphericalangle PDQ$ is equal to γ or $180° - \gamma$, where $\gamma = \sphericalangle ACB$.

Similarly, $\sphericalangle QDR$ is equal to α or $180° - \alpha$, where $\alpha = \sphericalangle CAB$. Then, by the law of sines, we have $PQ = CD \sin\gamma$ and $QR = AD \sin\alpha$. Hence the condition $PQ = QR$ is equivalent with $CD/AD = \sin\alpha/\sin\gamma$.

On the other hand, $\sin\alpha/\sin\gamma = CB/AB$ by the law of sines again,. Thus $PQ = QR$ if and only if $CD/AD = CB/AB$, i.e. $AB \cdot CD = CB \cdot AD$. □

4. **A probably new non-standard characterization**

Let $\mathcal{C}(\mathcal{O})$ be a given circle and X, Y two points in its plane. It is well-known that there exist exactly two circles, ρ_1 and ρ_2, passing through X, Y tangent to the initial circle $\mathcal{C}$. Consider A_1 and A_2 the mentioned tangency points. Draw a third mobile circle, ρ_3, through X, Y touching the reference circle $\mathcal{C}$ at B_1 and B_2, respectively. Then, the quadrilateral $A_1B_1A_2B_2$ is **harmonic**.

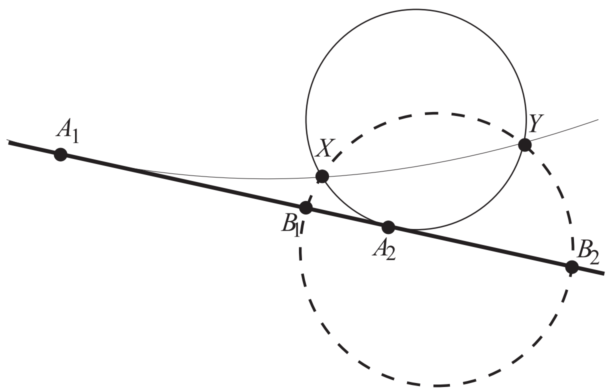


Proof. For example, perform an inversion with pole X. Hence, $\mathcal{C}$ is mapped into an other circle $\mathcal{C}^*$ and ρ_1, ρ_2 into the lines ρ_1^* and ρ_2^*, respectively, not passing through X, representing the tangents from Y^* to $\mathcal{C}^*$, i.e. A_1^*, A_2^* are the tangency points of $\mathcal{C}^*$ with ρ_1^*, ρ_2^*.

Since ρ_3 is mapped into the line ρ_3^* through Y^* cutting $\mathcal{C}^*$ at B_1^*, B_2^*, we deduce that the harmonicity of $A_1B_1A_2B_2$ is equivalent with the harmonicity of $A_1^*B_1^*A_2^*B_2^*$, which is clear, because $A_1^*A_2^*$ is the polar of Y^* w.r.t. $\mathcal{C}^*$. □

If in the above configuration we consider $\mathcal{C}$ a circle centered at infinity, we deduce the following consequence:

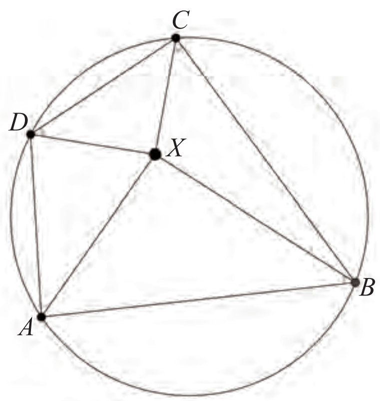
Corollary. Consider d a line and X, Y two points in its plane. Denote by A_1, A_2 the points where the two circles passing through X, Y and tangent to d, touch the respective line. Let ω be an arbitrary circle through X, Y, which touches d at B_1 and B_2. Then, the quadruple (A_1, B_1, A_2, B_2) forms a harmonic division.



Their converses are also true. Precisely, if we consider A_1, A_2 similarly as above and this time B_1, B_2 are two points on $\mathcal{C}$ such that $A_1B_1A_2B_2$ is a harmonic quadrilateral (degenerated or not), then the points X, Y, B_1, B_2 lie on a same circle.

5. **The Brocard points of a harmonic quadrilateral**

In this section we present the existence of the Brocard points of a harmonic quadrilateral, proved by *F. G. W. Brown* [2], more as an other characterization of this particular cyclic quadrilateral.



Consider $ABCD$ a cyclic quadrilateral. If there exists a point X such that the angles $\sphericalangle XAD$, $\sphericalangle XBA$, $\sphericalangle XCB$, $\sphericalangle XDC$ are equal, then quadrilateral $ABCD$ is **harmonic**.

Proof. Denote the sides BC, CD, DA, AB by a, b, c, d; the diagonals DB, AC by e, f; the area of the quadrilateral by Q; the circumradius by R and each of the angles XAD, XBA, XCB, XDC by ω. Then

$$\sphericalangle AXB = \pi - \omega - (A - \omega) = \pi - A.$$

Similarly, $\sphericalangle BXC = \pi - B$, $\sphericalangle CXD = \pi - C$, $\sphericalangle DXA = \pi - D$.

Now, since

$$\frac{AX}{\sin\omega} = \frac{d}{\sin AXB} = \frac{d}{\sin(\pi - A)} = \frac{d}{\sin A}$$

and

$$\frac{AX}{\sin(D - \omega)} = \frac{c}{\sin AXD} = \frac{c}{\sin(\pi - D)} = \frac{c}{\sin D},$$

we deduce that

$$\frac{\sin(D - \omega)}{\sin\omega} = \frac{d\sin D}{c\sin A},$$

or

$$\cot\omega = \frac{d}{c\sin A} + \cot D.$$

Similarly,

$$\cot w = \frac{c}{b\sin D} + \cot C, \tag{$*$}$$

and the cyclic analogues.

Hence,

$$\cot w = \frac{d}{c\sin A} + \cot D = \frac{c}{b\sin D} + \cot C = \frac{b}{a\sin C} + \cot B = \frac{a}{d\sin B} + \cot A$$

or

$$\frac{d}{c\sin A} - \frac{a}{d\sin B} + \frac{b}{a\sin C} - \frac{c}{b\sin D} = \cot A - \cot B + \cot C - \cot D.$$

Since $ABCD$ is cyclic, $1/\sin A = 1/\sin C$, $\cot A = -\cot C$, etc; therefore

$$\left(\frac{d}{c} + \frac{b}{a}\right)\frac{1}{\sin A} = \left(\frac{c}{b} + \frac{a}{d}\right)\frac{1}{\sin B}.$$

But

$$(ab + cd)\sin A = (ad + bc)\sin B = 2Q;$$

hence $ac = bd$, i.e. $ABCD$ is harmonic. $\square$

Its converse is also valid and can be proved on the same idea. Moreover, there is a second Brocard point X', such that $\sphericalangle X'AB = \sphericalangle X'BC =$

$= \sphericalangle X'CD = \sphericalangle X'DA = \omega'$. By simple manipulations on $(\star)$ and using *Ptolemy*'s theorem, we obtain

$$\cot\omega = \frac{8R^2Q}{e^2f^2}.$$

Similarly, we deduce that

$$\cot\omega' = \frac{8R^2Q}{e^2f^2}.$$

Hence, $\omega = \omega'$.

Further, we will compute the distance XX'. For this, let P be the foot of the perpendicular from X to BC and $CP = x$, $PX = y$. Then

$$x = CX\cos\omega = \frac{b\sin\omega\cos\omega}{\sin C} = \frac{b(ab+cd)\cot\omega}{2Q(1+\cot^2\omega)} = \frac{bR}{c}\cdot\sin 2\omega$$

and

$$y = x\tan\omega = \frac{b(ab+cd)}{2Q(1+\cot^2\omega)} = \frac{2bR}{c}\sin^2\omega.$$

Similarly, if $x' = P'B$, $y' = P'X'$, where P' is the foot of the perpendicular from X' to BC, then

$$x' = \frac{d(ad+bc)\cot\omega}{2Q(1+\cot^2\omega)}$$

and

$$y' = \frac{d(ad+bc)}{2Q(1+\cot^2\omega)}.$$

Since by the law of cosines $XX'^2 = XB^2+X'B^2-2XB\cdot X'B\cdot\cos XBX'$ and by

$$\sin B = \frac{ab+cd}{2bd}\tan\omega = \frac{a^2+d^2}{2ad}\cdot\tan\omega,$$

using that $ac = bd$, i.e. $ef = 2ac = 2bd$, it follows that

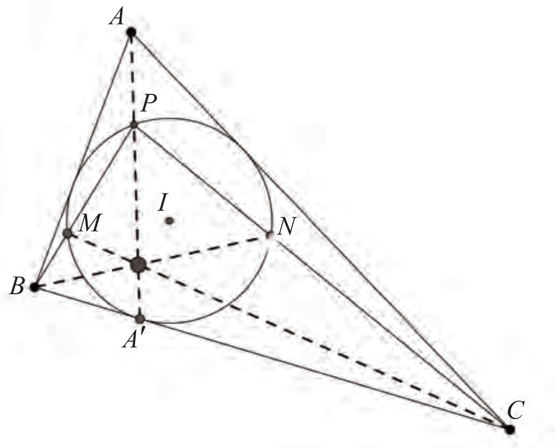
$$XX' = \frac{efbd\cos\omega\cdot\sqrt{\cos 2\omega}}{2RQ} = \frac{b^2d^2\cos\omega\cdot\sqrt{\cos 2\omega}}{RQ}.$$

6. **Remarks on an Iranian concurrence problem**

This problem firstly became popular on the Mathlinks forum for its difficulty. It is a nice result involving a concurrence of three lines in triangle.

Let ABC be a triangle. The incircle of $\triangle ABC$ touches the side BC at A', and the line AA' meets the incircle again at a point P. Let the lines CP and BP meet the incircle of triangle ABC again at N and M, respectively. Prove that the lines AA', BN and CM are concurrent.

Iran National Olympiad 2002



Proof. Consider Q the intersection of MN with BC. By Ceva and Menelaus theorem, it is clear that PA', BN and CM concur if and only if the quadruple (Q, B, A', C) is harmonic. So, the problem reduces to proving that $Q \equiv Q'$, where Q' is the intersection of the tangent in P to the incircle with BC.

Since (Q', B, D, C) is a harmonic division, we deduce that the pencil $P(Q', B, D, C)$ is harmonic and by intersecting it with the incircle, it follows that the quadrilateral $PMA'N$ is harmonic. Hence, the lines MN, the tangent in P, respectively, the tangent in A' to the incircle are concurrent, i.e. $Q \equiv Q'$. □

We leave the readers a more difficult extended result (Figure 8.):

I. *Suppose that in the configuration described above the lines AA', BN and CM are concurrent in a point X. Similarly, one can prove the excircle related problem, meaning, the lines AA'', BN_1, CM_1 are concurrent in a point X_1, where by A'' we denoted the tangency point of the A-excircle with the side BC and by N_1, M_1 the second intersections of CP', respectively BP' with the A-excircle, where P' is the second intersection of the line AA'' with the A-excircle. Prove that the lines PP', BC and XY are concurrent.*

Moreover, we can go further and observe the next concurrence, which remains as an other proposed problem for those who are interested (Figure 9.):

II. *Consider that in the above configuration the lines PP', BC and XY concur in a point X_a. Similarly, we deduce the existence of points Y_b, Z_c, with same right as X_a, on the lines CA, respectively AB. Then, the triangles ABC and $X_aY_bZ_c$ are perspective (i.e. the lines AX_a, BY_b, CZ_c concur); their perspector is X(75), i.e. the isotomic conjugate of the incenter.*

The last remark, regarding the determination of the perspector, was communicated by Darij Grinberg, using computer dynamic geometry.

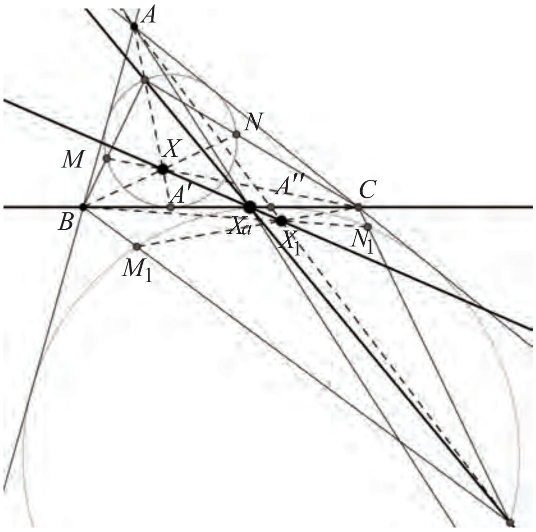


Figure 8.

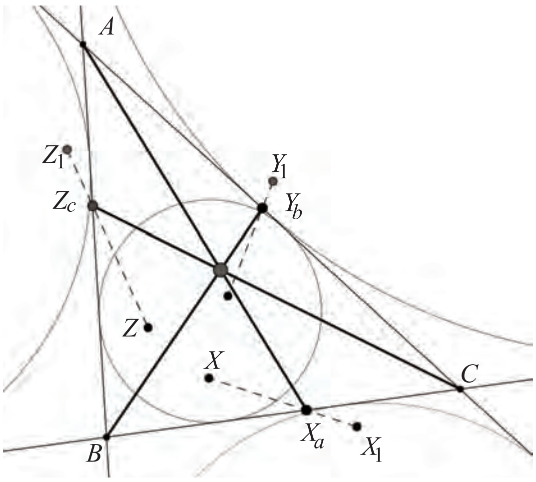


Figure 9.

In the following we leave again the readers, this time, a slightly more general statement of the original problem:

III. *Let ABC be a triangle. The incircle of $\triangle ABC$ touches the side BC at A', and* **let P be an arbitrary point on (AA')**. *Let the lines CP and BP meet the incircle of triangle ABC again at N, N' and M, M', respectively. Prove that the lines AA', BN and CM, respectively AA', BN', CM', are concurrent.* (i.e. the lines BC, MN, $M'N'$, $B'C'$ are concurrent, where B', C' are the tangency points of the incircle with CA, respectively AB.)

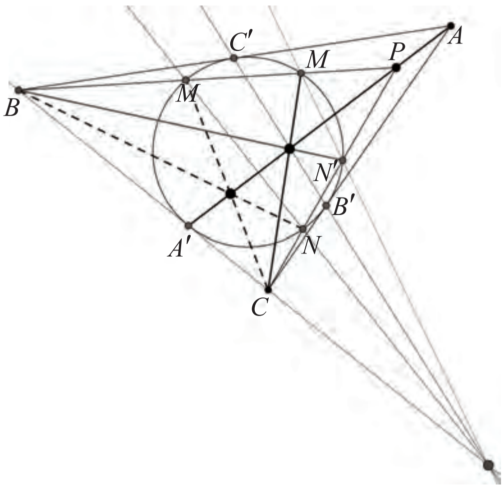


Figure 10

As a remark, it is clear that an extension similar to **II** of the above result is usually **not** true.

7. On a perpendicularity deduced projectively

This problem is a result involving a perpendicularity as a consequence of a projective fact. Moreover, we will give a pure by synthetic proof, based on a simple angle chasing.

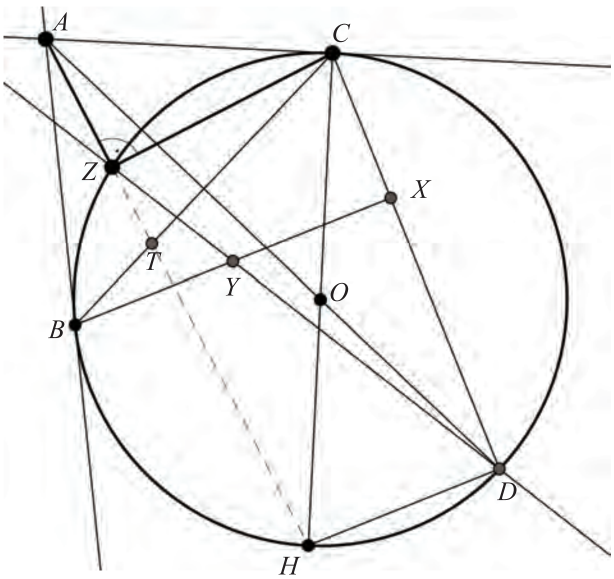


Figure 11

Let $\rho(O)$ be a circle and A a point outside it. Denote by B, C the points where the tangents from A w.r.t to $\rho(O)$ meet the circle, D the point on $\rho(O)$ for which $O \in (AD)$, X the foot of the perpendicular from B to CD, Y the midpoint of the line segment BX and by Z the second intersection of DY with $\rho(O)$. Prove that $ZA \perp ZC$ (Figure 11).

BMO Shortlist 2007, proposed by the author of this paper

First proof. Let H the second intersection of CO with $\rho(O)$. Thus $DC \perp DH$, so $DH \| BX$.

Because Y is the midpoint of (BX), we deduce that the division (B, Y, X, ∞) is harmonic, so also is the pencil $D(B, Y, X, H)$, and by intersecting it with $\rho(O)$, it follows that the quadrilateral $HBZC$ is harmonic.

Hence the pencil $C(HBZC)$ is harmonic, so by intersecting it with the line HZ, it follows that the division $(A'ZTH)$ is harmonic, where A', T are the intersections of HZ with the tangent in C, respectively with BC.

So, the line CH is the polar of A' w.r.t. $\rho(O)$, but $CH \equiv BC$ is the polar of A as well, so $A \equiv A'$, hence the points H, Z, A are collinear, therefore $ZA \perp ZC$. □

Second proof. Let E be the midpoint of BC. So $EY \parallel CD$ and $EY \perp BX$. Since $\sphericalangle YEB = \sphericalangle DCB = \sphericalangle DZB$, we deduce that Y, E, Z, B lie on a same circle. Thus $\sphericalangle ZBC = \sphericalangle ZEA$ (OA is tangent to the circle $YEZB$).

But because $\sphericalangle ZBC = \sphericalangle ZCA$, we have that $\sphericalangle ZCA = \sphericalangle ZEA$, i.e. the points C, A, Z, E are concyclic. Hence $\sphericalangle CZA = \sphericalangle CEA = 90$, i.e. $ZA \perp ZC$. □

On the idea of the second solution, we give a more general statement:

IV. *The tangents from a point A to the circle $\rho(O)$ touch it at the points B, C. For a point $E \in (BC)$ denote: the point D on ρ for which $E \in (AD)$; the second intersection X of the line CD with the circumcircle of the triangle BED; the point Y on the line BX for which $EY \parallel CD$; the second intersection Z of DY with the circle ρ; the intersection T of the line AC with BX; the second intersection H of the line AZ with the circle ρ. Prove that the quadrilaterals $AZEC$, $AZYT$ are cyclic and $BH \parallel \overline{AED}$.* (Figure 12)

Proof. Since $\sphericalangle ZBE = \sphericalangle ZBC = \sphericalangle ZDC = \sphericalangle ZYE$, we have that $\sphericalangle ZBE = \sphericalangle ZYE$, i.e. the quadrilateral $ZBYE$ is cyclic. Hence $\sphericalangle EBY = $ $= \sphericalangle EBX = \sphericalangle EDX = \sphericalangle EDC = \sphericalangle YED$. So $\sphericalangle EBY = \sphericalangle YED$, meaning that the line $\overline{AED}$ is tangent to the circumcircle of the quadrilateral $ZBYE$.

Thus, $\sphericalangle ZEA = \sphericalangle ZBE = \sphericalangle ZBC = \sphericalangle ZCA$, i.e. the quadrilateral $AZEC$ is cyclic. So, $\sphericalangle ZYB = \sphericalangle ZEB = \sphericalangle ZAC = \sphericalangle ZAT$, whence the quadrilateral $AZYT$ is also cyclic.

Finally, we have $\sphericalangle AEC = \sphericalangle AZC = \sphericalangle CDH$. Hence, $BH \| \overline{AED}$. □

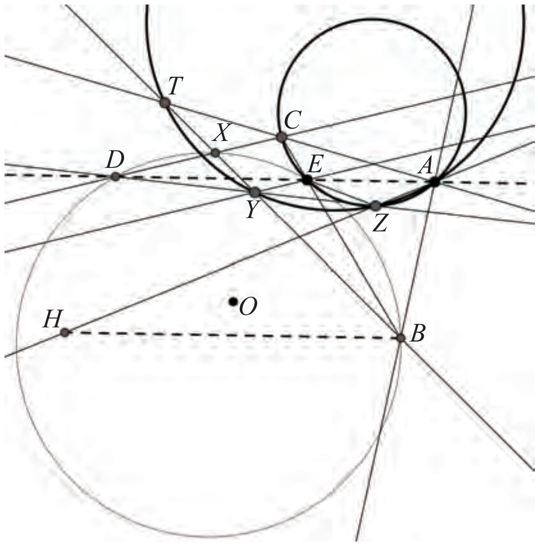


Figure 12

Observe that when E is the midpoint of the segment BC we obtain the original proposed problem.

8. On a classic locus problem as a recent IMO Team Preparation Test exercise

Consider the isosceles triangle ABC with $AB = AC$, and M the midpoint of BC. Find the locus of the points P interior to the triangle, for which $\sphericalangle BPM + \sphericalangle CPA = \pi$.

IMAR Contest 2006

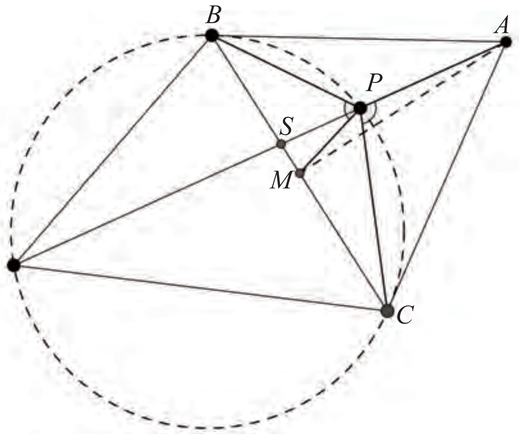


Figure 13

First proof (**by Dan Schwarz**). We will start with the following claim:

Lemma. *Prove that if, for a point P interior to the triangle, $\sphericalangle ABP = \sphericalangle BCP$, then $\sphericalangle BPM + \sphericalangle CPA = \pi$.*

Proof. Notice that the configuration is fully symmetrical:

$$\sphericalangle ABP = \sphericalangle BCP \quad \text{iff} \quad \sphericalangle ACP = \sphericalangle CBP$$

$$\sphericalangle BPM + \sphericalangle CPA = \pi \quad \text{iff} \quad \sphericalangle CPM + \sphericalangle BPA = \pi$$

and also that $P \in AM$ guarantees the result, therefore we may assume w.l.o.g. (for construction's sake) that P is interior to the triangle ABM. The given angle equality is readily seen to be equivalent with $P \in \Gamma$, where Γ is the arc interior to ABC of the circumcircle $\mathcal{K}$ of triangle BCI, with I the incenter of ABC. It is also immediately seen that AB and AC are tangent to $\mathcal{K}$.

Consider the *Apollonius* circle $\mathcal{A}$ for points A, M and ratio BA/BM; denote by U and V the points where AM meets $\mathcal{A}$. Clearly, $B, C \in \mathcal{A}$. Then $BA/BM = UA/UM = VA/VM$, from which follows $BA^2/BM^2 = UA \cdot VA/UM \cdot VM = UA \cdot VA/BM^2$, by symmetry and the power of a point relation, so $BA^2 = UA \cdot VA$, hence AB is tangent to $\mathcal{A}$, again by the power of a point relation. Therefore the two circles $\mathcal{K}$ and $\mathcal{A}$ coincide. Now, prolong AP until it meets $\mathcal{K}$ again at Q, and prolong PM until it meets $\mathcal{K}$ again at R. Then $BA/BM = PA/PM = QA/QM$, therefore $BA^2/BM^2 = PA \cdot QA/PM \cdot QM = BA^2/PM \cdot QM$, by the power of a point relation, so $BM^2 = PM \cdot QM$. But $BM^2 = PM \cdot RM$, again by the power of a point relation, so $QM = RM$, hence $BQ = CR$. It follows that $\sphericalangle BPQ = \sphericalangle CPR$, and this is enough to yield $\sphericalangle BPM + \sphericalangle CPA = \pi$.

Alternatively, one can calculate ratios using the symmetrical points P' and Q' with respect to AM, and denoting by N the meeting point of PQ' and $P'Q$. It can be obtained that $AM = AN$, therefore $M \equiv N$ and the rest easily follows as above.

Returning to the problem, we claim the locus is the arc Γ (defined in the above), together with the (open) segment (AM). Clearly $P \in (AM)$ fills the bill, so from now on we will assume $P \notin (AM)$. Also, as above, we will assume P interior to the triangle ABM (otherwise we work with the symmetrical relations).

Assume $P \notin \Gamma$, equivalent to $\sphericalangle BPC \neq \pi - \sphericalangle ABC$; then AB and AC are not tangent to $\mathcal{K}$. Take B' and C' to be the tangency points on $\mathcal{K}$ from A, and M' to be the midpoint of $B'C'$. We are now under the conditions from the lemma (for triangle $AB'C'$), therefore $\sphericalangle B'PM' + \sphericalangle C'PA = \pi$. But $\sphericalangle B'PM' = \sphericalangle BPM + \delta(\sphericalangle B'PB + \sphericalangle MPM')$ and $\sphericalangle C'PA = \sphericalangle CPA - \delta \sphericalangle C'PC$, where $\delta = 1$ if $\sphericalangle BPC < \pi - \sphericalangle ABC$, respectively $\delta = -1$ if $\sphericalangle BPC > \pi - \sphericalangle ABC$. We have $\sphericalangle B'PB = \sphericalangle C'PC$ from the symmetry of the configuration, and $\sphericalangle BPM + \sphericalangle CPA = \pi$ given; these relations therefore imply $\sphericalangle MPM' = 0$, which can only happen when M, M' and P are collinear, i.e. $P \in AM$, which was ruled out from the start in this part of the proof. The contradiction thus reached confirms our claim. $\square$

Second proof. Denote the point D as the intersection of the line AP with the circumcircle of BPC and $S = DP \cap BC$.

Since $\sphericalangle SPC = 180^\circ - \sphericalangle CPA$, it follows that $\sphericalangle BPS = \sphericalangle CPM$.

From the *Steiner* theorem applied to triangle BPC for the isogonals PS and PM,

$$\frac{SB}{SC} = \frac{PB^2}{PC^2}.$$

On other hand, using the law of sines, we obtain

$$\frac{SB}{SC} = \frac{DB}{DC} \cdot \frac{\sin SDB}{\sin SDC} = \frac{DB}{DC} \cdot \frac{\sin PCB}{\sin PBC} = \frac{DB}{DC} \cdot \frac{PB}{PC}.$$

Thus by the above relations, it follows that $DB/DC = PB/PC$, i.e. the quadrilateral $PBDC$ is harmonic. Therefore the point $A' = BB \cap CC$ lies on the line PD (where by XX we denoted the tangent in X to the circumcircle of BPC).

If $A' = A$, then lines AB and AC are always tangent to the circle BPC, and so the locus of P is the circle BIC, where I is the incenter of ABC. Otherwise, if $A' \neq A$, then $A' = AM \cap PS \cap BB \cap CC$, due to the fact that A' lies on PD and $A = PS \cap AM$, and by maintaining the condition that $A' \neq A$, we obtain that $PS \equiv AM$, therefore P lies on (AM). $\square$

Next, after two synthetic approaches, we continue with a direct trigonometric solution, actually the one which all three contestants who solved the problem used.

Third solution. Consider $m(\sphericalangle BPM) = \alpha$ and $m(\sphericalangle CPM) = \beta$ so $m(\sphericalangle APC) = 180^\circ - \alpha$ and $m(\sphericalangle APB) = 180^\circ - \beta$. Also let $m(\sphericalangle CBP) = u$, $m(\sphericalangle BCP) = v$ and denote $AB = AC = l$.

By the law of sines, applied to triangles ABP and ACP, $\dfrac{l}{\sin(180 - \beta)} = \dfrac{AP}{\sin(B - u)}$ and $\dfrac{l}{\sin(180 - \alpha)} = \dfrac{AP}{\sin(C - v)}$. Hence

$$\frac{\sin \alpha}{\sin \beta} = \frac{\sin (C - v)}{\sin (B - u)}. \qquad (\star)$$

Again, by the law of sines, this time applied in to the triangles BPM and CPM, $\dfrac{PM}{\sin u} = \dfrac{BM}{\sin \alpha}$ and $\dfrac{PM}{\sin v} = \dfrac{CM}{\sin \beta}$. Hence

$$\frac{\sin \alpha}{\sin \beta} = \frac{\sin u}{\sin v}. \qquad (\star\star)$$

From ($\star$) and ($\star\star$) we deduce that $\dfrac{\sin u}{\sin v} = \dfrac{\sin(C-v)}{\sin(B-u)}$, i.e. $\cos(B-2u)-$ $-\cos B = \cos(C-2v) - \cos C$. Since $B = C$, it follows that $\cos(B-2u) =$ $= \cos(B-2v)$.

If $B - 2u = B - 2v$, then we have $u = v$ i.e. P lies on (AM). Alternatively, if $B - 2u = 2v - B$, then we have $B = u + v$, i.e. P lies on the arc BIC. $\square$

9. The *orthotransversal* line of a triangle refreshed as a Mathematical Reflections problem

Let ABC be a triangle and P be an arbitrary point inside the triangle. Let A', B', C' be respectively the intersections of AP and BC, BP and CA, CP and AB. Through P we draw a line perpendicular to PA that intersects BC at A_1. We define B_1 and C_1 analogously. Let P' be the isogonal conjugate of the point P with respect to triangle $A'B'C'$. Then A_1, B_1, C_1 all lie on a same line l that is perpendicular to PP'.

Math. Reflections, 4/2006, problem O13, proposed by Khoa Lu Nguyen

In [7], *Bernard Gibert* names the line $\overline{A_1B_1C_1}$ the *orthotransversal* line of P.

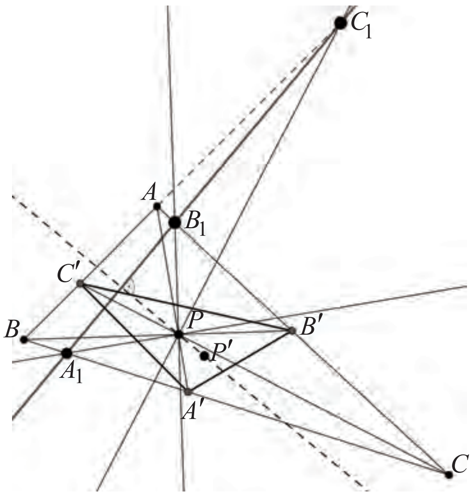


Figure 14

First proof. Given four collinear points X, Y, Z, T, let (X, Y, Z, T) denote, we mean the cross-ratio of four points X, Y, Z, T. Given four concurrent lines x, y, z, t, let (x, y, z, t) denote, we mean the cross-ratio of four lines x, y, z, t. We first introduce the following claim:

Lemma. *Let ABC be a triangle and P' be the isogonal conjugate of an arbitrary point P with respect to ABC. Then the six projections from P and P' to the sides of triangle ABC lie on a circle with center the midpoint of PP'.*

Proof. Let P_a, P_b, P_c be the projections from P to the sides BC, CA, AB. Similarly, let P'_a , P'_b, P'_c be the projections from P' to the sides BC, CA, AB. Call O the midpoint of PP'. We need to show that P_a, P_b, P_c, P'_a, P'_b, P'_c lie on a circle with center O.

Consider the trapezoid $PP'P'_aP_a$ that has $\mathrm{m}(PP_aP'_a) = \mathrm{m}(P'P'_aP_a) = 90°$ and O the midpoint of (PP'). Hence, O must lie on the perpendicular bisector of $P_aP'_a$. By a similar argument, we obtain that O also lies on the perpendicular bisector of $P_bP'_b$ and $P_cP'_c$.

Because P' is the isogonal conjugate of P with respect to ABC, we have $\sphericalangle BAP = \sphericalangle P'AC$, or $\sphericalangle P_cAP = \sphericalangle P'AP'_b$. Hence, we obtain $\sphericalangle APP_c = \sphericalangle P'_bP'A$. On the other hand, since quadrilaterals AP_cPP_b and $AP'_cP'P'_b$ are cyclic, it follows that $\sphericalangle APP_c = \sphericalangle AP_bP_c$ and $\sphericalangle P'_bP'A = \sphericalangle P'_bP'_cA = \sphericalangle P'_bP'_cP_c$. Thus, $\sphericalangle AP_bP_c = \sphericalangle P'_bP'_cP_c$. This means that $P_bP_cP'_cP'_b$ is inscribed in a circle. We notice that the center of this circle is the intersection of the perpendicular bisectors of $P_bP'_b$ and $P_cP'_c$, which is O. In the same manner, we obtain that $P_cP_aP'_cP'_a$ is inscribed in a circle with center O. Thus these two circles are congruent as they have the same center and pass through a common point P_c. Therefore, these six projections all lie on a circle with center O.

Back to the problem, let P_a, P_b, P_c be the projections from P to the sides $B'C'$, $C'A'$, $A'B'$ and O be the midpoint of PP'. Then (O) is the circumcircle of triangle $P_aP_bP_c$. We will show a stronger result. In fact, l is the polar of P w.r.t. the circle (O). We will prove that A_1 lies on the polar of P w.r.t. the circle (O), and by a similar argument so does B_1, respectively C_1.

Let B_2 and C_2 be the intersections of the line PA_1 with $A'C'$ and $A'B'$, respectively. Denote by M and N the intersections of the line PA_1 with the circle (O). It is suffice to prove that $(A_1, P, M, N) = -1$.

Consider X the projection from P to BC. Then five points P, P_b, P_c, X, A' lie on the circle (a) with diameter PA'. Since $(A'A_1, A'A, A'C', A'B') = -1$, we obtain that PP_cXP_b is a harmonic quadrilateral. This yields that P_bP_c and the tangents at P and X of the circle (a) are concurrent at a point U. Since the tangent at P of the circle (a) is PA_1, it follows that U is the concurrence point of P_bP_c, PA_1 and the perpendicular bisector of (PX). Consider the right triangle PXA_1 at X. Since U lies on PA_1 and the perpendicular bisector of (PX), we deduce that U is the midpoint of (PA_1). Therefore U lies on the line P_bP_c.

Since $A'P_b\Delta \cdot A'B_2 = A'P_c \cdot A'C_2 = A'P_2$, it follows that $P_bP_cC_2B_2$ is cyclic. Hence, $UB_2 \cdot UC_2 = UP_b \cdot UP_c$. On the other hand, we have $UP_b \cdot UP_c = UM \cdot UN$, because P_bP_cMN is cyclic.

Thus, we obtain $UB_2 \cdot UC_2 = UM \cdot UN$.

Again, since $(A'A_1, A', A'C', A'B') = -1$, we have $(A_1, P, B_2, C_2) = -1$. But, because U is the midpoint of A_1P, it follows that $UB_2 \cdot UC_2 = UA_1^2$. Hence $UA_1^2 = UM \cdot UN$, i.e. $(A_1, P, M, N) = -1$. □

Second proof for the collinearity of A_1, B_1, C_1. (**by Darij Grinberg, after Jacques Hadamard [6]**) Let $\mathcal{K}$ be an arbitrary circle centered at P; the polars of the points A, B, C, A_1, B_1, C_1 w.r.t. $\mathcal{K}$ are called a, b, c, a_1, b_1, c_1. After the construction of polars, we have $a \perp PA$, $b \perp PB$, $c \perp PC$, $a_1 \perp PA_1$, $b_1 \perp PB_1$ and $c_1 \perp PC_1$. Since the point A_1 lies on BC, the polar a_1 passes through the intersection $b \cap c$. From $a_1 \perp PA_1$ and $PA_1 \perp PA$, we have $a_1 \| PA$; from $a \perp PA$, thus we get $a_1 \perp a$. Hence, a_1 is the line passing through the intersection $b \cap c$ and orthogonal to a. Analogously, b_1 is the line passing through the intersection $c \cap a$ and orthogonal to b, and c_1 is the line passing through the intersection $a \cap b$ and orthogonal to c. Therefore, a_1, b_1 and c_1 are the altitudes of the triangle formed by the lines a, b and c; consequently, the lines a_1, b_1 and c_1 concur. From this, we derive that the points A_1, B_1 and C_1 are collinear. □

Following the ideas presented above, we leave the readers a similar collinearity:

V. *Consider ABC a triangle and P a point in its plane. Let R, S, T, X, Y, Z be the midpoints of the segments BC, CA, AB, PA, PB and PC, respectively. Draw the lines through the X, Y, Z, orthogonal to PA, PB, respectively PC; these lines touch ST, TR, RS at A_1, B_1 and C_1, respectively. Prove that the points A_1, B_1, C_1 lie on the same line.*

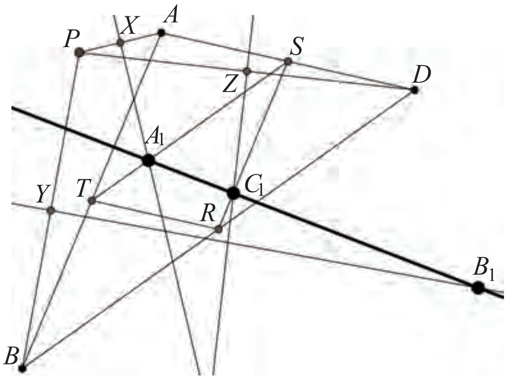


Figure 15

10. **Bellavitis' theorem on *balanced* quadrilaterals**

We call $ABCD$ a *balanced* [8] quadrilateral if and only if $AB \cdot CD = BC \cdot DA$. Particulary, if $ABCD$ is cyclic we come back to the harmonic quadrilateral. However, this class contains other quadrilaterals, for example all kites.

Let the lengths of the sides AB, BC, CD and DA of a (convex) quadrilateral $ABCD$ be denoted by a, b, c and d respectively. Similarly, the lengths of the quadrilateral's diagonals AC and BD will be denoted by e and f. Let E be the point of intersection of the two diagonals. The magnitude of $\sphericalangle DAB$ will be referred to as α, with similar notation for the other angles of the quadrilateral. The magnitudes of $\sphericalangle DAC$, $\sphericalangle ADB$ etc. will be denoted by α_B, δ_C a.s.o. (see Figure 16.). Finally, the magnitude of $\sphericalangle CED$ will be referred to as ϵ.

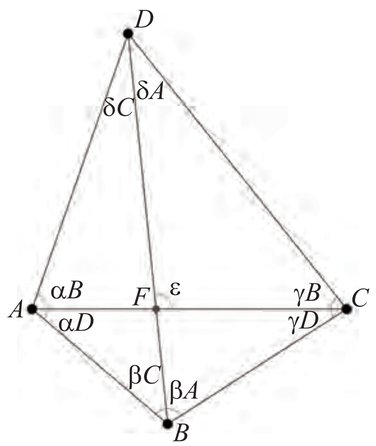


Figure 16.

Theorem. (Bellavitis 1854) If a (convex) quadrilateral $ABCD$ is *balanced*, then

$$\alpha_B + \beta_C + \gamma_D + \delta_A = \beta_A + \gamma_B + \delta_C + \alpha_D = 180^\circ.$$

Note that the convexity condition is a necessary one. The second equality sign does not hold for non-convex quadrilaterals.

Proof. (**by Eisso J. Atzema [8]**) Although in literature it is known that Giusto Bellavitis himself gave a proof using complex numbers, a trigonometric proof of his theorem follows from the observation that by the law of sines for any balanced quadrilateral we have

$$\sin \gamma_B \cdot \sin \alpha_D = \sin \alpha_B = \sin \gamma_D,$$

or

$$\cos(\gamma_B + \alpha_D) - \cos(\gamma_B - \alpha_D) = \cos(\alpha_B + \gamma_D) - \cos(\alpha_B - \gamma_D).$$

That is,

$$\cos(\gamma_B + \alpha_D) - \cos(\gamma_B - \alpha + \alpha_B) = \cos(\alpha_B + \gamma_D) - \cos(\alpha_B - \gamma + \gamma_B),$$

or

$$\cos(\gamma_B + \alpha_D) + \cos(\delta + \alpha) = \cos(\alpha_B + \gamma_D) + \cos(\delta + \gamma).$$

By cycling through, we also have

$$\cos(\delta_C + \beta A) + \cos(\alpha + \beta) = \cos(\beta_C + \delta_A) + \cos(\alpha + \delta).$$

Since $\cos(\alpha + \beta) = \cos(\delta + \gamma)$, adding these two equations gives

$$\cos(\gamma_B + \alpha_D) + \cos(\delta_C + \beta_A) = \cos(\alpha_B + \gamma_D) + \cos(\beta_C + \delta_A),$$

or

$$\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos\frac{1}{2}(\gamma_B + \alpha_D - \delta_C - \beta_A)$$
$$= \cos\frac{1}{2}(\alpha_B + \beta_C + \gamma_D + \delta_A) \cdot \cos\frac{1}{2}(\alpha_B + \gamma_D - \beta_C - \delta_A).$$

Now, note that

$$\gamma_B + \alpha_D - \delta_C - \beta_A = 360^\circ - 2\epsilon - \delta - \beta$$

and likewise

$$\alpha_B + \gamma_D - \beta_C - \delta_A = 2\epsilon - \beta - \delta.$$

Finally,

$$\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) + \frac{1}{2}(\alpha_B + \beta_C + \gamma_D + \delta_A) = 180^\circ.$$

It follows that

$$\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos(\epsilon + \frac{1}{2}(\beta + \delta)) =$$
$$= -\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos(\epsilon - \frac{1}{2}(\beta + \delta)),$$

or

$$\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos(\epsilon)\cos\frac{1}{2}(\delta + \beta) = 0.$$

Clearly, if neither of the last two factors is equal to zero, the first factor has to be zero and we are done. The last factor, however, will be zero if and only if $ABCD$ is cyclic. It is easy to see that any such quadrilateral has the angle property of *Bellavitis*' theorem. Therefore, in the case that $ABCD$ is cyclic, *Bellavitis*' theorem is true. Consequently, we may assume that $ABCD$ is not cyclic and that the third term does not vanish. Likewise, the second factor only vanishes in case $ABCD$ is orthodiagonal. For such quadrilaterals, we know that $a^2 + c^2 = b^2 + d^2$. In combination with the initial condition $ac = bd$, this implies that each side has to be congruent to an adjacent side. In other words, $ABCD$ has to be a kite. Again, it is easy to see that in that case *Bellavitis*' theorem is true. We can safely assume that $ABCD$ is not a kite and that the second term does not vanish either. $\square$

Moreover, for a synthetic solution one can see Nikolaos Dergiades' proof in [9]. Also, a solution using inversion exists. See [11].

References

[1] E. J. Atzema, *A Theorem by Giusto Bellavitis on a Class of Quadrilaterals*, Forum Geometricorum, (2006), 181 – 185.

[2] F.G.W. Brown, *Brocard points for a quadrilateral*, The Mathematical Gazette, Vol. 9, No. 129. (May, 1917), 83 – 85.

[3] N. Dergiades, *A synthetic proof and generalization of Bellavitis' theorem*, Forum Geometricorum, (2006) 225 – 227.

[4] D. Djukić, V. Janković, I. Matić, N. Petrović, *The IMO Compendium*, Springer, (May 12, 2005).

[5] B. Gibert, *Orthocorrespondence and Orthopivotal Cubics*, Forum Geometricorum, (2003), 1 – 27.

[6] D. Grinberg, *Problem: The ortho-intercepts line.*

[7] C. Kimberling, *Triangle centers and central triangles*, Congressus Numerantium, (1998) vol. 129, 1 – 285.

[8] E. M. Langley, *A Note on Tucker's Harmonic Quadrilateral*, The Mathematical Gazette, Vol. 11, No. 164. (May, 1923), 306 – 309.

[9] V. Nicula, C. Pohoaţă, *Diviziune armonică*, GIL Publishing House, Zalău (2007).

[10] C. Pohoaţă, *Harmonic Division and Its Applications*, Mathematical Reflections, No. 4 (2007).

[11] See also http://www.faculty.evansville.edu/ck6/encyclopedia/ETC.html; http://www.mathlinks.ro/viewtopic.php?t=88077, *Mathlinks Forum*.

Problems and Solutions from SEEMOUS 2011 Competition

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Abstract. In this note, we present the problems with solutions and comments from the 5th South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2011, organized by the MASEE and Romanian Mathematical Society between March 2 and March 6, 2011.

Keywords: Chebyshef's integral inequality, eigenvalues of a matrix, characteristic polynomial, vectors, Lagrange multipliers, Riemann sum, Taylor expansion.

MSC : 15A24, 15A26, 26A42, 26D15, 51D20.

1. Introducere

Cea de-a cincea ediţie a Olimpiadei de Matematică pentru Studenţi din Sud-Estul Europei, SEEMOUS 2011, a fost organizată de Mathematical Society of South Eastern Europe (MASSEE) şi de Societatea de Ştiinţe Matematice din Romania (S.S.M.R.). Gazda competiţiei a fost Universitatea Politehnica din Bucureşti (U.P.B.). Trebuie menţionat sprijinul pe care conducerea acestei universităţi îl acordă în mod constant şi eficient concursurilor studenţeşti de matematică. În acest sens, reamintim faptul că reluarea Concursului de matematică pentru studenţi „Traian Lalescu“ acum patru ani s-a datorat cooperării dintre Ministerul Educaţiei şi Cercetării, Societatea de Ştiinţe Matematice din România şi Universitatea Politehnica din Bucureşti. Organizatorii SEEMOUS 2011 au beneficiat şi de ajutorul colegilor din Facultatea de Matematică şi Informatică din Universitatea Bucureşti şi al celor din Institutul de Matematică „Simion Stoilow“ al Academiei Romane. Majoritatea participanţilor au fost de acord că această ediţie a fost cea mai bine organizată, atât din punct de vedere ştiinţific cât şi din punct de vedere logistic.

Concursul propriu-zis s-a bucurat de un număr record de participanţi. Au participat 85 de studenţi organizaţi în 20 de echipe de la universităţi din ţări precum Bulgaria, Columbia, Grecia, Macedonia, Moldova, România, Rusia şi Ucraina.

Studenţii români au avut din nou o comportarea remarcabilă reuşind să obţină 2 medalii de aur prin *Pădureanu Victor* (locul 2 după punctaj) (Academia Tehnică Militară, Bucureşti) şi *Cocalea Andrei* (Universitatea din

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Bucureşti), 14 medalii de argint: *Ţiţiu Radu* (Universitatea din Bucureşti), *Barzu Mihai* (Departamentul de Informatică, Universitatea „Alexandru Ioan Cuza", Iaşi), *Burtea Cosmin* (Universitatea „Alexandru Ioan Cuza", Iaşi), *Filip Laurian* (Universitatea din Bucureşti), *Fodor Dan* (Universitatea „Alexandru Ioan Cuza", Iaşi), *Hlihor Petru* (Universitatea din Bucureşti), *Sârbu Paul* (Universitatea Tehnică „Gheorghe Asachi", Iaşi), *Popescu Roxana-Irina* (Universitatea din Bucureşti), *Mesaros Ionuţ* (Universitatea Tehnică, Cluj-Napoca), *Ţucă Laurenţiu* (Universitatea Politehnica Bucureşti), *Gîlcă Dragoş* (Departamentul de Informatică, Universitatea „Alexandru Ioan Cuza", Iaşi), *Beltic Marius Jimy Emanuel* (Universitatea „Alexandru Ioan Cuza", Iaşi), *Sasu Robert* (Universitatea Politehnica Bucureşti), *Pleşca Iulia* (Universitatea „Alexandru Ioan Cuza", Iaşi) şi 19 medalii de bronz: *Cervicescu Virgil* (Academia Tehnică Militară, Bucureşti), *Raicea Marina* (Academia Tehnică Militară, Bucureşti), *Mihăilă Ştefan* (Departamentul de Informatică, Universitatea „Alexandru Ioan Cuza", Iaşi), *Munteanu Alexandra-Irina* (Universitatea din Bucureşti), *Vasile Mihaela Andreea* (Universitatea Politehnica Bucureşti), *Mocanu Maria-Cristina* (Departamentul de Informatică, Universitatea „Alexandru Ioan Cuza", Iaşi), *Petre Luca* (Universitatea Politehnica Bucureşti), *Rublea Alina* (Universitatea Politehnica Bucureşti), *Bobeş Maria Alexandra* (Universitatea „Babeş Bolyai", Cluj-Napoca), *Genes Cristian* (Universitatea Tehnică „Gheorghe Asachi", Iaşi), *Kolumban Jozsef* (Universitatea „Babeş-Bolyai", Cluj-Napoca), *Craus Sabina* (Universitatea „Alexandru Ioan Cuza", Iaşi), *Damanian Lavinia-Mariana* (Departamentul de Informatică, Universitatea „Alexandru Ioan Cuza", Iaşi), *Moldovan Dorin Vasile* (Universitatea Tehnică, Cluj-Napoca), *Mincu Diana* (Universitatea Politehnica Bucureşti), *Birghila Corina* (Universitatea „Ovidius", Constanţa), *Vlad Ilinca* (Universitatea Tehnică „Gh. Asachi", Iaşi), *Sav Adrian-Gabriel* (Universitatea „Alexandru Ioan Cuza", Iaşi), *Alban Andrei* (Universitatea din Bucureşti).

Echipa Universităţii Politehnica din Bucureşti a fost formată din 8 studenţi, dintre care 6 au reuşit să obţină medalii.

Proba de concurs a constat în rezolvarea a 4 probleme pe durata a 5 ore. Vom prezenta mai jos soluţiile problemelor, precum şi comentariile de rigoare.

2. **Probleme, Soluţii şi Comentarii**

Problema 1. *Pentru un întreg dat* $n \geq 1$*, fie* $f : [0,1] \to \mathbb{R}$ *o funcţie crescătoare. Demonstraţi că*

$$\int_0^1 f(x)\mathrm{d}x \leq (n+1)\int_0^1 x^n f(x)\mathrm{d}x.$$

Găsiţi toate funcţiile continue crescătoare pentru care egalitatea are loc.

* * *

Soluţia 1. Pentru n dat şi $x, y \in [0,1]$ integrăm pe intervalul $[0,1]$ în raport cu x şi în raport cu y inegalitatea evidentă

$$(x^n - y^n)(f(x) - f(y)) \geq 0, \tag{1}$$

şi obţinem

$$\int_0^1 \left(\int_0^1 (x^n - y^n)(f(x) - f(y)) \mathrm{d}x \right) \mathrm{d}y \geq 0,$$

sau

$$\int_0^1 x^n f(x) \mathrm{d}x - \int_0^1 y^n \mathrm{d}y \int_0^1 f(x) \mathrm{d}x - \int_0^1 x^n \mathrm{d}x \int_0^1 f(y) \mathrm{d}y + \int_0^1 y^n f(y) \mathrm{d}y \geq 0,$$

care ne dă inegalitatea cerută.

Dacă f este continuă, când egalitatea din enunţ are loc, în (1) are loc de asemenea egalitate, deci funcţia f trebuie să fie constantă.

Soluţia a 2*-a.* Prin schimbare de variabilă avem

$$(n+1) \int_0^1 x^n f(x) \mathrm{d}x = \int_0^1 f(\sqrt[n+1]{t}) \mathrm{d}t.$$

Cum f este crescătoare, rezultă că $f(x) \leq f(\sqrt[n+1]{x})$, $x \in [0,1]$, de unde prin integrare obţinem inegalitatea cerută.

Soluţia a 3*-a.* Problema este cazul particular al inegalităţii lui *Cebâşev*: *Fie $f_1, f_2, \ldots, f_n : [a,b] \to \mathbb{R}$ funcţii integrabile, pozitive, monotone.*

1) *Dacă $f_1, f_2, \ldots, f_n$ sunt fie toate monoton crecătoare, fie toate monoton descrescătoare, atunci*

$$\int_a^b f_1(x) \mathrm{d}x \cdot \int_a^b f_2(x) \mathrm{d}x \cdot \int_a^b f_n(x) \mathrm{d}x \leq (b-a)^{n-1} \int_a^b f_1(x) f_2(x) \ldots f_n(x) \mathrm{d}x.$$

2) *Dacă $f_1, f_2, \ldots, f_n$ sunt de monotonii diferite, atunci inegalitatea este de semn contrar.*

În cazul nostru, pentru $n = 2$ şi $f_1(x) = f(x)$, $f_2(x) = x^n$ obţinem concluzia din enunţ.

Soluţia a 4*-a.* Notăm $C = \displaystyle\int_0^1 f(x) \mathrm{d}x$ şi fie $g : [0,1] \to \mathbb{R}$,

$g(x) = f(x) - C$. Deci $\displaystyle\int_0^1 g(x) \mathrm{d}x = 0.$

Astfel, inegalitatea noastră este echivalentă cu $0 \leq \int_0^1 x^n g(x)\mathrm{d}x$. Cum g este crescătoare, rezultă că există $d = \sup\{x \in [0,1] : g(x) \leq 0\}$. Vom obţine

$$\int_0^1 x^n g(x)\mathrm{d}x = -\int_0^d x^n |g(x)|\mathrm{d}x + \int_d^1 x^n |g(x)|\mathrm{d}x.$$

Cum $\int_0^1 g(x)\mathrm{d}x = 0$, avem $\int_0^d |g(x)|\mathrm{d}x = \int_d^1 |g(x)|\mathrm{d}x$. Înmulţind cu d^n, rezultă următorul şir de inegalităţi evidente

$$\int_0^d x^n |g(x)|\mathrm{d}x \leq \int_0^d d^n |g(x)|\mathrm{d}x = \int_d^1 d^n |g(x)|\mathrm{d}x \leq \int_d^1 x^n |g(x)|\mathrm{d}x,$$

de unde concluzia problemei noastre. □

Comentariu. Problema a fost considerată de mulţi concurenţi drept una foarte uşoară. Cu toate acestea, au existat destui care n-au reuşit să obţină punctajul maxim. De asemenea trebuie remarcat şi faptul că au existat concurenţi care nu şi-au mai amintit sau pur şi simplu nu ştiau inegalitatea lui *Cebâşev*.

Problema 2. *Fie $A = (a_{ij})$ o matrice reală cu n linii şi n coloane astfel încât $A^n \neq 0$ şi $a_{ij}a_{ji} \leq 0$ pentru orice i, j. Demonstraţi că există două numere nereale printre valorile proprii ale lui A.*

Ivan Feshchenko, Kiev, Ucraina

Soluţie. Fie $\lambda_1, \ldots, \lambda_n$ toate valorile proprii ale acestei matrice. Polinomul caracteristic asociat matricei A este

$$P(\lambda) = \det(\lambda I_n - A) = \lambda^n - a_1\lambda^{n-1} + \ldots + (-1)^n a_n,$$

unde $a_1 = \sum_{i=1}^{n} a_{ii}, a_n = \det A$ şi a_i suma minorilor principali de ordin i ai lui A ; de exemplu, $a_2 = \sum_{i,j} \begin{vmatrix} 0 & a_{ij} \\ a_{ji} & 0 \end{vmatrix}$.

Din ipoteză rezultă că $a_{ii} = 0$, deci $\sum_{k=1}^{n} \lambda_k = 0$. Suma $\sum_{i<j} \lambda_i\lambda_j = a_2$, i. e. $\sum_{i<j} \lambda_i\lambda_j = -\sum_{i<j} a_{ij}a_{ji}$. Aşadar, suma $\sum_{k=1}^{n} \lambda_k^2 = 2\sum_{i<j} a_{ij}a_{ji} \leq 0$.

Cum $A^n \neq 0$, A are cel puţin o valoarea proprie nenulă, de unde rezultă că cel puţin o valoare proprie nu este reală. Deoarece polinomul caracteristic al lui A are coeficienţi reali şi o valoare proprie este complexă, rezultă că şi conjugata complexă a acestei valori este valoare proprie pentru A, deci ceea ce trebuia demonstrat. $\square$

Comentariu. Problema a fost considerată de dificultate medie. Mai mult de jumătate dintre concurenţi n-au reuşit să o rezolve complet. Din păcate mulţi dintre studenţii români au întâmpinat dificultăţi mari în abordarea acestei probleme. Acest lucru arată că este nevoie de mai multă Algebră Liniară în programele noastre din universităţi.

Problema 3. *Fie vectorii* $\overline{a}, \overline{b}, \overline{c} \in \mathbb{R}^n$. *Arătaţi că*

$$(||\overline{a}||\left\langle \overline{b}, \overline{c}\right\rangle)^2 + (||\overline{b}||\left\langle \overline{a}, \overline{c}\right\rangle)^2 \leq ||\overline{a}||||\overline{b}||(||\overline{a}||||\overline{b}|| + |\left\langle \overline{a}, \overline{b}\right\rangle|)||\overline{c}||^2,$$

unde $\langle \overline{x}, \overline{y}\rangle$ *este produsul scalar al vectorilor* $\overline{x}, \overline{y}$ şi $\|\overline{x}\|^2 = \langle \overline{x}, \overline{x}\rangle$.

Dan Schwarz, Bucureşti, România

Soluţia 1. (*Soluţia trigonometrică.*) Vom demonstra mai întâi că, daţi vectorii $u, v \in \mathbb{R}^n$, cu $||u|| = ||v|| = 1$ (deci $u, v \in S^{n-1}$), atunci

$$\sup_{||x||=1} \left(\left\langle u, x\right\rangle^2 + \left\langle v, x\right\rangle^2\right) = 1 + |\left\langle u, v\right\rangle|.$$

Considerăm reprezentarea vectorială unică $x = x' + x''$ cu $x' \in \operatorname{span}(u, v)$ şi $x'' \perp \operatorname{span}(u, v)$. Atunci

$$1 = ||x||^2 = \left\langle x' + x'', x' + x''\right\rangle =$$

$$= ||x'||^2 + 2\left\langle x', x''\right\rangle + ||x''||^2 = ||x'||^2 + ||x''||^2 \text{ (relaţia lui } Pitagora\text{)},$$

deci $||x'|| \leq 1$. Avem $|\left\langle u, x\right\rangle| = |\left\langle u, x'\right\rangle| \leq |\left\langle u, y\right\rangle|$, unde $y = 0$ dacă $x' = 0$ şi $y = \dfrac{x'}{||x'||}$ dacă $x' \neq 0$ (deci $||y|| = 1$). Analog $|\left\langle v, x\right\rangle| = |\left\langle v, x'\right\rangle| \leq |\left\langle v, y\right\rangle|$. Maximul este aşadar obţinut când $x \in \operatorname{span}(u, v)$.

Deci problema noastră vectorială a fost transferată în spaţiul de dimensiune 2, cu vectorii unitari u, v şi x. Capetele vectorilor $\pm u$ şi $\pm v$ împart cercul S^1 în patru arce, fiecare măsurând cel mult π (şi posibil 0, când $u = \pm v$); capătul lui x va fi într-unul din ele. Fie ω măsura acelui arc şi α, β, cu $\alpha + \beta = \omega$, măsurile arcelor dintre capătul lui x şi capetele acelui arc. Atunci $\left\langle u, x\right\rangle^2 + \left\langle v, x\right\rangle^2 = \cos^2\alpha + \cos^2\beta$, din binecunoscuta interpretare geometrică a produsului scalar (independent de dimensiunea spaţiului). Atunci

$$\left\langle u, x\right\rangle^2 + \left\langle v, x\right\rangle^2 = \cos^2\alpha + \cos^2\beta = 1 + \frac{1}{2}(\cos 2\alpha + \cos 2\beta) =$$

$$= 1 + \cos(\alpha + \beta)\cos(\alpha - \beta) \leq 1 + |\cos\omega| = 1 + |\left\langle u, v\right\rangle|,$$

cu egalitate în cazurile evidente:

- când $\alpha = \beta = \omega/2$, deci când $|\left\langle u, x\right\rangle| = |\left\langle v, x\right\rangle|$, aşadar

$$x = (\pm u \pm v)/|| \pm u \pm v||,$$

unde semnele sunt astfel încât $0 \leq \omega < \dfrac{\pi}{2}$;

- când $\omega = \pi/2$, deci când $\langle u, v\rangle = 0$, i.e. $u \perp v$, pentru orice

$$x \in \operatorname{span}(u, v).$$

Atunci, pentru orice $\bar{a}$, $\bar{b}$, $\bar{c}$ nenuli, luăm $u = \dfrac{\bar{a}}{||\bar{a}||}$, $v = \dfrac{\bar{b}}{||\bar{b}||}$, $x = \dfrac{\bar{c}}{||\bar{c}||}$, aşadar

$$\langle u, x\rangle^2 = \frac{1}{||\bar{a}|| \cdot ||\bar{c}||^2} \langle \bar{a}, \bar{c}\rangle^2,\ \langle v, x\rangle^2 = \frac{1}{||\bar{b}|| \cdot ||\bar{c}||^2} \langle \bar{b}, \bar{c}\rangle^2,\ \langle u, v\rangle = \frac{\langle \bar{a}, \bar{b}\rangle}{||\bar{a}|| \cdot ||\bar{b}||},$$

deci relaţia demonstrată devine

$$\frac{1}{||\bar{a}||} \langle \bar{a}, \bar{c}\rangle^2 + \frac{1}{||\bar{b}||} \langle \bar{b}, \bar{c}\rangle^2 \le \left(1 + \frac{|\langle \bar{a}, \bar{b}\rangle|}{||\bar{a}|| \cdot ||\bar{b}||}\right) ||\bar{c}||^2,$$

echivalent cu inegalitatea cerută, de asemenea adevărată pentru $\bar{a}, \bar{b}$ sau $\bar{c}$ egale cu zero.

Soluţia a 2-*a.*(*Soluţie cu forme pătratice*) Pentru $||x|| = ||u|| = ||v|| = 1$ avem

$$\begin{aligned} 0 &\le ||\lambda x + \mu u + \nu v||^2 = \langle \lambda x + \mu u + \nu v, \lambda x + \mu u + \nu v\rangle = \\ &= \lambda^2 + \mu^2 + \nu^2 + 2\lambda\mu \langle x, u\rangle + 2\lambda\nu \langle x, v\rangle + 2\mu\nu \langle u, v\rangle, \end{aligned}$$

o formă pătratică care ia valori pozitive pentru orice parametri reali λ, μ, ν, deci corespunzând unei matrice pozitiv semidefinită

$$\begin{bmatrix} 1 & \langle x, u\rangle & \langle x, v\rangle \\ \langle x, u\rangle & 1 & \langle u, v\rangle \\ \langle x, v\rangle & \langle u, v\rangle & 1 \end{bmatrix}.$$

Minorii principali de ordinul 1 sunt aşadar pozitivi, ceea ce ne dă semipozitivitatea normei; minorii principali de ordinul 2 sunt pozitivi, ceea ce ne dă inegalitatea *Cauchy-Buniakowski-Schwarz*, $1 \ge \langle u, v\rangle^2$; de asemenea, determinatul matricii este pozitiv

$$\Delta = 1 - (\langle u, v\rangle^2 + \langle x, u\rangle^2 + \langle x, v\rangle^2) + 2\langle u, v\rangle \langle x, u\rangle \langle x, v\rangle \ge 0,$$

care poate fi scris

$$\langle x, u\rangle^2 + \langle x, v\rangle^2 \le 1 + |\langle u, v\rangle| - |\langle u, v\rangle|(1 + |\langle u, v\rangle| - 2|\langle x, u\rangle \langle x, v\rangle|).$$

Dar $\langle x, u\rangle^2 + \langle x, v\rangle^2 \ge 2|\langle x, u\rangle \langle x, v\rangle|$, care înlocuit în relaţie ne dă

$$(1 - |\langle u, v\rangle|)(1 + |\langle u, v\rangle| - 2|\langle x, u\rangle \langle x, v\rangle|) \ge 0.$$

Apoi, fie $1 = |\langle u, v\rangle|$, când $1 + |\langle u, v\rangle| = 2 \ge 2|\langle x, u\rangle \langle x, v\rangle|$ (din inegalitatea *Cauchy-Buniakowski-Schwarz*), fie $1 > |\langle u, v\rangle|$, ceea ce implică $1 + |\langle u, v\rangle| \ge 2|\langle x, u\rangle \langle x, v\rangle|$. Aşadar, întotdeauna

$$\langle x, u\rangle^2 + \langle x, v\rangle^2 \le 1 + |\langle u, v\rangle|.$$

Soluţia a 3*-a* (*Soluţia cu multiplicatorii lui Lagrange*). Definim

$$L(x,\lambda) = \langle u,x\rangle^2 + \langle v,x\rangle^2 - \lambda(||x||^2 - 1)$$

şi considerăm sistemul

$$\frac{\partial L}{\partial x_i} = 2u_i\langle u,x\rangle + 2v_i\langle v,x\rangle - 2\lambda x_i = 0, \text{ pentru } 1 \le i \le n,$$

şi

$$\frac{\partial L}{\partial \lambda} = ||x||^2 - 1 = 0.$$

Avem

$$0 = \frac{1}{2}\sum_{i=1}^{n} x_i \frac{\partial L}{\partial x_i} = \langle u,x\rangle \sum_{i=1}^{n} u_i x_i + \langle v,x\rangle \sum_{i=1}^{n} v_i x_i - \lambda \sum_{i=1}^{n} x_i^2 =$$

$$= \langle u,x\rangle^2 + \langle v,x\rangle^2 - \lambda||x||^2 = \langle u,x\rangle^2 + \langle v,x\rangle^2 - \lambda.$$

Pe de altă parte,

$$0 = \frac{1}{2}\sum_{i=1}^{n} u_i \frac{\partial L}{\partial x_i} = \langle u,x\rangle \sum_{i=1}^{n} u_i^2 + \langle v,x\rangle \sum_{i=1}^{n} v_i u_i - \lambda \sum_{i=1}^{n} x_i u_i =$$

$$= \langle u,x\rangle ||u||^2 + \langle v,x\rangle \langle u,v\rangle - \lambda\langle u,x\rangle = \langle u,x\rangle + \langle u,v\rangle\langle v,x\rangle - \lambda\langle u,x\rangle,$$

şi analog pentru v, deci obţinem sistemul de două ecuaţii (în variabilele $\langle u,x\rangle$ şi $\langle v,x\rangle$)

$$\begin{cases} (1-\lambda)\langle u,x\rangle + \langle u,v\rangle\langle v,x\rangle = 0, \\ \langle u,v\rangle\langle u,x\rangle + (1-\lambda)\langle v,x\rangle = 0. \end{cases}$$

Determinantul Δ al matricei acestui sistem este $\Delta = (1-\lambda)^2 - \langle u,v\rangle^2$, şi dacă este nenul, unica soluţie este cea trivială $\langle u,x\rangle = \langle v,x\rangle = 0$, când expresia noastră atinge un minim evident egal cu zero (atunci $\lambda = 0$, aceasta înseamnă $\langle u,v\rangle \neq \pm 1$, care s-ar traduce în $u \neq \pm v$ şi $x \perp \operatorname{span}(u,v)$). Aşadar, suntem interesaţi de $\Delta = 0$ (pentru toate celelalte puncte critice), ducând la $\lambda = 1 \pm \langle u,v\rangle$, deci $\lambda = 1 + |\langle u,v\rangle|$ într-un punct de maxim şi $\lambda = 1 - |\langle u,v\rangle|$ într-un punct critic. Există câteva cazuri particulare.

Când $u = \pm v$, deci $\langle u,v\rangle = \pm 1$, situaţia este simplă. Avem maximul 2 când $x = \pm u$ şi minimul 0 când $x \perp u$.

Când $u \perp v$, deci $\langle u,v\rangle = 0$, atunci λ (deci şi expresia) este 1 în punctele de maxim, pentru orice $x \in \operatorname{span}(u,v)$, şi λ (deci şi expresia) este 0 în punctele de minim, pentru orice $x \perp \operatorname{span}(u,v)$.

Soluţia a 4*-a.* Vom reduce problema la cazul $\mathbb{R}^3$. Considerăm o bază ortonormală pentru care primii trei vectori genereză subspaţiul vectorial generat de $\{\overline{a}, \overline{b}, \overline{c}\}$. O transformare ortogonală păstrează norma şi produsul scalar, deci este suficient să demonstrăm inegalitatea în noua bază în care $\overline{a}$, $\overline{b}$, $\overline{c}$ au primele trei componente nenule.

În $\mathbb{R}^3$ vom folosi un argument de geometrie. Mai întâi, observăm că orice schimbare de semn a vectorilor $\overline{a}$, $\overline{b}$, $\overline{c}$ nu schimbă semnul inegalităţii iniţiale.

Fie $\alpha = \sphericalangle(\overline{b}, \overline{c})$, $\beta = \sphericalangle(\overline{c}, \overline{a})$ şi $\gamma = \sphericalangle(\overline{a}, \overline{b})$. Aranjăm semnele astfel încât $\alpha, \beta \in \left[0, \frac{\pi}{2}\right]$ şi considerăm triedrul $OABC$ având laturile OA, OB, OC determinate de vectorii $\overline{a}$, $\overline{b}$, $\overline{c}$ astfel încât proiectând vectorul $\overline{c}$ pe planul OAB, proiecţia sa să rămână în interiorul unghiului OAB.

Este cunoscut faptul că, dacă x, y, z satisfac $x + y = z$, atunci $\cos^2 x + \cos^2 y + \cos^2 z - 2\cos x \cos y \cos z = 1$. Astfel, avem de arătat că $\cos^2\alpha + \cos^2\beta \leq 1 + |\cos\gamma|$. Se observă că, dacă unul dintre unghiurile α sau β este egal cu $\frac{\pi}{2}$, atunci inegalitatea este evidentă .

Proiectăm punctul C pe planul OAB în punctul P şi considerăm $\alpha_1 = \sphericalangle POB$, $\beta_1 = \sphericalangle POA$. Proiectăm P pe OA, respectiv OB în punctele A_1, respectiv B_1. În triunghiurile dreptunghice OB_1C şi OB_1P avem

$$\cos^2\alpha = \frac{OB_1^2}{OC^2} \leq \frac{OB_1^2}{OP^2} = \cos^2\alpha_1.$$

Analog avem $\cos^2\beta \leq \cos^2\beta_1$. Din $\alpha_1 + \beta_1 = \gamma$ obţinem

$$\cos^2\alpha_1 + \cos^2\beta_1 = 1 + 2\cos\alpha_1\cos\beta_1\cos\gamma - \cos^2\gamma.$$

Arătăm că $1 + 2\cos\alpha_1\cos\beta_1\cos\gamma - \cos^2\gamma \leq 1 + \cos\gamma$.

Dacă $\cos\gamma = 0$, atunci inegalitatea este evidentă. Dacă $\cos\gamma \neq 0$, inegalitatea este echivalentă cu

$$2\cos\alpha_1\cos\beta_1 \leq 1 + \cos\gamma = \cos(\alpha_1 + \beta_1) = 1 + \cos\alpha_1\cos\beta_1 - \sin\alpha_1\sin\beta_1,$$

care în final va da $\cos(\alpha_1 - \beta_1) \leq 1$.

Soluţia a 5-a (*Nicolae Beli*). Asemănător cu soluţia 4, presupunem $\alpha, \beta \in \left[0, \frac{\pi}{2}\right]$.

Trebuie arătat că

$$|\cos\gamma| \geq \cos^2\alpha + \cos^2\beta - 1 = \frac{1}{2}(\cos 2\alpha + \cos 2\beta) = \cos(\alpha + \beta)\cos(\alpha - \beta).$$

Cum $|\alpha - \beta| \leq \frac{\pi}{2}$, avem $\cos(\alpha - \beta) \geq 0$. Prin urmare, putem presupune că $\cos(\alpha - \beta), \cos(\alpha + \beta) > 0$. Însă α, β, γ sunt unghiurile triedrului $OABC$ (care poate fi şi degenerat), deci avem $0 \leq \gamma \leq \alpha + \beta \leq \pi$. Rezultă că $0 < \cos(\alpha + \beta) \leq \cos\gamma$, care împreună cu $0 < \cos(\alpha - \beta) \leq 1$ implică $\cos(\alpha + \beta)\cos(\alpha - \beta) \leq \cos\gamma$. $\square$

Comentariu. Această problemă s-a dovedit a fi cea mai grea problemă din concurs, doar câţiva reuşind să o rezolve complet. Dintre români, doar *Victor Pădureanu* a rezolvat problema. În rezolvarea problemei de faţă s-au folosit aceleaşi ingrediente ca într-o altă problemă, un pic mai veche, dată la ultimul test de selecţie a lotului olimpic al României din anul 2007. Ea suna cam aşa:

Problemă înrudită. *Pentru $n \geq 2$ întreg pozitiv, se consideră numerele reale a_i, b_i, $i = \overline{1,n}$, astfel încât*

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2 = 1, \sum_{i=1}^{n} a_i b_i = 0.$$

Să se arate că

$$\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 \leq n.$$

Pentru soluţii şi comentarii recomandăm cititorilor Romanian Mathematical Competitions 2007, paginile 97–100.

Problema 4. *Fie $f : [0,1] \to \mathbb{R}$ o funcţie crescătoare de clasă $\mathcal{C}^2$. Definim şirurile date de $L_n = \frac{1}{n}\sum\limits_{k=0}^{n-1} f\left(\frac{k}{n}\right)$ şi $U_n = \frac{1}{n}\sum\limits_{k=1}^{n} f\left(\frac{k}{n}\right)$, $n \geq 1$. Intervalul $[L_n, U_n]$ se împarte în trei segmente egale. Demonstraţi că, pentru n suficient de mare, numărul $I = \int\limits_0^1 f(x)\mathrm{d}x$ aparţine segmentului din mijloc dintre aceste trei segmente egale.*

Alexander Kukush, Kiev, Ucraina

Soluţia 1. Enunţăm şi demonstrăm mai întâi următoarea lemă :

Lema. *Pentru $f \in C^2[0,1]$:* $L_n = I - \dfrac{f(1)-f(0)}{2n} + O\Big(\dfrac{1}{n^2}\Big)$, $\quad n \to \infty$.

Demonstraţie. Notăm $C = f(1) - f(0)$. Considerăm

$$I - L_n = \sum_{k=0}^{n-1} \int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} \Big(f(x) - f\Big(\frac{k}{n}\Big)\Big)\mathrm{d}x =$$

$$= \sum_{k=0}^{n-1} \int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} \Big(f'\Big(\frac{k}{n}\Big)\Big(x - \frac{k}{n}\Big) + \frac{1}{2}f''(\theta_{kn}^x)\Big(x - \frac{k}{n}\Big)^2\Big)\mathrm{d}x =$$

$$= \frac{1}{2n^2}\sum_{k=0}^{n-1} f'\Big(\frac{k}{n}\Big) + r_n, \tag{1}$$

unde

$$|r_n| \leq \frac{1}{2}\max|f''| \cdot \sum_{k=0}^{n-1} \frac{1}{3}\Big(x - \frac{k}{n}\Big)^3\Bigg|_{\frac{k}{n}}^{\frac{k+1}{n}} \leq \frac{\text{const}}{n^2}.$$

Aici $\theta^x_{kn} \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$ sunt punctele intermediare din teorema lui *Taylor* în vecinătatea punctului $\frac{k}{n}$.

Analog

$$\int_0^1 f'\mathrm{d}x - \frac{1}{n}\sum_{k=0}^{n-1} f'\left(\frac{k}{n}\right) = \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f''(\widetilde{\theta}^x_{kn})\left(x - \frac{k}{n}\right)\mathrm{d}x = O\left(\frac{1}{n}\right).$$

Deci în partea dreaptă a relaţiei (1) avem

$$\frac{1}{n}\sum_{k=0}^{n-1} f'\Big(\frac{k}{n}\Big) \to \int_0^1 f'\mathrm{d}x = C, \quad n \to \infty,$$

şi eroarea în partea dreaptă a relaţiei (1) este de ordin $O\left(\frac{1}{n^2}\right)$.

Deci din (1) avem

$$I - L_n = \frac{C}{2n} + O\Big(\frac{1}{n^2}\Big), \quad n \to \infty. \qquad \square$$

Acum ne întoarcem la soluţia problemei noastre şi avem $U_n = L_n + \frac{C}{n}$. Fie $x_n = L_n + \frac{kC}{3n}$, $k = 1, 2$. Atunci

$$x_n = I + \frac{C}{n}\Big(\frac{k}{3} - \frac{1}{2}\Big) + O\Big(\frac{1}{n^2}\Big).$$

Pentru $k = 1$ avem $\frac{k}{3} - \frac{1}{2} < 0$, deci $x_n < I$ pentru n suficient de mare; pentru $k = 2$ avem $\frac{k}{3} - \frac{1}{2} > 0$, deci $x_n > I$ pentru n suficient de mare. Aşadar, pentru n suficient de mare

$$L_n + \frac{C}{3n} < I < L_n + \frac{2C}{3N}.$$

Soluţia a 2-a. Fie $F(t) = \int_0^t f(x)\mathrm{d}t$. Atunci avem

$$I = \int_0^1 f(x)\mathrm{d}x = F(1) - F(0) =$$

$$= F(1) - F\left(\frac{n-1}{n}\right) + F\left(\frac{n-1}{n}\right) - F\left(\frac{n-2}{n}\right) + \ldots + F\left(\frac{1}{n}\right) - F(0).$$

Din formula lui *Taylor* de ordinul 2, vom obţine (pentru punctele $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$):

$$F(x) = F\left(\frac{k}{n}\right) + \left(x - \frac{k}{n}\right) F'\left(\frac{k}{n}\right) + \frac{1}{2}\left(x - \frac{k}{n}\right)^2 F''(\theta),$$

cu $\theta \in \left[\frac{k}{n}, x\right]$, $k = \overline{0, n-1}$, sau

$$F(x) = F\left(\frac{k}{n}\right) + \left(x - \frac{k}{n}\right) f\left(\frac{k}{n}\right) + \frac{1}{2}\left(x - \frac{k}{n}\right)^2 f'(\theta).$$

Pentru $x = \frac{1}{n}, \ldots, 1$ avem

$$F\left(\frac{k+1}{n}\right) - F\left(\frac{k}{n}\right) = \frac{1}{n} f\left(\frac{k}{n}\right) + \frac{1}{2n^2} f'(\theta_k),$$

cu $\theta_k \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$, $k = \overline{0, n-1}$.

Însumând aceste relaţii pentru $k = 0, 1, n-1$ obţinem

$$I = L_n + \frac{1}{2n}\sigma_n,$$

unde σ_n este suma *Riemann* pentru $\int\limits_0^1 f'(x)\mathrm{d}x = f(1) - f(0)$.

Intervalul din mijloc este $\left[\frac{2}{3}L_n + \frac{1}{3}U_n, \frac{1}{3}L_n + \frac{2}{3}U_n\right] = [u_n, v_n]$.

Dacă $f(0) = f(1)$, atunci f este constantă, ceea ce nu se poate.

Presupunem că $f(1) > f(0)$.

Avem $n(I - u_n) = \frac{n}{3}(L_n - U_n) + \frac{1}{2}\sigma_n = \frac{1}{3}(f(0) - f(1)) + \frac{1}{2}\sigma_n$ şi trecând la limită când $n \to \infty$, obţinem $n(I - u_n) \to \frac{1}{6}(f(1) - f(0)) > 0$, de unde pentru n suficient de mare, $I > u_n$.

În acelaşi mod, avem

$$n(v_n - I) = \frac{2n}{3}(U_n - L_n) - \frac{1}{2}\sigma_n \to \frac{1}{6}(f(1) - f(0)) > 0. \qquad \square$$

Remarcă. Relaţia $I = L_n + \frac{1}{2n}\sigma_n$ mai poate fi dedusă, din următorul fapt:

Fie $f : [0,1] \to \mathbb{R}$ *derivabilă cu derivata integrabilă pe* $[0,1]$. *Atunci*

$$\lim_{n\to\infty} n\left[\int\limits_0^1 f(x)\mathrm{d}x - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right] = \frac{f(0) - f(1)}{2}.$$

Demonstraţie. Fie $\Delta = \left(0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right)$ o diviziune a intervalului $[0,1]$. Notăm

$$M_k = \sup_{x \in \left[\frac{k-1}{n}, \frac{k}{n}\right]} f'(x), \qquad m_k = \inf_{x \in \left[\frac{k-1}{n}, \frac{k}{n}\right]} f'(x), \ \ k = \overline{1,n}.$$

Atunci avem

$$E_n = n\left[I - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right] = n\sum_{k=1}^{n}\left[\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\mathrm{d}x - \frac{1}{n} f\left(\frac{k}{n}\right)\right] =$$

$$= n\sum_{k=1}^{n}\left[\int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(f(x) - f\left(\frac{k}{n}\right)\right)\mathrm{d}x\right].$$

Aplicând teorema lui *Lagrange* pe intervalul $\left[x, \frac{k}{n}\right]$, rezultă că există $c_k(x) \in \left(\frac{k-1}{n}, \frac{k}{n}\right)$ astfel încât $f(x) - f\left(\frac{k}{n}\right) = \left(x - \frac{k}{n}\right) f'(c_k(x))$.

Atunci $E_n = n\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(x - \frac{k}{n}\right) f'(c_k(x))\mathrm{d}x$.

Dar $m_k \le f'(c_k(x)) \le M_k$, deci avem $-\frac{1}{2n}\sum_{k=1}^{n} M_k \le E_n \le -\frac{1}{2n}\sum_{k=1}^{n} m_k$.

Trecând la limită obţinem $\lim_{n\to\infty} E_n = \frac{f(0) - f(1)}{2}$.

Comentariu. Problema a fost considerată una foarte dificilă, iar puţini concurenţi au reuşit să o rezolve în întregime, cu toate că problema putea fi abordată şi de un elev foarte bine pregătit de clasa a XII-a. Dintre concurenţii români, doar câţiva au reuşit să obţină mai mult de jumătate din punctaj.

3. **Concluzii**

Olimpiada pentru studenţi SEEMOUS din acest an a fost o reuşită pentru studenţii români, care s-au descurcat onorabil. Mulţi dintre concurenţi au arătat destul de bine pregătiţi, dovada fiind medaliile obţinute. Pe de altă parte trebuie remarcat şi faptul că parcurgerea unor noţiuni din anul I de facultate în liceu poate constitui un real avantaj pentru cei care erau foarte buni la matematică în liceu.

În ceea ce priveşte studenţii Universităţii Politehnica din Bucureşti, putem spune că au avut o comportare decentă. Din păcate niciunul dintre ei n-a reuşit să obţină medalie de aur, cu toate că *Laurenţiu Ţucă* a fost foarte aproape dacă nu ar fi existat abordarea superficială a ultimei probleme. De asemenea, trebuie tras un semnal de alarmă în ceea ce priveşte rezultatele la prima problemă. Doar doi din cei 8 studenţi români şi-au amintit de inegalitatea lui *Cebâşev* (forma discretă sau integrală), deşi ea fusese făcută la una din pregătiri într-o formă sau alta. La prima problemă, o altă soluţie corectă a fost dată de *Alina Rublea*, care a obţinut aproape punctajul maxim având mici scăpări în tratarea cazului de egalitate.

Problema 2 a fost cea mai abordată de studenţii Politehnicii. *Laurenţiu Ţucă* a fost cel mai aproape de o soluţie completă. Din păcate, el nu a văzut un argument simplu care l-ar fi ajutat să finalizeze. Ceilalţi au abordat problema, însă au existat mici deficienţe în înţelegerea noţiunilor de polinom caracteristic şi valoare proprie. Din păcate, trebuie să semnalăm din nou faptul că din cauza comprimării materiei în anul I studenţii, fie ei chiar şi cei mai buni, nu reuşesc să-şi însuşească noţiunile importante din anumite capitole ale algebrei liniare.

Problema 3 a fost cea mai dificilă din concurs şi a fost rezolvată complet doar de 3 concurenţi, printre care şi *Victor Pădureanu*, fost component al lotului olimpic şi al celui lărgit pe perioada liceului. O altă încercare, dar nefinalizată, a avut *Robert Sasu.* La fel, parcă se pune prea puţin accent la cursul de Algebră pe aceste noţiuni de vectori, respectiv forme pătratice, determinanţi *Gram*, inegalităţile *Cauchy-Schwarz*, *Bessel*, etc.

Problema 4 a fost a doua din concurs ca dificultate. Unii dintre studenţii români s-au descurcat bine, iar dintre cei ai Politehnicii doar *Laurenţiu Ţucă* a fost aproape de a lua punctaj maxim, dacă ar fi demonstrat lema de mai sus.

În concluzie, suntem de părere că, din cauza cantităţii mari de materie care a fost comprimată aproape într-un singur semestru, studenţii nu mai au posiblitatea să mai simtă importanţa şi mai ales gustul unor noţiuni peste care se trece cu mult prea mare uşurinţă la orele de curs şi seminar. Graba predării materiei nu va avea ca rezultat decât familiarizarea insuficientă chiar şi a celor mai buni studenţi cu noţiunile noi de la cursurile Analiză Matematică, Algebră Liniară şi Superioară, Geometrie. Din păcate, aceasta este o meteahnă care datează de mult prea mulţi ani în învăţământul matematic din România.

PROBLEMS

Authors should submit proposed problems to office@rms.unibuc.ro or to gmaproblems@gmail.com. Files shold be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail adress. For this issue, solutions should arrive before **November 15, 2011**.

PROPOSED PROBLEMS

323. Let $\mathcal{C}$ be the set of the circles in the plane and $\mathcal{L}$ be the set of the lines in the plane. Show that there exist bijective maps $f, g : \mathcal{C} \to \mathcal{L}$ such that for any circle $C \in \mathcal{C}$, the line $f(C)$ is tangent at C and the line $g(C)$ contains the center of C.

Proposed by Marius Cavachi, Ovidius University of Constanţa, Romania.

324. Consider the set

$$K := \left\{ f\left(\sqrt[4]{20}\,, \sqrt[6]{500}\right) \mid f(X,Y) \in \mathbb{Q}[X,Y] \right\}.$$

(a) Show that K is a field with respect to the usual addition and multiplication of real numbers.
(b) Find all the subfields of K.
(c) If one considers K as a vector space ${}_{\mathbb{Q}}K$ over the field $\mathbb{Q}$ in the usual way, find the dimension of ${}_{\mathbb{Q}}K$.
(d) Exhibit a vector space basis of ${}_{\mathbb{Q}}K$.

Proposed by Toma Albu, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

325. We call *toroidal chess board* a regular chess board (of arbitrary dimension) in which the opposite sides are identified in the same direction. Show that the maximum number of kings on a toroidal chess board of dimensions $m \times n$ $(m, n \in \mathbb{N})$ such that each king attacks no more than six other kings is less than or equal to $\dfrac{4mn}{5}$ and the inequality is sharp.

Proposed by Eugen Ionaşcu, Columbus State University, Columbus, GA, USA.

326. For $t > 0$ define $H(t) = \displaystyle\sum_{n=0}^{\infty} \frac{t^n}{n!(n+1)!}$. Show that

$$\lim_{t\to\infty} \frac{t^{3/4} H(t)}{\exp(2\sqrt{t})} = \frac{1}{2\sqrt{\pi}}.$$

Proposed by Moubinool Omarjee, Jean Lurçat High School, Paris, France.

327. Let $f : [a, b] \to \mathbb{R}$ be a convex and continuous function. Prove that:

a) $\mathcal{M}(a; b) + (b-a) f\left(\dfrac{a+b}{2}\right) \geq \mathcal{M}\left(\dfrac{3a+b}{4}; \dfrac{3b+a}{4}\right)$;

b) $3\mathcal{M}\left(\dfrac{2a+b}{3}; \dfrac{2b+a}{3}\right) + \mathcal{M}(a; b) \geq 4\mathcal{M}\left(\dfrac{3a+b}{4}; \dfrac{3b+a}{4}\right)$.

Here $\mathcal{M}(x, y) = \dfrac{1}{y-x} \displaystyle\int_x^y f(t) \mathrm{d}t$.

Proposed by Cezar Lupu, Polytechnic University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal High School, Constanţa, Romania.

328. Given any positive integers m, n, prove that the set $\{1, 2, 3, \ldots, m^{n+1}\}$ can be partitioned into m subsets $A_1, A_2, \ldots, A_m$, each of size m^n, such that

$$\sum_{a_1 \in A_1} a_1^k = \sum_{a_2 \in A_2} a_2^k = \ldots = \sum_{a_m \in A_m} a_m^k, \quad \text{for all} \quad k = 1,\ 2,\ \ldots,\ n.$$

Proposed by Cosmin Pohoaţă, student Princeton University, Princeton, NJ, USA.

329. Let $p \geq 11$ be prime number. Show that, if

$$\sum_{j=1}^{(p-1)/2} \frac{1}{j^6} = \frac{a}{b},$$

with a, b relatively prime, then p divides a.

Proposed by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bârlad, Romania.

330. Determine all nonconstant monic polynomials $f \in \mathbb{Z}[X]$ such that $\varphi(f(p)) = f(p-1)$ for all natural prime numbers p. Here $\varphi(n)$ is the *Euler* totient function.

Proposed by Vlad Matei, student Cambridge University, Cambridge, UK.

331. Let $\mathcal{B}_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \leq x_{i+2} \text{ for } 1 \leq i \leq n-2\}$ and let $\mathcal{B} = \bigcup_{n \geq 1} \mathcal{B}_n$. On $\mathcal{B}$ we define the relation $\leq$ as follows. If $x, y \in \mathcal{B}$, $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$, we say that $x \leq y$ if $m \geq n$ and for any $1 \leq i \leq n$ we have either $x_i \leq y_i$ or $1 < i < m$ and $x_i + x_{i+1} \leq y_{i-1} + y_i$. Prove that $(\mathcal{B}, \leq)$ is a partially ordered set.

Proposed by Nicolae Constantin Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

332. For a positive integer $n = \prod_{i=1}^{s} p_i^{\alpha_i}$ denote by $\Omega(n) := \sum_{i=1}^{s} \alpha_i$ the total number of prime factors of n (counting multiplicities). Of course, by default $\Omega(1) = 0$. Define now $\lambda(n) := (-1)^{\Omega(n)}$, and consider the sequence $\mathfrak{S} := (\lambda(n))_{n\geq 1}$. You are asked to prove the following claims on $\mathfrak{S}$:

a) It contains infinitely many terms $\lambda(n) = -\lambda(n+1)$;

b) It is not ultimately periodic;

c) It is not ultimately constant over an arithmetic progression;

d) It contains infinitely many pairs $\lambda(n) = \lambda(n+1)$;

d) It contains infinitely many terms $\lambda(n) = \lambda(n+1) = 1$;

e) It contains infinitely many terms $\lambda(n) = \lambda(n+1) = -1$.

Proposed by Dan Schwarz, Bucharest, Romania.

333. Show that there do not exist polynomials $P, Q \in \mathbb{R}[X]$ such that

$$\int_0^{\log\log n} \frac{P(x)}{Q(x)} \mathrm{d}x = \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_n}, n \geq 1,$$

where p_n is the nth prime number.

Proposed by Cezar Lupu, Polytechnic University of Bucharest, Bucharest and Cristinel Mortici, Valahia University of Târgovişte, Târgovişte, Romania.

334. Let a, b be two positive integers with a even and $b \equiv 3 \pmod 4$. Show that $a^m + b^m$ does not divide $a^n - b^n$ for any even m, $n \geq 3$.

Proposed by Octavian Ganea, student École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland.

335. Let m and n be positive integers with $m \leq n$ and let $A \in \mathcal{M}_{m,n}(\mathbb{R})$ and $B \in \mathcal{M}_{n,m}(\mathbb{R})$ be matrices such that $\operatorname{rank} A = \operatorname{rank} B = m$. Show that there exists $C \in \mathcal{M}_n(\mathbb{R})$ such that $A \cdot C \cdot B = I_m$, where I_m denotes the m by m unit matrix.

Proposed by Vasile Pop, Technical University Cluj-Napoca, Cluj-Napoca, Romania.

336. Show that the sequence $(a_n)_{n\geq 1}$ defined by $a_n = [2^n\sqrt{2}] + [3^n\sqrt{3}]$, $n \geq 1$, contains infinitely many odd numbers and infinitely many even numbers. Here $[x]$ is the integer part of x.

Proposed by Marius Cavachi, Ovidius University of Constanţa, Constanţa, Romania.

SOLUTIONS

295. Determine all nonconstant polynomials $P \in \mathbb{Z}[X]$ such that $P(p)$ is square-free for all prime numbers p.

`Proposed by Vlad Matei, student University of Bucharest, Bucharest, Romania.`

Solution by the author. We will need the following observation.

Lemma. *For all nonconstant polynomials $f \in \mathbb{Z}[X]$ the set*

$$A = \{q \text{ prime } \mid \exists\, p \text{ prime}, q|f(p)\}$$

is infinite.

Proof of the lemma. Assume the contrary. Let $A = \{q_1, q_2, \ldots, q_n\}$ and let $M = q_1q_2 \cdots q_n$. Let k be an arbitrary natural number. According to *Dirichlet*'s theorem, the arithmetic progression $M^k r+1$ with $r \in \mathbb{N}^*$ contains an infinity of prime numbers.

Let us note that $f(M^k r + 1) \equiv f(1) (\text{mod } q_i^k)$, $\forall\, i = 1, \ldots, n$. Below we will assume that the leading coefficient of f is positive, the other case is similar considering $-f$. We know that $\lim\limits_{x\to\infty}(f(x) - x) = \infty$, thus for r sufficiently large $f(M^k r + 1) \geq M^k r$. Since A is finite, it follows that there is an index j with $q_j^k | f(M^k r + 1)$. We would have that $q_j^k | f(1)$. Now for k sufficiently large $q_i^k > |f(1)|$, $\forall i = 1, \ldots, n$, and we are done unless $f(1) = 0$. In this case the lemma is obvious, since $f(p) = (p - 1)g(p)$, with $g \in \mathbb{Z}[X]$. $\square$

First of all, we claim that $f(0) = 0$. Assume the contrary $f(0) \neq 0$. We can choose infinitely many primes q such that $f(q) = d \cdot p_1p_2 \cdots p_m$ with $p_i \neq p_j$ for $1 \leq i \neq j \leq m$ and $(f(0), f(q)) = d$. We know from *Taylor* expansion of polynomials that

$$f(q+tp_i) = f(q)+f'(q)\cdot tp_i+f''(q)\cdot\frac{(tp_i)^2}{2!}+\cdots \equiv f(q)+f'(q)\cdot tp_i \pmod{p_i^2}$$

for $1 \leq i \leq m$.

If we would have $p_n \nmid f'(q)$ for a certain index n, then we could choose t such that $f(q) + f'(q) \cdot tp_n \equiv 0 \pmod{p_i^2}$, since this is equivalent to $f'(q) \cdot t \equiv -\dfrac{f(q)}{p_n} \pmod{p_i^2}$ and $f'(q)$ is invertible modulo p_n. This means that there is $m \in \mathbb{Z}$ such that $p_n^2 | f(m)$. Now $m = q + tp_n$, and if $(m, p_n) > 1$ it follows that $p_n | q$. Since $f(q) = d \cdot p_1p_2 \ldots p_m$, we have that $p_n | f(q)$, so $p_n | (q, f(q))$. But $(q, f(q)) | f(0)$, and we deduce that $p_n | d$, a contradiction. Thus $(m, p_n) = 1$, and by Dirichlet's theorem we could find a prime r in the arithmetic progression $m + ap_n^2$, $a \in \mathbb{N}^*$. We would get that $f(r) \equiv f(m) \equiv 0 \pmod{p_i^2}$, in contradiction with the hypothesis.

Therefore there is no such index and it follows $p_1p_2\cdots p_m|f'(q)$. This means $|f'(q)|\geq |p_1p_2\cdots p_m|$, which is equivalent to $\dfrac{|f'(q)|}{|f(q)|}\geq \dfrac{1}{d}\geq \dfrac{1}{f(0)}$.

To finish the proof, we make q arbitrarily large and it results that $\lim\limits_{x\to\infty}\dfrac{f'(x)}{f(x)}\geq \dfrac{1}{f(0)}$. This contradicts the well known fact that $\lim\limits_{x\to\infty}\dfrac{f'(x)}{f(x)}=0$. This contradiction proves that our initial claim is true, so that $f(0)=0$. If we write $f(X)=X^ig(X)$, $g(0)\neq 0$, we get immediately from the hypothesis that $i=1$ and g is constant. So $f(X)=cX$. Morever, if c has a prime factor l, then $f(l)$ is not square-free. Thus $c=\pm 1$.

We conclude that the only such polynomials are $f(X)=-X$ and $f(X)=X$. $\square$

296. Let $a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_n>0$ and $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n$ be real numbers. Show that

$$\left(\sum_{1\leq i,j\leq n}x_ix_j\min(a_i,a_j)\right)\left(\sum_{1\leq i,j\leq n}y_iy_j\min(b_i,b_j)\right)\geq \sum_{1\leq i,j\leq n}x_iy_j\min(a_i,b_j).$$

`Proposed by Alin Gălăţan, student University of Bucharest, Bucharest, Romania.`

Solution by the editors. Let λ_A be the characteristic function of an arbitrary set A. Consider the functions $f,g:[0,\infty)\to\mathbb{R}$ defined by

$$f(x)=\sum_{i=1}^n x_i\lambda_{[0,a_i]}(x),\quad g(x)=\sum_{i=1}^n y_i\lambda_{[0,b_i]}(x).$$

We have

$$\int_0^\infty f^2(x)\mathrm{d}x=\sum_{1\leq i,j\leq n}x_ix_j\int_0^\infty \lambda_{[0,a_i]}(x)\lambda_{[0,b_i]}(x)\mathrm{d}x=\sum_{1\leq i,j\leq n}x_ix_j\min(a_i,a_j).$$

Analogously, we obtain

$$\int_0^\infty g^2(x)\mathrm{d}x=\sum_{1\leq i,j\leq n}y_iy_j\min(b_i,b_j),\ \int_0^\infty f(x)g(x)\mathrm{d}x=\sum_{1\leq i,j\leq n}x_iy_j\min(a_i,b_j).$$

Finally, our inequality reduces to

$$\int_0^\infty f^2(x)\mathrm{d}x\int_0^\infty g^2(x)\mathrm{d}x\geq\left(\int_0^\infty f(x)g(x)\mathrm{d}x\right)^2,$$

which is nothing else than the celebrated integral version of the *Cauchy-Schwarz* inequality. $\square$

297. Let S^2 be the bidimensional sphere and $\alpha > 0$. Show that for any positive integer n and $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n$ and $C_1, C_2, \ldots, C_n$ arbitrary points on S^2, there exists $P_n \in S^2$ such that

$$\sum_{i=1}^{n} P_nA_i^{\alpha} = \sum_{i=1}^{n} P_nB_i^{\alpha} = \sum_{i=1}^{n} P_nC_i^{\alpha}$$

if and only if $\alpha = 2$.

Proposed by Marius Cavachi, Ovidius University of Constanţa, Constanţa, Romania.

Solution by the author. We will divide the proof in two cases $\alpha = 2$ and $\alpha \neq 2$. In the first case let us define the function $f : S^2 \to \mathbb{R}^2$,

$$f(P) = \left(\sum_{i=1}^{n} PA_i^2 - \sum_{i=1}^{n} PB_i^2, \sum_{i=1}^{n} PA_i^2 - \sum_{i=1}^{n} PC_i^2 \right).$$

Since $f(-P) = -f(P)$, $\forall P \in S^2$, according to *Borsuk-Ulam*' s Theorem, there is a $P_n \in S^2$ such that $f(P_n) = (0, 0)$.

We can provide an elementary argument for this fact, as it follows.

Let H_1, H_2 be the geometrical locus of the points P in space for which $\sum_{i=1}^{n} PA_i^2 = \sum_{i=1}^{n} PB_i^2$, respectively $\sum_{i=1}^{n} PA_i^2 = \sum_{i=1}^{n} PC_i^2$. H_1 and H_2 are planes which both pass trough the center of the sphere. In consequence, $H_1 \cap H_2$ contains a line, which passes through the center of the sphere, and we can choose P_n as one of its intersections with the surface of the sphere.

For the second case, when $\alpha \neq 2$, we will use standard cartesian coordinates, and let $N = (0, 0, 1)$ and $S = (0, 0, -1)$, the standard north and south pole of the sphere. We denote with S^1 the standard plane section given by the ecuation $z = 0$.

For n arbitrary and $i \in \{1, 2, \ldots, n\}$ we choose $A_i = N$, $B_i = S$, and C_i are the vertices of a regular polygon with n sides circumscribed around S_1. A point P_n which satisfies the statement of the problem must be on S^1 and satisfies $\sum_{i=1}^{n} P_nC_i^{\alpha} = n(\sqrt{2})^{\alpha}$.

Also, since $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} P_nC_i^{\alpha} = \frac{1}{2\pi} \int\limits_{P\in S^1} QP^{\alpha} \text{ ds} = I$, where Q is for example the point $(1, 0, 0)$, we deduce that

$$I = (\sqrt{2})^{\alpha}. \tag{1}$$

Now

$$I = \frac{1}{2\pi} \int_0^{2\pi} \left(2 \sin\left(\frac{t}{2}\right) \right)^{\alpha} \text{ dt} = \frac{2^{\alpha}}{\pi} \int_0^{\pi} \sin^{\alpha} u \text{ du} =$$

$$= \frac{2^\alpha}{\pi} \int_0^{\frac{\pi}{2}} (\sin^\alpha v + \cos^\alpha v)\ \mathrm{dv} = \frac{2^\alpha}{2} (\sin^\alpha v_0 + \cos^\alpha v_0),$$

using the mean theorem, and $v_0 \in \left(0; \frac{\pi}{2}\right)$.

We deduce that

$$\frac{I}{2^{\frac{\alpha}{2}}} = \frac{1}{2} \left[(2\sin^2(v_0))^{\frac{\alpha}{2}} + (2\cos^2(v_0))^{\frac{\alpha}{2}} \right] = \frac{1}{2}[(1+w)^\beta + (1-w)^\beta],$$

unde $\beta = \frac{\alpha}{2}$, $w \in (0; 1)$.

If $\alpha > 2$, then $\beta > 1$, and using the Bernoulli inequality we have $(1+w)^\beta + (1-w)^\beta > 1 + \beta w + 1 - \beta w > 2$. Then $\frac{I}{2^{\frac{\alpha}{2}}} > \frac{1}{2} \cdot 2 = 1$, a contradiction with (1).

Similarly for $\alpha \in (0; 2)$, we obtain the contradiction $\frac{I}{2^{\frac{\alpha}{2}}} < 1$. $\square$

Remark. *Marius Olteanu* had a geometrical approach based on Leibniz relations.

298. Let t be an odd number. Find all monic polynomials $P \in \mathbb{Z}[X]$ such that for all integers n there exists an integer m for which

$$P(m) + P(n) = t.$$

Proposed by Octavian Ganea, student École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland and Cristian Tălău, student Polytechnic University of Bucharest, Bucharest, Romania.

Solution by the authors. Firstly we prove that the degree of P is odd. If P would have even degree, then it would be bounded from below, but from our statement taking $n \to \infty$ we have that $\lim_{n\to+\infty} P(n) = +\infty$, it follows that there is a sequence of values of m for which $P(m) \to -\infty$, which contradicts the fact that P is bounded from below.

So we have proven that degree of P is odd. Now let M be large enough such that P is strictly increasing on $[M, +\infty)$ (it is possible since P' has a finite number of roots and $\lim_{x\to+\infty} P'(x) = +\infty$). So from our hypothesis, it follows that P is strictly increasing on $(-\infty, N]$ $(*)$, where N is such $P(M) + P(N) = t$.

Let $m \geq M$ and $n \in \mathbb{Z}$ such that $P(m) + P(-n) = t$. From the hypothesis there is a $k \in \mathbb{Z}$ such that $P(m+1) + P(-n-k) = t$. It is clear from $(*)$ that $k \geq 1$. Also from the hypothesis there is a $r \geq 1$ with the property that $P(m+r) + P(-n-1) = t$. If $r \geq 2$ or $k \geq 2$, then $P(m+1) + P(-n-k) < P(m+r) + P(-n-1)$, a contradiction. Thus $P(m+1) + P(-n-1) = t$, so by induction $P(m+k) + P(-n-k) = t$, $\forall k \in \mathbb{N}$.

Let $a = m - n$ and $Q(X) = P\left(\frac{a}{2} + X\right) + P\left(\frac{a}{2} - X\right) - t$. Let us notice that it has infinitely many roots of the form $x = \frac{m+n}{2} + k$, $\forall\, k \in \mathbb{N}$. It follows that $Q = 0$. For $x = 0$ we have $P\left(\frac{a}{2}\right) = \frac{t}{2} \notin \mathbb{Z}$, so a is odd. Let us notice that if we denote $k = \deg(P)$ we have $2^{k-1}P\left(\frac{a}{2}\right) = 2^{k-2}t$ and for $k \geq 3$ we know that $2^{k-2}t \in \mathbb{Z}$, so $2^{k-1}P\left(\frac{a}{2}\right) \in \mathbb{Z}$ and if we write the expression of P, it would follow that $\frac{a^k}{2} \in \mathbb{Z}$, which is in contradiction with a odd. Thus $k = 1$ so $P(X) = X + b$, which verifies the hypothesis. $\square$

299. Find all functions $f : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$ satisfying the following properties:
i) $a - b$ divides $f(a) - f(b)$ for all $a, b \in \{1, 2, \ldots\}$;
ii) if a, b are relatively prime, so are $f(a)$ and $f(b)$.

Proposed by Gabriel Dospinescu, École Normale Supérieure de Paris, Paris, France and Fedor Nazarov, University of Wisconsin, Madison, WI, USA.

Solution by the authors. First, we claim that any prime factor of $f(n)$ divides n. Assume that p divides $f(n)$ and does not divide n. Since p divides $f(n+p) - f(n)$, p also divides $f(n+p)$, so it divides $\gcd(f(n), f(n+p))$. As $\gcd(n, n+p) = 1$, we also have $\gcd(f(n), f(n+p)) = 1$, which makes the last result impossible. The first claim is thus proved.

Write $f(p) = p^{g(p)}$ for each prime p, for some $g(p) \geq 0$. We will prove that g is constant. Fix odd primes p, q and a positive integer m and define $u = \frac{p^{2^m}+1}{2}$ and $v = \frac{q^{2^n}+1}{2}$ for some n such that $\gcd(u, v) = 1$ (this is possible since classically any prime factor of v is at least 2^n, so if n is large enough, v will be relatively prime to u). The Chinese Remainder Theorem combined with *Dirichlet*'s theorem give us a prime r such that $r \equiv p \pmod{u}$, $r \equiv q \pmod{v}$. Therefore, u divides $f(r) - f(p) = r^{g(r)} - p^{g(p)}$. Since u also divides $r^{g(r)} - p^{g(r)}$, it follows that u divides $p^{|g(r)-g(p)|} - 1$. Recalling the definition of u, it immediately follows that 2^{m+1} divides $g(r) - g(p)$, and doing the same with v yields that 2^{m+1} also divides $g(q) - g(r)$, so that 2^{m+1} divides $g(p) - g(q)$. Since m was arbitrary, we must have that g is constant, say $g(p) = d$ for any prime p.

Finally, if n is a positive integer we have $n - p | f(n) - f(p) = f(n) - p^d$ and $n - p | n^d - p^d$, so that $n - p | f(n) - n^d$. Since this holds for any prime p, we must have $f(n) = n^d$ for all n. Obviously, all these functions are solutions to the problem. $\square$

300. Consider the sequence $(a_n)_{n\geq 1}$ defined by $a_1 = 2$ and $a_{n+1} = 2a_n + \sqrt{3(a_n^2-1)}$, $n \geq 1$. Show that the terms of a_n are positive integers and any odd prime p divides $a_p - 2$.

Proposed by Alin Gălăţan, student University of Bucharest and Cezar Lupu, student University of Bucharest, Bucharest, Romania.

Solution by the authors. It is well-known that the function $\mathrm{ch} : \mathbb{R} \to [1,\infty)$, defined by $\mathrm{ch}(x) = \dfrac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}$, is onto and thus there exists $\alpha \geq 1$ such that $\mathrm{ch}(\alpha) = 2$, which implies that $\mathrm{sh}(\alpha) = \sqrt{3}$, with $\mathrm{sh} : \mathbb{R} \to [1,\infty)$ defined by $\mathrm{sh}(x) = \dfrac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}$. It follows that

$$a_{n+1} = \mathrm{ch}(\alpha)a_n + \mathrm{sh}(\alpha)\sqrt{a_n^2-1}, \quad \forall\, n \geq 1.$$

We shall prove by induction that $a_n = \mathrm{ch}(n\alpha)$, $\forall n \geq 1$. The case $n = 1$ is obvious. We assume that $a_n = \mathrm{ch}(n\alpha)$ for some $n \geq 1$ and we prove that $a_{n+1} = \mathrm{ch}((n+1)\alpha)$. Indeed, since $\mathrm{ch}(n\alpha+\alpha) = \mathrm{ch}\,\alpha\cdot\mathrm{ch}(n\alpha) + \mathrm{sh}\,\alpha\cdot\mathrm{sh}(n\alpha)$, $\forall n \geq 1$, we obtain that $a_n = \mathrm{ch}(n\alpha)$, $\forall n \geq 1$.

In what follows, we prove by induction that all terms a_n are positive integers. Indeed, we assume that for some $n \geq 2$ one has $a_k \in \mathbb{N}$ for all integers $1 \leq k < n$, and we prove that $a_n \in \mathbb{N}$, too. Assume by contradiction that $a_n \notin \mathbb{N}^*$. This means that there exists a square-free number $d > 1$ such that $a_n = M + N\sqrt{d}$, so that $a_n \notin \mathbb{Q}$. We have

$$2^{2n} = \mathrm{ch}^n(\alpha) = \left(\frac{e^\alpha + e^{-\alpha}}{2}\right)^n,$$

which is equivalent to

$$2^{2n} = \sum_{k=0}^{n}\binom{n}{k}e^{(n-k)\alpha - k\alpha} = \sum_{k=0}^{n}\binom{n}{k}\left(e^{(n-2k)\alpha} + e^{-(n-2k)\alpha}\right) = \sum_{k=0}^{n}\binom{n}{k}a_{|n-2k|}.$$

Since a_k is integer for all $k < n$, it follows that $a_n \in \mathbb{Q}$, which contradicts the initial assumption. Thus we get $a_n \in \mathbb{N}$ for all $n \geq 1$.

Finally, consider a prime number $p > 2$. We have

$$4^p = \sum_{k=0}^{p}\binom{k}{p}\cdot e^{(p-2k)\alpha} = \sum_{k=0}^{\frac{p-1}{2}}\binom{p}{k}\left(e^{(p-2k)\alpha} + e^{-(p-2k)\alpha}\right) = \sum_{k=0}^{\frac{p-1}{2}}\binom{p}{k}\cdot 2a_{p-2k}.$$

Since p divides $\dbinom{p}{k}$ and, from *Fermat*'s Little Theorem, $4^p \equiv 4 \pmod p$, it follows that $2 \equiv a_p \pmod p$ and hence the conclusion follows immediately having in view that p is odd. $\square$

Solution by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bârlad, Romania. That a_n is positive for every $n \geq 1$ immediately follows by induction, and this readily implies $a_{n+1} > a_n$ for all n (thus, the sequence is strictly increasing).

By squaring $a_{n+1}-2a_n = \sqrt{3(a_n^2-1)}$ one gets $a_{n+1}^2-4a_{n+1}a_n+a_n^2 = -3$ for all positive integers n; therefore,

$$a_{n+1}^2 - 4a_{n+1}a_n + a_n^2 = a_n^2 - 4a_na_{n-1} + a_{n-1}^2, \ \ \forall\, n \geq 2.$$

This one can be read as

$$(a_{n+1} - a_{n-1})(a_{n+1} + a_{n-1} - 4a_n) = 0,$$

hence the terms of $(a_n)_{n\geq 1}$ also verify the recurrence

$$a_{n+1} - 4a_n + a_{n-1} = 0, \ \ \forall\, n \geq 2$$

(as $a_{n+1} - a_{n-1}$ is always nonzero – actually positive). A canonical induction shows now that a_n is an integer for each $n \geq 1$ (as it starts with $a_1 = 2$, $a_2 = 7$, and $a_{n+1} = 4a_n - a_{n-1}$, for $n \geq 2$).

Also canonical is to find the formula of the general term; we have

$$a_n = \frac{1}{2}\left(\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n\right), \ \ \forall\, n \geq 1.$$

Now we have the following general result: if $x_1, x_2, \ldots, x_m$ are the zeros of a monic polynomial with integer coefficients, and p is a positive prime number, then

$$(x_1 + x_2 + \cdots + x_m)^p - (x_1^p + x_2^p + \cdots + x_m^p)$$

is an integer which is divisible by p.

Indeed, the first part follows easily via the fundamental theorem of symmetric polynomials. By the multinomial formula and the fact that each multinomial coefficient $\dfrac{p!}{i_1!i_2!\cdots i_m!}$ is divisible by p whenever none of $i_1, i_2, \ldots, i_m$ equals p,

$$\frac{1}{p}\left((x_1 + x_2 + \cdots + x_m)^p - (x_1^p + x_2^p + \cdots + x_m^p)\right)$$

is a sum of products of powers of $x_1, x_2, \ldots, x_m$ with integer coefficients, therefore it is an algebraic integer. Being rational and algebraic integer, the above number is an integer, which was to be proven.

(See also *Example 6* at page 191 of *Problems from the Book* by T. Andreescu and G. Dospinescu, XYZ Press, 2008. And note that yet another beautiful solution, which uses matrices, is given to that problem starting with page 192.)

In our case one can consider $x_1 = 2+\sqrt{3}$, and $x_2 = 2-\sqrt{3}$; the above theorem implies that

$$4^p - \left(\left(2+\sqrt{3}\right)^p + \left(2-\sqrt{3}\right)^p\right)$$

is divisible by p for every positive prime number p. If p is odd, from here we infer that

$$x_p - 2^{2p-1} = \frac{1}{2}\left(\left(2+\sqrt{3}\right)^p + \left(2-\sqrt{3}\right)^p\right) - 2^{2p-1}$$

is divisible by p. Then

$$x_p - 2 = \left(x_p - 2^{2p-1}\right) + \left(2^{2p-1} - 2\right)$$

is also divisible by p, because $2^{2p-1} - 2$ is divisible by p, too, according to *Fermat*'s Little Theorem (which is, somehow, generalized by the above mentioned result).

Note that the fact that $x_p - 2$ is divisible by p can also be obtained by using the binomial development in the formula for x_p. □

Remark. Similar solutions with the second approach were also given by *Nicuşor Minculete, Braşov* and *Marius Olteanu, Râmnicu Vâlcea.*

301. Determine the greatest prime number $p = p(n)$ such that there exists a matrix $X \in \mathrm{SL}(n, \mathbb{Z})$ with $X^p = I_n$ and $X \neq I_n$.

Proposed by Victor Vuletescu, University of Bucharest, Bucharest, Romania.

Solution by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bârlad, Romania. We show that $p(n)$ is the greatest prime which does not exceed $n + 1$.

First we prove that $p(n) \leq n + 1$. Suppose p is a prime, and that there exists $X \in SL(n, \mathbb{Z})$ with $X^p = I_n$. Thus X is a root of the polynomial

$$T^p - 1 = (T - 1)(T^{p-1} + \cdots + T + 1)$$

and then the minimal polynomial of X over $\mathbb{Q}$ is one of $T-1$, $T^{p-1}+\cdots+T+1$ (both factors are irreducible), or $T^p - 1$ itself. Since we look for $X \neq I_n$, the first case is excluded. Because the degree of the minimal polynomial of a matrix of order n is at most n, we find that either $p - 1 \leq n$ or $p \leq n$; in both cases $p \leq n + 1$ follows.

Now we show that for $p = p(n)$, the greatest prime which does not exceed $n + 1$, there is a matrix $X \in SL_n(\mathbb{Z})$ with $X^p = I_n$. As $p - 1 \leq n$, we can consider $X = (x_{ij})_{1\leq i,j\leq n}$ with $x_{11} = x_{12} = \cdots = x_{1,p-1} = -1$, $x_{21} = x_{32} = \cdots = x_{p-1,p-2} = 1$, and all other entries equal to 0. For this matrix we have $X^{p-1} + \cdots + X + I_n = 0_n$ (it is built starting from the companion matrix of $T^{p-1}+\cdots+T+1$), therefore $X^p - I_n = 0_n \Leftrightarrow X^p = I_n$, too (just multiply the previous equality with $X - I_n$).

Note that when $p(n) \leq n$ (that is, when $n + 1$ is not a prime), a permutation matrix corresponding to a cycle of order p can be used as yet an example. □

302. Let $n \geq 2$ and denote by $\mathcal{D} \subset M_n(\mathbb{C})$ and $\mathcal{C} \subset M_n(\mathbb{C})$ the set of diagonal and circulant matrices, respectively. Consider $\mathcal{V} = [D, C] =$

$= \operatorname{span}\{dc - cd \mid d \in \mathcal{D}, c \in \mathcal{C}\}$. Prove that $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{V}$ span $M_n(\mathbb{C})$ if and only if n is prime.

Proposed by Remus Nicoară, University of Tennessee, Knoxville, TN, USA.

Solution by the author. We begin with a standard notation; $E_{i,j}$ is the matrix with 0 or 1 as entries and which has the value 1 only on the position (i, j). Let us note firstly that the set $\mathcal{D}$ is generated by the matrices $E_{h,h} = D_h$, where $1 \leq h \leq n$. The set $\mathcal{C}$ is generated by the matrices X^l, with $l \leq n$, where $X = \begin{pmatrix} 0 & 1 & 0 & ... & ... & 0 \\ 0 & 0 & 1 & ... & .... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 & 0 \\ 1 & 0 & 0 & ... & ... & 0 \end{pmatrix}$, so $X^l = \sum_{i=1}^{n} E_{i,i+l}$, where the indices are taken modulo n.

For $1 \leq k \leq n$ and $l \leq n$, we have that $[D_k, X^l] = E_{k,k+l} - E_{k-l,k}$ generate the space $\mathcal{V}$. It is easy to verify that $\mathcal{C}, \mathcal{D} \perp \mathcal{V}$, where the scalar product is given by $< X, Y >= \operatorname{Tr}(\overline{Y^t} X)$.

Since $\operatorname{span}(\mathcal{C}, \mathcal{D})$ has dimension $2n - 1$, it follows that $\operatorname{span}(\mathcal{C}, \mathcal{D}, \mathcal{V}) = M_n(\mathbb{C})$ if and only if $\dim_{\mathbb{C}} V = (n-1)^2$.

Let $V_{h,l} = E_{h,h+l} - E_{h-l,h}$. They span the space $\mathcal{V}$. We must look at the dependence relations between $V_{h,l}$. Let $c_{h,l} \in \mathbb{C}$ such that $\sum_{h,l} c_{h,l} V_{h,l} = O_n$. This is equivalent to $\sum_{i,j} (c_{i,j-i} - c_{j-i,j}) E_{i,j} = O_n$, so $c_{i,j-i} = c_{j-i,j}$.

If we denote $d = j - i$, this means $c_{i,d} = c_{i+d,d}$, for any i, d, so we deduce $c_{i,d} = c_{i+md,d}$, for any $m \geq 1$. Now if $d \neq 0$ and n is prime the set $\{i + md\}$, where $m \geq 1$, contains all the residues modulo n, besides i, and since $i \neq j$ we can find an m_0 such that $i + m_0 d \equiv j \pmod{p}$. Thus $c_{i,d} = c_{j,d}$, $\forall i, j, d \neq 0$.

Now let us look at the linear transformation $c_{h,l} \to \sum_{h,l} c_{h,l} V_{h,l}$. From what we obtained, for n prime, we know that its kernel is determined, on the first $n - 1$ columns, by the values on the first row, and the last column is arbitrary. Thus we have $2n - 1$ „degrees of freedom“, so the kernel has dimension $2n - 1$. This means that the image has dimension $n^2 - 2n + 1 = (n-1)^2$.

Finally, let us see what happens when n is not prime. We prove that the kernel has a bigger dimension, and the proof ends. This is easy to see since $\{i + md\}$ cannot span the whole residues when d is not coprime with n, and we can choose such a $d \neq 0$ and $d < n$. In this case on the d-th column we have at least one more „degree of freedom“ so the dimension of the kernel is bigger than $2n - 1$. $\square$

303. Prove that

$$\sum_{n=0}^{\infty} \frac{1}{\Gamma(n+3/2)} = \frac{2e}{\sqrt{\pi}} \int_0^1 e^{-x^2} \mathrm{d}x$$

and

$$\sum_{n=0}^{\infty} \frac{\Gamma(-n+1/2)\,\Gamma(n+1/2)}{\Gamma(n+3/2)} = \frac{\sqrt{\pi}}{2e} \int_0^1 e^{x^2} \mathrm{d}x,$$

where $\Gamma(x) = \int\limits_0^{\infty} \mathrm{e}^{-t} t^{x-1} \mathrm{d}t$ is *Euler*'s Gamma function.

`Proposed by Cezar Lupu, student University of Bucharest, Bucharest and Tudorel Lupu, Decebal High School, Constanţa,Romania.`

Solution by the authors. On the first serie, we use Series expansion of Incomplete Gamma function, namely

$$\Gamma(a-1) - \Gamma(a-1,x) = x^a \cdot \left(\frac{1}{(a-1)x} - \frac{1}{a} + \frac{x}{2(a+1)} - \frac{x^2}{6(a+2)} + \right. \\ \left. + \frac{x^3}{24(a+3)} - \frac{x^4}{120(a+4)} + \frac{x^5}{720(a+5)} + O\left(x^6\right) \right).$$

We multiply the equation by $x^{(1-a)}$ to make things more polynomial-like and thus we have

$$x^{1-a}\big(\Gamma(a-1) - \Gamma(a-1,x)\big) = \frac{1}{a-1} - \frac{x}{a} + \frac{x^2}{2(a+1)} - \frac{x^3}{6(a+2)} \\ + \frac{x^4}{24(a+3)} - \frac{x^5}{120(a+4)} + O\left(x^6\right).$$

The magic happens when multiplying by e^x, it will make a translation in the x terms, or in series form

$$\left(\sum_{j=0}^{\infty} \frac{x^j}{j!} \right) \left(\sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!(a+i-1)} \right) = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{(-1)^i x^k}{i!(a+i-1)(k-i)!}.$$

So, we obtain

$$\mathrm{e}^x x^{1-a}\big(\Gamma(a-1) - \Gamma(a-1,x)\big) = \frac{1}{a-1} + \frac{x}{(a-1)a} + \frac{x^2}{(a-1)a(a+1)} + \\ + \frac{x^3}{(a-1)a(a+1)(a+2)} + \frac{x^4}{(a-1)a(a+1)(a+2)(a+3)} + O\left(x^5\right).$$

Finally eliminating the term $a-1$, we have

$$(a-1)e^x x^{1-a}\left(\Gamma(a-1) - \Gamma(a-1,x)\right) = 1 + \frac{x}{a} + \frac{x^2}{a(a+1)} + \frac{x^3}{a(a+1)(a+2)} +$$

$$+\frac{x^3}{a(a+1)(a+2)}+\frac{x^4}{a(a+1)(a+2)(a+3)}+O\left(x^5\right)=\sum_{i=0}^{\infty}\frac{x^i(a-1)!}{(i+a-1)!}.$$

Dividing by $\Gamma(a)$ and using Gamma function instead of factorial, we deduce

$$\sum_{i=0}^{\infty}\frac{x^i}{\Gamma(a+i)}=\frac{(-1+a)\mathrm{e}^x x^{1-a}\big(\Gamma(-1+a)-\Gamma(-1+a,x)\big)}{\Gamma(a)}.$$

The left and right hand side are valid for any positive real number a. Making $x=1$, $a=\frac{3}{2}$ and using erf function, i.e., $\operatorname{erf}(x)=\frac{2}{\pi}\int\limits_0^x e^{-t^2}\mathrm{d}t$, instead of the incomplete Gamma, we conclude the solution.

For the second series, we just have to put $x\to -x$ and the solution is almost the same as above. □

Remark. A similar solution, but a little bit more laborious, was given by *Marius Olteanu, Râmnicu Vâlcea.*

304. Let $f,g:[0,1]\to\mathbb{R}$ be two functions such that f is continuous and g is increasing and differentiable, with $g(0)\geq 0$. Prove that if for any $t\in[0,1]$

$$\int\limits_t^1 f(x)\mathrm{d}x\geq\int\limits_t^1 g(x)\mathrm{d}x,$$

then

$$\int\limits_0^1 f^2(x)\mathrm{d}x\geq\int\limits_0^1 g^2(x)\mathrm{d}x.$$

`Proposed by Andrei Ciupan, student Harvard University, Boston, MA, USA.`

Solution by the author. Let F and G be fixed, but otherwise arbitrary, primitives of f and g, respectively. From the AM-GM inequality we have

$$\int\limits_0^1 f^2(x)\mathrm{d}x+\int\limits_0^1 g^2(x)\mathrm{d}x\geq 2\cdot\int\limits_0^1 f(x)g(x)\mathrm{d}x. \tag{1}$$

By integration by parts we obtain

$$\int\limits_0^1 f(x)g(x)\mathrm{d}x=F(x)\cdot g(x)\Big|_0^1-\int\limits_0^1 F(x)g'(x)\mathrm{d}x. \tag{2}$$

The hypothesis tells us that $F(1)-F(x)\geq G(1)-G(x)$ for any $x\in[0,1]$, which can be rewritten as $-F(x)\geq G(1)-F(1)-G(x)$. Since g

is differentiable and increasing, we have $g'(x) \geq 0$ for any x in $[0,1]$. From these two facts we obtain

$$-\int_0^1 F(x)g'(x)\mathrm{d}x \geq \int_0^1 (G(1)-F(1))\,g'(x)\mathrm{d}x - \int_0^1 G(x)g'(x)\mathrm{d}x \Leftrightarrow$$

$$\Leftrightarrow -\int_0^1 F(x)g'(x)\mathrm{d}x \geq (G(1)-F(1))(g(1)-g(0)) - G(x)\cdot g(x)\Big|_0^1 + \int_0^1 g^2(x)\mathrm{d}x.$$

By combining this last relation with (2), we obtain

$$\int_0^1 f(x)g(x)\mathrm{d}x \geq F(1)g(1) - F(0)g(0) + (G(1)-F(1))\,(g(1)-g(0))$$

$$-G(1)g(1) + G(0)g(0) + \int_0^1 g^2(x)\mathrm{d}x.$$

By reducing and grouping, we finally obtain

$$\int_0^1 f(x)g(x)\mathrm{d}x \geq g(0)\,(F(1)-F(0)+G(0)-G(1)) + \int_0^1 g^2(x)\mathrm{d}x.$$

Therefore, by taking into account relation (1), we obtain

$$\int_0^1 f^2(x)\mathrm{d}x \geq \int_0^1 g^2(x)\mathrm{d}x + 2g(0)\left(\int_0^1 f(x)\mathrm{d}x - \int_0^1 g(x)\mathrm{d}x\right) \geq \int_0^1 g^2(x)\mathrm{d}x,$$

since $g(0) \geq 0$ and $\displaystyle\int_0^1 f(x)\mathrm{d}x \geq \int_0^1 g(x)\mathrm{d}x$. □

Remark. A similar solution, but a little bit more complicated, was given by *Marius Olteanu, Râmnicu Vâlcea.*

305. Let K be a field and $f \in K[X]$ with $\deg(f) = n \geq 1$ having distinct roots $x_1, x_2, \ldots, x_n$. For $p \in \{1,2,\ldots,n\}$ let $S_1, S_2, \ldots, S_p$ be the symmetric fundamental polynomials in $x_1, x_2, \ldots, x_p$. Show that

$$[K(S_1, S_2, \ldots, S_p) : K] \leq \binom{n}{p}.$$

Proposed by Marius Cavachi, Ovidius University of Constanţa, Constanţa, Romania.

Solution by the author. We consider firstly the following fields, $F = K(S_1, \ldots, S_p)$, $L = K(x_1, \ldots, x_p)$, $M = K(x_1, x_2, \ldots, x_n)$. We have the inclusions $K \subset F \subset L \subset M$. The extension $K \hookrightarrow M$ is *Galois* and let

$G = \mathrm{Gal}(M/K)$ be the *Galois* group, and finally let us denote with G_1 the image of the group $\mathrm{Gal}(M/F)$ through the canonical inclusion $\mathrm{Gal}(M/F) \hookrightarrow G$.

The group G acts on the set of subfields of M and $\left|\dfrac{G}{\mathrm{Stab}(L)}\right| = |\mathrm{Orb}(L)|$. Since for $\sigma \in G$, $\sigma(L)$ is uniquely determined by the subset $\{\sigma(x_1), \ldots, \sigma(x_p)\}$ of $\{x_1, \ldots, x_n\}$, we deduce that $|\operatorname{Orb}(L)| \leq \dbinom{n}{p}$.

On the other hand, $\sigma \in \mathrm{Stab}(L)$ is equivalent to $\{\sigma(x_1), \ldots, \sigma(x_p)\} =$ $= \{x_1, \ldots, x_n\}$, which is further equivalent to $(X - \sigma(x_1)) \ldots (X - \sigma(x_p)) =$ $= (X - x_1) \ldots (X - x_p)$. Thus $\{\sigma(S_1), \ldots, \sigma(S_p)\} = \{S_1, \ldots, S_p\}$ so $\sigma(F) = F$, hence $\sigma \in G_1$. We conclude that $|G/\operatorname{Stab}(L)| = |G/G_1| = [F : K]$.
Thus, it follows that $[F : K] \leq \dbinom{n}{p}$.

Solution by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bârlad, Romania. For $1 \leq i_1 < i_2 < \cdots < i_p \leq n$ let us denote by

$$S_1(x_{i_1}, x_{i_2}, \ldots, x_{i_p}),\ S_2(x_{i_1}, x_{i_2}, \ldots, x_{i_p}), \ldots,\ S_p(x_{i_1}, x_{i_2}, \ldots, x_{i_p})$$

the symmetric fundamental polynomials in $x_{i_1}, x_{i_2}, \ldots, x_{i_p}$.

Thus, $S_1 = S_1(x_1, x_2, \ldots, x_p)$ and so on.

Let x be an arbitrary element from $K(S_1, S_2, \ldots, S_n)$; that is,

$$x = \sum a_{j_1 j_2 \ldots j_p} S_1^{j_1} S_2^{j_2} \cdots S_p^{j_p},$$

for some $a_{j_1 j_2 \ldots j_p} \in K$ (the sum being extended over a finite number of indices $0 \leq j_1 \leq n$, $0 \leq j_2 \leq n$, $\ldots$, $0 \leq j_p \leq n$). Denote this x by $x = x_{12\ldots p}$ and define, similarly, $x_{i_1 i_2 \ldots i_p}$ to be

$$\sum a_{j_1 j_2 \ldots j_p} \left(S_1\left(x_{i_1}, x_{i_2}, \ldots, x_{i_p}\right)\right)^{j_1} \left(S_2\left(x_{i_1}, x_{i_2}, \ldots, x_{i_p}\right)\right)^{j_2} \cdots$$
$$\cdots \left(S_p\left(x_{i_1}, x_{i_2}, \ldots, x_{i_p}\right)\right)^{j_p}$$

with the same coefficients as in x, for all $1 \leq i_1 < i_2 < \cdots < i_p \leq n$.

Now observe that the polynomial

$$f = \prod_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} (X - x_{i_1 i_2 \ldots i_p})$$

has degree $\dbinom{n}{p}$, has coefficients in K (due to the fundamental theorem of symmetric polynomials), and has $x = x_{12\ldots p}$ as a root.

Thus, any element of the algebraic extension $K \subseteq K(S_1, S_2, \ldots, S_p)$ has degree over K at most $\dbinom{n}{p}$. This means that the degree of this extension is at most $\dbinom{n}{p}$, too, finishing the proof. $\square$

Remark. It seems that the condition about $x_1, x_2, \dots, x_n$ to be distinct is unnecessary.

306. Let $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ be positive real numbers such that $x_1 + x_2 + \cdots + x_n = 1$, and denote by $\mathcal{A}$ the set of pairs (i, j) such that $y_1 + \cdots + y_j \le x_1 + \cdots + x_i$. Prove that one has

$$\sum_{(i,j)\in\mathcal{A}} \frac{x_{i+1}y_j}{1 + (x_1 + \cdots + x_{i+1})(y_1 + \cdots + y_j)} \le \frac{\pi^2}{24}.$$

Prove that the constant $\dfrac{\pi^2}{24}$ is the best satisfying the property.

`Proposed by Radu Gologan, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.`

Solution by the author. In the Cartesian plane consider the triangle T with vertices $O(0,0)$, $A(1,0)$, $B(1,1)$. For $(i,j) \in \mathcal{A}$ put $a_i = x_1 + \cdots + x_i$, $b_j = y_1 + \cdots + y_j$, and denote by R_{ij} the rectangle with vertices of coordinates

$$(a_i, b_{j-1}),\ (a_i, b_j),\ (a_{i+1}, b_{j-1}),\ (a_{i+1}, b_j).$$

It is clear that $\bigcup\limits_{\mathcal{A}} R_{ij}$ is contained in T and the rectangles have disjoint interiors.

Then the lower *Darboux* sum for $f(x,y) = \dfrac{1}{1+xy}$ on T is less than

$$\sum_{R_{ij}} \min_{R_{ij}} f(x,y) x_{i+1} y_j \ge \sum_{(i,j)\in\mathcal{A}} \frac{x_{i+1}y_j}{1 + (x_1 + \cdots + x_{i+1})(y_1 + \cdots + y_j)}$$

by the fact that f is decreasing in each variable.

In conclusion, the last sum is less than the double integral

$$c = \iint\limits_T \frac{1}{1+xy} \mathrm{d}x\mathrm{d}y.$$

By *Fubini* calculation $c = \displaystyle\int_0^1 \frac{\ln(1+x^2)}{x}\mathrm{d}x$, so the existence part of the result is proven. To prove that $c = \dfrac{\pi^2}{24}$, use the power series for $\ln(1+t)$, and the uniform convergence by the Abel theorem, on $[0,1]$. Thus

$$\int_0^1 \frac{\ln(1+x^2)}{x}\mathrm{d}x = \int_0^1 \sum_{n=1}^{\infty} (-1)^{n+1} x^n \mathrm{d}x = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = \frac{\pi^2}{24},$$

concluding the proof.

To prove the fact that the found constant is the best, it suffices to consider in the inequality, for every n, the numbers $x_i = \frac{1}{n} = y_j$, and remark that the sum in the inequality is a Riemannian sum for the double integral, so for n going to infinity the sum tends to $\frac{\pi^2}{24}$. □

307. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable with f'' continuous and $\lim_{x\to\pm\infty} f(x) = \infty$ such that $f(x) > f''(x)$ for all $x \in \mathbb{R}$. Show that $f(x) > 0$ for any real number x.

`Proposed by Beniamin Bogoşel, student West University of Timişoara, Timişoara, Romania.`

Solution by Richard Stevens, Columbus State University, GA, USA. This solution follows also in large lines the solution of the author. Suppose that $f(a) \leq 0$ for some real number a. From the given conditions we see that there is an interval containing a for which f has a minimum value that does not occur at either end point of the interval. Denoting this minimum point as $(b, f(b))$, it follows that $f(b) \leq 0$, $f'(b) = 0$ and $f''(b) = \lim_{x\to b} \frac{f'(x)}{x-b} < 0$. Thus, for some $\varepsilon > 0$, $f'(x) < 0$ for $b < x < b + \varepsilon$ and $f'(x) > 0$ for $b - \varepsilon < x < b$. This indicates that $(b, f(b))$ is a relative maximum point and that f is not constant on an interval containing b. Therefore, $f(x) > 0$ for all x. □

308. Let $M_n(\mathbb{Q})$ be the ring of square matrices of size n and $X \in M_n(\mathbb{Q})$. Define the adjugate (classical adjoint) of X, denoted $\mathrm{adj}(X)$, as follows. The (i, j)-minor M_{ij} of X is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j of X, and the (i, j) − cofactor of X is $C_{ij} = (-1)^{i+j} M_{ij}$. The adjugate of X is the transpose of the 'cofactor matrix' C_{ij} of X. Consider $A, B \in M_n(\mathbb{Q})$ such that

$$(\mathrm{adj}(A))^3 - (\mathrm{adj}(B))^3 = 2((\mathrm{adj}(A)) - (\mathrm{adj}(B))) \neq O_n.$$

Show that

$$\mathrm{rank}(AB) \in \{\mathrm{rank}(A), \mathrm{rank}(B)\}.$$

`Proposed by Flavian Georgescu, student University of Bucharest, Bucharest, Romania.`

Solution by the author. We have to prove that either A or B is invertible. We shall argue by contradiction, so assume neither is invertible.

Firstly let us note that for non-invertible matrix $Y \in M_n(\mathbb{Q})$, we have $\mathrm{rank}(\mathrm{adj}(Y)) \in \{0, 1\}$. This is true since if $\mathrm{rank}(Y) \leq n-2$, then $\mathrm{adj}(Y) = O_n$, otherwise if $\mathrm{rank}(Y) = n-1$, using *Sylvester*'s inequality we have

$$\mathrm{rank}(Y \cdot \mathrm{adj}(Y)) + n \geq \mathrm{rank}(Y) + \mathrm{rank}(\mathrm{adj}(Y)),$$

and since $Y \cdot \mathrm{adj}(Y) = O_n$, it follows that $\mathrm{rank}(\mathrm{adj}(Y)) \leq 1$. From above we get for A and B that $\mathrm{rank}(\mathrm{adj}(A)), \mathrm{rank}(\mathrm{adj}(B)) \in \{0, 1\}$, so we can deduce that

$$(\mathrm{adj}(A))^2 = \mathrm{tr}(\mathrm{adj}(A)) \cdot \mathrm{adj}(A),$$

respectively

$$(\mathrm{adj}(B))^2 = \mathrm{tr}(\mathrm{adj}(B)) \cdot \mathrm{adj}(B).$$

For more ease, let us denote $\alpha = \mathrm{tr}(\mathrm{adj}(A))$ and $\beta = \mathrm{tr}(\mathrm{adj}(A))$.

We can rewrite our hyphotesis as

$$\alpha^2 \cdot \mathrm{adj}(A) - \beta^2 \cdot \mathrm{adj}(B) = 2(\mathrm{adj}(A) - \mathrm{adj}(B)),$$

and passing to traces we get $\alpha^3 - \beta^3 = 2(\alpha - \beta)$, so

$$(\alpha - \beta)(\alpha^2 + \beta^2 + \alpha \cdot \beta - 2) = 0.$$

If $\alpha = \beta$, then $(\alpha^2 - 2)(\mathrm{adj}(A) - \mathrm{adj}(B)) = 0$, and since $\mathrm{adj}(A) \neq \mathrm{adj}(B)$, it would lead to $\alpha^2 - 2 = 0$, so $\sqrt{2} \in \mathbb{Q}$, a contradiction. If $\alpha^2 + \beta^2 + \alpha \cdot \beta = 2$, we can rewrite it as $(\alpha + 2\beta)^2 + 3\alpha^2 = 8$. We can reduce to the following equation in integers, $a^2 + 3b^2 = 8c^2$. This implies $3 \mid a^2 + c^2$ whence, since -1 is not a quadratic residue modulo 3, it follow that $3|a$ and $3|c$, thus $3|b$. Next we proceed by infinite descent, to obtain that the only solution is $a = b = c = 0$, a contradiction.

Thus our assumption is false, so one of the matrices A or B is invertible. $\square$

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