2025 TEST 1 seniors— Solution to Problems 2, 3 and 4

Problem 2. Let ABC be a scalene acute triangle with incentre I and circumcentre O. Let AI cross BC at D. On circle ABC, let X and Y be the mid-arc points of ABC and BCA, respectively. Let DX cross CI at E and let DY cross BI at F. Prove that the lines FX, EY and IO are concurrent on the external bisector of $\angle BAC$.

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Solution. The argument hinges on the claim below:

Claim. The lines AE and BI are perpendicular; similarly, AF and CI are perpendicular

Proof. Let $\alpha = \angle BAC$, $\beta = \angle CAB$ and $\gamma = \angle ACB$. Let DX cross the circle ADC again at D'. Note that $\angle ECX = \angle ACX - \angle ACE = 90° - \beta/2 - \gamma/2 = \alpha/2 = \angle DAC = \angle DD'C = \angle XD'C$. As $\angle CXE = \angle CXD'$, triangles XCD' and XEC are similar, so $XD' \cdot XE = XC^2$.

As $XA = XC$, it follows that $XD' \cdot XE = XA^2$, so triangles $XD'A$ and XAE are similar, so $\angle XAE = \angle AD'X = \angle AD'D = \angle ACD = \gamma$.

Finally, note that $\angle XAD = \angle XAC - \angle DAC = 90° - \beta/2 - \alpha/2 = \gamma/2$, so $\angle IAE = \angle DAE = \angle XAE - \angle XAD = \gamma/2$. As $\angle AIB = 90° + \gamma/2$, the claim follows.

Let W be the mid-arc point of CAB and let I' be the reflection of I across O. As W, X, Y are the mid-arc points of CAB, ABC, BCA, respectively, their reflections across O are the mid-arc points opposite. These latter form a triangle with orthocentre I, so I' is the orthocentre of triangle WXY.

Reflection across O maps lines XV, WY and WX to the perpendicular bisectors of AI, BI and CI, respectively, so $XY \perp AI$, $WY \perp BI$ and $WX \perp CI$. By the claim, $AE \perp IF$ and $AF \perp IE$, so I is the orthocentre of triangle AEF and hence $EF \perp AI$ as well.

Triangles AEF and WXY have therefore corresponding parallel sides, so they are homothetic from some point R. This homothety maps I to I', as they are corresponding orthocentres. Hence the lines AW, EY, FY and II' are concurrent at R. As I, O and I' are collinear and AW is the external bisector of $\angle BAC$, the conclusion follows.

Problem 3. Determine all polynomials P with integer coefficients, satisfying $0 \le P(n) \le n!$ for all non-negative integers n.

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Solution. The required polynomials are $P = 0$, $P = 1$, $P = (X-1)^2$, $P = X(X-1)\cdots(X-k)$ and $P = X(X-1)\cdots(X-k)(X-k-2)^2$ for some non-negative integer k. The verification is routine and is hence omitted.

Let P be a polynomial satisfying the condition in the statement. Clearly, $P(0) = 0$ or $P(0) = 1$.

We first deal with the case $P(0) = 1$. The polynomials $P_1 = 1$ and $P_2 = (X-1)^2$ both satisfy the condition in the statement and $P_1(0) = P_2(0) = 1$.

We will prove that either $P = P_1$ or $P = P_2$. Consider an index i such that $P(1) = P_i(1)$ and let $\tilde{P} = P - P_i$.

Induct on n to show that $\tilde{P}(n) = 0$ for all non-negative integers n. The base cases $n = 0$ and $n = 1$ are clear. For the inductive step, assume $\tilde{P}(m) = 0$ for all non-negative integers $m < n$. Then $X(X-1)\cdots\big(X-(n-1)\big)$ divides $\tilde{P}$, so $n!$ divides $\tilde{P}(n)$. As $0 < P_i(n) < n!$, it follows that $|\tilde{P}(n)| = |P(n) - P_i(n)| < n!$, so $\tilde{P}(n) = 0$.

Consequently, $\tilde{P}$ has infinitely many roots, so it vanishes identically; that is, $P = P_i$, as desired.

Finally, we deal with the case $P(0) = 0$. Assume P is non-zero. Let $P(X) = XQ(X-1)$, where Q has integer coefficients. Then $0 \le Q(n) \le n!$ for all non-negative integers n. If $Q(0) = 0$, repeat the argument for Q and so on and so forth, all the way down to some polynomial with a non-zero constant term — this is clearly the case, as P is non-zero and degrees strictly decrease in the process. By the preceding, such a polynomial is either 1 or $(X-1)^2$. An obvious induction then shows that P has one of the last two forms mentioned in the beginning.

Problem 4. Determine the sets S of positive integers satisfying the following two conditions:
(a) For any positive integers a, b, c, if $ab + bc + ca$ is in S, then so are $a + b + c$ and abc; and

(b) The set S contains an integer $N \ge 160$ such that $N - 2$ is not divisible by 4.

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Solution. We will prove that S is the set of all positive integers. The argument hinges on the three facts below:

(1) The set S contains an integer $M \ge 40$ divisible by 4.

(2) If $4k$ belongs to S for some integer $k \ge 2$, then so does $4m$ for all positive integers $m < k$.

(3) The set S contains $4k$ for all integers $k \ge 10$.

Assume the three for the moment and argue as follows: By **(1)** and **(2)**, S contains all positive multiples of 4 at most 40, and by **(3)** it contains all multiples of 4 at least 40, so S contains all positive multiples of 4.

Let a be any positive integer and let $b = c = 2$. By the preceding, S contains $4a + 4$, so it contains $a + 4$, by **(a)**. Hence S also contains all positive integers congruent to a modulo 4. Combine with the preceding paragraph to deduce that S contains all integers at least 4.

Let $a = 3$ and let $b = c = 1$. As 7 is in S, so is 3, by **(a)**. Repeat the argument for $a = b = c = 1$ to deduce that S contains 1.

Finally, let $a = 2$ and let again $b = c = 1$. As 5 lies in S, so does 2. Combining with the previous paragraphs, if follows that S exhausts all positive integers, as stated.

Proof of (1). If N is divisible by 4, choose $M = N$. If $N = 4k + 1$, set $a = 2k$ and $b = c = 1$ in **(a)** to deduce that $2k$ and $2k + 2$ are both in S. As $N \ge 160$, the numbers $2k$ and $2k + 2$ are both at least $80 > 40$. Note that exactly one of $2k$ and $2k + 2$ is divisible by 4 and let M be that number.

If $N = 4k + 3$, set $a = 2k + 1$ and $b = c = 1$ in **(a)** to deduce that $2k + 1$ and $2k + 3$ both lie in S. Next, set $a = k$ or $k + 1$ and $b = c = 1$ in **(a)** to deduce that k, $k + 1$, $k + 2$ and $k + 3$ are all in S. Exactly one of these numbers is divisible by 4. As $k \ge 40$, choose M to be that number.

Proof of (2). Let $b = c = 2$. By **(a)**, if $4a + 4$ is in S, then so is $4a$. Beginning with M provided by **(1)**, statement **(2)** now follows by backward recursion.

Proof of (3). Let $b = c = 4$. By **(a)**, if $8a + 16$ is in S, then so is $16a$. Note that $16a = 8(2a - 2) + 16$ with $2a - 2 > a$ for $a \ge 3$.

As $40 \in S$, then we S contains a strictly increasing subsequence of multiples of 8 (and thus of multiples of 4.

Reference to **(2)** concludes the proof and completes the solution.