# ON THE FACTORIZATION OF MAXIMAL SECTORIAL EXTENSIONS

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## 1. Abstract

If $S$ is a nonnegative linear relation in a Hilbert space $\mathfrak{H}$, then there are two extreme nonnegative selfadjoint extensions, namely the Kreĭn-von Neumann extension $S_N$ and the Friedrichs extension $S_F$, and all nonnegative selfadjoint extensions of $S$ are between $S_N$ and $S_F$. These extreme extensions and, in fact, all extremal nonnegative extensions (see [7]) can be factorized by means of the Kreĭn-von Neumann extension $S_N$; see [6, 8, 9]. The purpose of the present paper is to develop a similar theory for sectorial relations. The general theory of extensions of sectorial operators and relations is due to Yu.M. Arlinskiĭ [3, 1, 2, 4, 5].

A linear relation $S$ in a Hilbert space $\mathfrak{H}$ is said to be sectorial with vertex at the origin and semi-angle $\alpha$, $\alpha \in [0, \pi/2)$, if

$$(\tan\alpha)\mathrm{Re}\,(f', f) \geq |\mathrm{Im}\,(f', f)|, \quad \{f, f'\} \in S. \tag{1.1}$$

A linear relation $S$ in a Hilbert space $\mathfrak{H}$ is said to be maximal sectorial if the existence of a sectorial relation $T$ in $\mathfrak{H}$ with $S \subset T$ implies $S = T$. Then there exist two maximal sectorial extensions of $S$ in $\mathfrak{H}$, namely the Kreĭn-von Neumann extension $S_N$ and the Friedrichs extension $S_F$, cf. [3]. It will be shown that the Kreĭn-von Neumann extension $S_N$ has a factorization $S_N = J^{**}(I + iB)J^*$, where $J$ is a relation from an auxiliary space $\mathfrak{H}_S$ to $\mathfrak{H}$, and $B$ is a bounded selfadjoint operator in $\mathfrak{H}_S$, and that the Friedrichs extension has a similar factorization $S_F = Q^*(I + iB)Q^{**}$, where $Q^{**}$ is a certain restriction of $J^*$. The factorizations of $S_N$ and $S_F$ in the general case provide a novel approach to notions such as disjointness, transversality, and equality of $S_N$ and $S_F$; and to the notion of positive closability of $S$ ($S_N$ being an operator). A maximal sectorial extension $H$ of $S$ is said to be *extremal* if

$$\inf\{\,\mathrm{Re}\,(f' - h', f - h) : \{h, h'\} \in S\,\} = 0 \quad \text{for all} \quad \{f, f'\} \in H. \tag{1.2}$$

This definition goes back to Yu.M. Arlinskiĭ [3]. In the present paper the extremal extensions are characterized by factorizations in the general case.

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