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Olimpiada Naţională de Matematică
Etapa locală, 11 februarie 2023
Clasa a XII– a

V3

BAREM ORIENTATIV de CORECTARE şi NOTARE:

Problema 1: soluție orientativă	**Punctaj**
a) Prin inducție obținem: $\underbrace{x\circ x\circ\ldots\circ x}_{\text{de } n \text{ ori } x}=a^{n-1}(x-b)^n+b$, oricare ar fi $n\in\mathbb{N}^*$.	**1p**
Ecuația din enunț se scrie: $a^{n-1}(x-b)^n+b=a^p+b\Leftrightarrow(x-b)^n=a^{1+p-n}$.	**1p**
Obținem $x_n=a^{\frac{1+p-n}{n}}+b$, n impar și $x_n=\pm a^{\frac{1+p-n}{n}}+b$, n par. Dacă n este număr natural par atunci subșirul x_{2k} este divergent, iar dacă n este număr natural impar atunci subșirul x_{2k+1} are limita $a^{-1}+b$.	**1p**
b) Avem $\lim\limits_{\substack{x\to 0\\x>0}}f(x)=-\frac{a}{b}$ şi $\lim\limits_{x\to\infty}f(x)=1$, f strict crescătoare, deci $G=\left(-\frac{a}{b},\,1\right)$. Funcția $f:(0,\infty)\to\left(-\frac{a}{b},\,1\right)$, $f(x)=\frac{x-a}{x+b}$ este inversabilă şi $f^{-1}:\left(-\frac{a}{b},\,1\right)\to(0,\infty)$, $f^{-1}(x)=\frac{a+bx}{1-x}$.	**2p**
Cum f^{-1} este morfism obţinem $f^{-1}(x\,\Delta y)=f^{-1}(x)\cdot f^{-1}(y)\Leftrightarrow x\Delta y=f\left(\frac{a+bx}{1-x}\cdot\frac{a+by}{1-y}\right)=$ $=f\left(\frac{a^2+abx+aby+b^2yx}{1-x-y+xy}\right)=\frac{\frac{a^2+abx+aby+b^2yx}{1-x-y+xy}-a}{\frac{a^2+abx+aby+b^2yx}{1-x-y+xy}+b}=$ $=\frac{a^2-a+a(b+1)x+a(b+1)y+(b^2-a)xy}{a^2+b+b(a-1)x+b(a-1)y+b(b+1)xy}$.	**2p**

Problema 2: soluție orientativă	**Punctaj**
Din $a^3b = ba \Rightarrow a^3b^2 = bab \Rightarrow a^3 = bab \Rightarrow ba^3 = ab \Rightarrow ba^3b = a.$	**1p**
$\Rightarrow (ba^3b)(ba^3b)(ba^3b) = a^3$ de unde $ba^9b = a^3 \Rightarrow a^9 = ba^3b \Rightarrow a^9 = a$ $\Rightarrow a^8 = e.$	**2p**
Din $(ba^3b)(ba^3b) = a^2 \Rightarrow ba^6b = a^2 \Rightarrow ba^6 = a^2b \Rightarrow$	**2p**
$\Rightarrow (a^2b)^2 = a^2b \cdot a^2b = ba^6 \cdot a^2b = ba^8b = b^2 = e.$	**2p**
Problema 3: soluție orientativă	
Pentru $x \in (-a, a)$ avem: $I = \int x\left(\sqrt{a+x}+\sqrt{a-x}\right)dx = \int x\sqrt{a+x}dx + \int x\sqrt{a-x}dx = I_1 + I_2.$	**2p**
Cu substituţia $\sqrt{a+x} = t$ avem: $I_1 = \int x\sqrt{a+x}dx = \int (t^2-a)t \cdot 2tdt = 2\int t^2(t^2-a)dt = 2\int (t^4 - at^2)dt = 2\left(\frac{t^5}{5} - a\frac{t^3}{3}\right) =$ $= \frac{2}{5}\left(\sqrt{a+x}\right)^5 - \frac{2}{3}a\left(\sqrt{a+x}\right)^3.$ Cu substituţia $\sqrt{a-x} = t$ avem: $I_2 = \int x\sqrt{a-x}dx = \int (a-t^2)\, t \cdot (-2t)dt = 2\int t^2(t^2-a)dt = 2\int (t^4 - at^2)dt =$ $2\left(\frac{t^5}{5} - a\frac{t^3}{3}\right) = \frac{2}{5}\left(\sqrt{a-x}\right)^5 - \frac{2}{3}a\left(\sqrt{a-x}\right)^3.$	**2p**
$I = I_1 + I_2 = \frac{2}{5}\left(\sqrt{a+x}\right)^5 - \frac{2}{3}a\left(\sqrt{a+x}\right)^3 + \frac{2}{5}\left(\sqrt{a-x}\right)^5 - \frac{2}{3}a\left(\sqrt{a-x}\right)^3 =$ $= \frac{2}{5}\left[\left(\sqrt{a+x}\right)^5 + \left(\sqrt{a-x}\right)^5\right] - \frac{2}{3}a\left[\left(\sqrt{a+x}\right)^3 + \left(\sqrt{a-x}\right)^3\right].$	**2p**
$\int x\left(\sqrt{a+x}+\sqrt{a-x}\right) = \frac{2}{5}\left[\left(\sqrt{a+x}\right)^5 + \left(\sqrt{a-x}\right)^5\right] - \frac{2}{3}a\left[\left(\sqrt{a+x}\right)^3 + \left(\sqrt{a-x}\right)^3\right] + C$.	**1p**

Problema 4: soluție orientativă	**Punctaj**
a). Fie $F_1:\boldsymbol{R}\to\boldsymbol{R}$ primitiva funcției $f_1:\boldsymbol{R}\to\boldsymbol{R}$ astfel încât $F_1(0)=0$. $\displaystyle F_1(x)=\int\frac{x+1}{x^2+x+1}dx=\frac{1}{2}\int\frac{2x+2}{x^2+x+1}dx=\frac{1}{2}\int\frac{2x+1}{x^2+x+1}dx+\frac{1}{2}\int\frac{1}{x^2+x+1}dx=$ $=\frac{1}{2}\cdot\ln(x^2+x+1)+\frac{1}{2}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx=\frac{1}{2}\cdot\ln(x^2+x+1)+\frac{1}{\sqrt{3}}\cdot arctg\frac{2x+1}{\sqrt{3}}+c.$	**2p**
Din $F_1(0)=0\Rightarrow\frac{1}{2}\cdot\ln 1+\frac{1}{\sqrt{3}}\cdot arctg\frac{1}{\sqrt{3}}+c=0\Rightarrow c=-\pi\cdot\frac{\sqrt{3}}{18}$. Deci $F_1(x)=\frac{1}{2}\cdot\ln(x^2+x+1)+\frac{1}{\sqrt{3}}\cdot arctg\frac{2x+1}{\sqrt{3}}-\pi\cdot\frac{\sqrt{3}}{18}$	**1p**
b). $\int f_n(x)dx=\int\frac{x^{2n-1}+x^{n-1}}{x^{2n}+x^n+1}dx=\frac{1}{2n}\int\frac{2n\cdot x^{2n-1}+2n\cdot x^{n-1}}{x^{2n}+x^n+1}dx=\frac{1}{2n}\int\frac{2n\cdot x^{2n-1}+n\cdot x^{n-1}}{x^{2n}+x^n+1}dx+$ $+\frac{1}{2n}\int\frac{n\cdot x^{n-1}}{x^{2n}+x^n+1}dx=\frac{1}{2n}\ln(x^{2n}+x^n+1)+\frac{1}{2n}\int\frac{n\cdot x^{n-1}}{\left(x^n+\frac{1}{2}\right)^2+\frac{3}{4}}dx=\frac{1}{2n}\ln(x^{2n}+x^n+1)+$ $+\frac{1}{n\sqrt{3}}\cdot arctg\frac{2x^n+1}{\sqrt{3}}+C.$	**2p**
Deci $\int_{\sqrt[n]{n}}^{\sqrt[n]{n+1}}f_n(x)dx=\frac{1}{2n}[\ln((n+1)^2+n+1+1)-\ln(n^2+n+1)]+$ $+\frac{1}{n\sqrt{3}}\left[arctg\left(\frac{2n+3}{\sqrt{3}}\right)-arctg\left(\frac{2n+1}{\sqrt{3}}\right)\right]=\frac{1}{2n}\cdot\ln\frac{n^2+3n+3}{n^2+n+1}+\frac{1}{n\sqrt{3}}\cdot arctg\frac{\sqrt{3}}{2n^2+4n+3}.$ $\displaystyle\lim_{n\to\infty}n^2\cdot\int_{\sqrt[n]{n}}^{\sqrt[n]{n+1}}f_n(x)dx=\lim_{n\to\infty}\left(\frac{n}{2}\cdot\ln\frac{n^2+3n+3}{n^2+n+1}\right)+\lim_{n\to\infty}\left(\frac{n}{\sqrt{3}}arctg\frac{\sqrt{3}}{2n^2+4n+3}\right)=$ $\displaystyle=\lim_{n\to\infty}\left(\ln\left(1+\frac{2n+2}{n^2+n+1}\right)^{\frac{n}{2}}\right)+\lim_{n\to\infty}\left(\frac{n}{\sqrt{3}}\cdot\frac{arctg\frac{\sqrt{3}}{2n^2+4n+3}}{\frac{\sqrt{3}}{2n^2+4n+3}}\cdot\frac{\sqrt{3}}{2n^2+4n+3}\right)=$ $\displaystyle=\ln e+\lim_{n\to\infty}\frac{n}{2n^2+4n+3}=1.$	**2p**

<u>Notă</u>: Orice altă soluţie corectă se punctează corespunzător.