OLIMPIADA DE MATEMATICĂ
ETAPA LOCALĂ - CLASA A XII-A

SOLUȚII ŞI BAREM ORIENTATIV

SUBIECTUL 1

a) Notăm $(a_n,b_n)=\underbrace{(9,4)*(9,4)*...*(9,4)}_{n\text{ ori}},n\geq 1\Rightarrow(9,4)*(a_n,b_n)=(9a_n+20b_n,9b_n+4a_n)(\mathbf{1p})\Rightarrow$

$$a_{n+1}=9a_n+20b_n\Rightarrow b_n=\frac{a_{n+1}-9a_n}{20},\ b_{n+1}=9b_n+4a_n\Rightarrow\frac{a_{n+2}-9a_{n+1}}{20}=9\frac{a_{n+1}-9a_n}{20}+4a_n\Rightarrow$$

$$a_{n+2}-9a_{n+1}=9a_{n+1}-81a_n+80a_n\Rightarrow a_{n+2}-18a_{n+1}+a_n=0(\mathbf{1p})\Rightarrow t^2-18t+1=0,\ \Delta=320\Rightarrow$$

$$\Rightarrow t_{1,2}=\frac{18\pm 8\sqrt{5}}{2}=9\pm 4\sqrt{5}\Rightarrow a_n=c_1\left(9+4\sqrt{5}\right)^n+c_2\left(9-4\sqrt{5}\right)^n\Rightarrow$$

$$\Rightarrow\begin{cases}9=a_1=c_1\left(9+4\sqrt{5}\right)+c_2\left(9-4\sqrt{5}\right),\\161=a_2=c_1\left(9+4\sqrt{5}\right)^2+c_2\left(9-4\sqrt{5}\right)^2\end{cases}(\mathbf{1p})\Rightarrow c_1=c_2=\frac{1}{2}\Rightarrow\begin{cases}a_n=\frac{1}{2}\left[\left(9+4\sqrt{5}\right)^n+\left(9-4\sqrt{5}\right)^n\right](\mathbf{1p})\\b_n=\frac{1}{2\sqrt{5}}\left[\left(9+4\sqrt{5}\right)^n-\left(9-4\sqrt{5}\right)^n\right](\mathbf{1p})\end{cases}$$

b) $(9,4)\in S;$ demonstrăm că dacă $(a,b),(c,d)\in S\Rightarrow(a,b)*(c,d)\in S;(ac+5bd)^2-5(ad+bc)^2=(a^2-5b^2)$ $(c^2-5d^2)=1\Rightarrow\underbrace{(9,4)*(9,4)*...*(9,4)}_{n\text{ ori}}\in S.(9,4)*(a_n,b_n)=(9a_n+20b_n,9b_n+4a_n)\Rightarrow$

$a_{n+1}=9a_n+20b_n>a_n(\mathbf{1p})\Rightarrow b_{n+1}=9b_n+4a_n>b_n\Rightarrow S$ are o infinitate de elemente distincte. $(\mathbf{1p})$

SUBIECTUL 2

Soluție: $$\left(\frac{x\cdot arcsinx}{\sqrt{1-x^2}}\right)'=\frac{(\arcsin x+\frac{x}{\sqrt{1-x^2}})\sqrt{1-x^2}-(x\cdot arcinx)\cdot\frac{-x}{\sqrt{1-x^2}}}{1-x^2},$$

$$\left(\frac{x\cdot arcinx}{\sqrt{1-x^2}}\right)'=\frac{(\arcsin x)\cdot(\sqrt{1-x^2}+\frac{x^2}{\sqrt{1-x^2}})+x}{1-x^2}=\frac{\arcsin x}{(1-x^2)\sqrt{1-x^2}}+\frac{x}{1-x^2}=\mathrm{f}(x),$$ ....2p. rezultă

$F(x)=\frac{x\cdot arcinx}{\sqrt{1-x^2}}+c,\ F(0)=c=0,\ F(x)=\frac{x\cdot arcinx}{\sqrt{1-x^2}}$, ...3p.

$$\int\frac{x\cdot\arcsin x}{\sqrt{1-x^2}}dx=-\int(\sqrt{1-x^2})'\arcsin x\,\mathrm{d}x=-\sqrt{1-x^2}\arcsin x+\int\sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}}dx,$$

$$\int\frac{x\cdot\arcsin x}{\sqrt{1-x^2}}dx=-\sqrt{1-x^2}\arcsin x+\int 1dx=-\sqrt{1-x^2}\arcsin x+x+c_1=\mathrm{F}_1(x),\ F_1(0)=0=c_1,$$

$\mathrm{F}_1(\mathrm{x})=-\sqrt{1-x^2}\arcsin x+x,$ ....2p.

SUBIECTUL 3

***Soluţie*:**

$y=\pi-x \Rightarrow dy=-dx; x=0 \Rightarrow y=\pi; x=\pi \Rightarrow y=0\,(\mathbf{1p})$

$I=\int_0^{\pi}\frac{x\cdot\sin x}{8+\sin^2 x}dx=-\int_{\pi}^{0}\frac{(\pi-y)\cdot\sin(\pi-y)}{8+\sin^2(\pi-y)}dy\,(\mathbf{1p})=\int_0^{\pi}\frac{(\pi-y)\cdot\sin y}{8+\sin^2 y}dy\,(\mathbf{1p})=\pi\int_0^{\pi}\frac{\sin y}{8+\sin^2 y}dy-$

$-\int_0^{\pi}\frac{y\cdot\sin y}{8+\sin^2 y}dy\,(\mathbf{1p})\Rightarrow 2I=\pi\int_0^{\pi}\frac{\sin y}{8+\sin^2 y}dy\Rightarrow I=\frac{\pi}{2}\int_0^{\pi}\frac{\sin y}{9-\cos^2 y}dy.\,(\mathbf{1p})$

$\cos y=t\Rightarrow \sin y\,dy=-dt; y=0\Rightarrow t=1; y=\pi\Rightarrow t=-1.\,(\mathbf{1p})$

$I=-\frac{\pi}{2}\int_1^{-1}\frac{dt}{9-t^2}dt=-\frac{\pi}{2}\int_{-1}^{1}\frac{dt}{t^2-9}dt=-\frac{\pi}{2\cdot2\cdot3}\ln\left|\frac{t-3}{t+3}\right|\Bigg|_{-1}^{1}=-\frac{\pi}{12}\left(\ln\frac{1}{2}-\ln 2\right)=\frac{\pi}{6}\ln 2.\,(\mathbf{1p})$

SUBIECTUL 4

***Soluţie*:**

1. Dacă $a^{-1}b^{-1}\notin Z(G)$, pentru $c=a^{-1}b^{-1}$,**(1p)** din ipoteză rezultă că $a\left(a^{-1}b^{-1}\right)b=b\left(a^{-1}b^{-1}\right)a$, **(1p)**adică $e=ba^{-1}b^{-1}a.\,(\mathbf{1p})$ Deducem că $b^{-1}a^{-1}=a^{-1}b^{-1}$, adică de unde rezultă că $ab=ba$.**(1p)**

2. Dacă $a^{-1}b^{-1}\in Z(G)\Rightarrow \left(a^{-1}b^{-1}\right)b=b\left(a^{-1}b^{-1}\right)(\mathbf{1p})\Rightarrow a^{-1}=ba^{-1}b^{-1}.$ Atunci

$a=\left(ba^{-1}b^{-1}\right)^{-1}=\left(b^{-1}\right)^{-1}\left(a^{-1}\right)^{-1}b^{-1}=bab^{-1}\,(\mathbf{1p})$, deci $ab=\left(bab^{-1}\right)b=ba.$ **(1p)**