Olimpiada Națională de Matematică

Etapa locală- clasa a XI-a

SI. Rezolvați ecuația $X^3 = \begin{pmatrix} 13 & 14 \\ 14 & 13 \end{pmatrix}, X \in M_2(\mathbb{R}).$

(S. Romașcu)

Barem de rezolvare:

Fie $A = \begin{pmatrix} 13 & 14 \\ 14 & 13 \end{pmatrix}, X^3 = A \Rightarrow XA = AX \Rightarrow X = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$................$2p$

$X^3 = A \Rightarrow \begin{cases} a^3 + 3ab^2 = 13 \\ b^3 + 3a^2b = 14 \end{cases}$................$2p$

$\begin{cases} (a+b)^3 = 27 \\ (a-b)^3 = -1 \end{cases}, a, b \in R, 3 - impar \Rightarrow \begin{cases} a+b=3 \\ a-b=-1 \end{cases}$................$2p$

$a=1, b=2 \Rightarrow X = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$......................$1p$

SII. Fie $A \in M_2(\mathbb{R})$ astfel încât $\det(A+I_2)=3$ și $\det(A-I_2)=1$.

Calculați suma $S = \det(A-2I_2) + \det(A^3+6A^2+8A) + \det(A^2+2A+2I_2).$

(prelucrare dupa SL23.264)

Barem de rezolvare:

Fie $t = TrA, d = \det A, f(x) = \det(A + xI_2)$.

Avem $f(x) = x^2 + tx + d, \forall x \in \mathbb{C}$.

$\begin{cases} f(1)=3 \\ f(-1)=1 \end{cases} \Rightarrow t = d = 1 \Rightarrow f(x) = x^2 + x + 1, \forall x \in \mathbb{C}$...........$1p$

$\det(A-2I_2) = f(-2) = 3$..............$1p$

$A^3 + 6A^2 + 8A = A(A+2I_2)(A+4I_2) \Rightarrow \det(A^3+6A^2+8A) =$

$= f(2)f(4) = 147$..........................$2p$

$$A^2 + 2A + 2I_2 = A - I_2 + 2A + 2I_2 = 3(A + \frac{1}{3}I_2) \Rightarrow$$

$$\Rightarrow \det(A^2 + 2A + 2I_2) = 9f\left(\frac{1}{3}\right) = 13 ....2p$$

$$S = 3 + 147 + 13 = 163 ....1p$$

SIII. Fie şirul $(a_n)_{n\in\mathbb{N}}$ definit prin $a_0 = -2$ şi $a_{n+1} = \frac{1}{2}\left(a_n + \frac{9}{a_n}\right), \forall n \in \mathbb{N}.$

a) Demonstraţi că şirul $(a_n)_{n\in\mathbb{N}}$ este convergent.

b) Calculaţi .$\lim_{n\to\infty} \sqrt[n]{|a_n + 3|}$

Barem de rezolvare:

a)

$$a_{n+1} = \frac{a_n^2 + 9}{2a_n}, \forall n \geq 0 \text{ si } a_0 < 0 \Rightarrow a_n < 0, \forall n \in \mathbb{N} ....1p$$

$$a_{n+1} + 3 = \frac{(a_n + 3)^2}{2a_n}, a_n < -3, \forall n \geq 1 ....1p$$

$$a_{n+1} - a_n = \frac{(a_n + 3)(3 - a_n)}{2a_n} > 0, \forall n \geq 1 \Rightarrow (a_n)_{n\geq 1} s. \nearrow, a_n < -3, \forall n \geq 1 ....2p$$

$$(a_n)_{n\geq 0} \, convergent ....1p$$

b)

$$a_n \to a \Rightarrow \begin{cases} a^2 = 9 \\ a < 0 \end{cases} \Rightarrow a = -3 ....1p$$

$$\lim_{n\to\infty} \sqrt[n]{a_n + 3} = \lim_{n\to\infty} \frac{a_{n+1} + 3}{a_n + 3} = \lim_{n\to\infty} \frac{a_n + 3}{2a_n} = 0 ....1p$$

SIV. Fie şirul $(x_n)_{n\in\mathbb{N}}$ definit prin $x_1 \in R$ şi $x_{n+1} = x_n + a^{x_n}, \forall n \geq 1$ unde $a \in (0,1)$ este fixat.

Demonstraţi că $$\lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right)^{a^{-x_n}\cdot\ln n} = \frac{1}{a}.$$

Barem de rezolvare:

$x_{n+1} - x_n = a^{x_n} > 0 \Rightarrow (x_n)_{n\geq 1}\, s. \nearrow$.

Presupunem şirul $(x_n)_{n\geq 1}$ mărginit superior $\Rightarrow x_n \to x \in R \Rightarrow a^x = 0$ contradicţie

şi şirul $(x_n)_{n\geq 1}$ este strict crescător şi nemărginit superior $\Rightarrow x_n \to \infty$ …………...2p

$$\frac{x_{n+1}}{x_n} = 1 + \frac{a^{x_n}}{x_n} \Rightarrow \frac{x_{n+1}}{x_n} \to 1; a^{-x_n}\cdot\ln n \to \infty \text{..............} 1p$$

$$\lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right)^{a^{-x_n}\cdot\ln n} = \lim_{n\to\infty}\left[\left(1 + \frac{x_{n+1}-x_n}{x_n}\right)^{\frac{x_n}{x_{n+1}-x_n}}\right]^{\frac{x_{n+1}-x_n}{x_n}\cdot a^{-x_n}\cdot\ln n} = e^{\lim\limits_{n\to\infty}\frac{\ln n}{x_n}} \text{...............} 1p$$

$$\lim_{x\to\infty}\frac{\ln n}{x_n} = \lim_{n\to\infty}\frac{\ln(n+1)-\ln n}{x_{n+1}-x_n} = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)^n}{n\cdot a^{x_n}} = \lim_{n\to\infty}\frac{a^{-x_n}}{n} = \lim_{n\to\infty}\frac{a^{-x_{n+1}} - a^{-x_n}}{1} =$$

$$= \lim_{n\to\infty}\frac{a^{x_n - x_{n+1}} - 1}{a^{x_n}} = \lim_{n\to\infty}\frac{a^{x_n - x_{n+1}} - 1}{x_{n+1} - x_n} = -\ln a \text{.......................................} 2p$$

Limita cerută este $e^{-\ln a} = \dfrac{1}{a}$...1p