Test 3 — Solutions

Problem 1. Given a positive integer n, determine all functions f from the first n positive integers to the positive integers, satisfying the following two conditions: **(1)** $\sum_{k=1}^{n} f(k) = 2n$; and **(2)** $\sum_{k \in K} f(k) = n$ for no subset K of the first n positive integers.

Solution. If n is odd, the required functions are the constant function $f_0 \equiv 2$ along with the n functions $f_i \colon \{1, \ldots, n\} \to \mathbb{N}^*$,

$$f_i(j) = \begin{cases} n+1, & \text{if } j = i, \\ 1, & \text{if } j \neq i, \end{cases} \qquad i = 1, \ldots, n;$$

notice that $f_0 = f_1$ if $n = 1$. If n is even, f_0 is to be removed from the list, by **(2)**. All these functions plainly satisfy the conditions in the statement.

Labelling the positive integers in the list $f(1), \ldots, f(n)$ increasingly, $a_1 \leq \cdots \leq a_n$, the problem amounts to determining all lists of positive integers $a_1 \leq \cdots \leq a_n$ such that $(\mathbf{1'})$ $\sum_{k=1}^{n} a_k = 2n$ and $(\mathbf{2'})$ $\sum_{k \in K} a_k = n$ for no subset K of the first n positive integers.

Notice that $a_n \leq n+1$, by $(\mathbf{1'})$, and $a_n \neq n$, by $(\mathbf{2'})$. With reference again to $(\mathbf{1'})$, notice further that if $a_n = n+1$, then the other a_k are all 1, and if $a_n = 2$, then so are the other a_k. If n is even, $(\mathbf{2'})$ rules out the latter case.

Leaving aside the trivial cases $n = 1$ and $n = 2$, let $n \geq 3$. To rule out $a_n \neq 2$, $n+1$, assume, if possible, this is the case, and notice that

$$a_1 - a_n, \quad 0, \quad a_1, \quad a_1 + a_2, \quad \ldots, \quad a_1 + a_2 + \cdots + a_{n-1}$$

are pairwise distinct integers, so at least two are congruent modulo n. Since $a_n \neq 2, n, n+1$, the first two cannot be congruent modulo n, and $(\mathbf{2'})$ rules out the remaining cases.

Problem 2. Given a positive integer k and an integer $a \equiv 3$ modulo 8, show that $a^m + a + 2$ is divisible by 2^k for some positive integer m.

Solution. Proceed by induction on k. Since $a \equiv 3 \pmod 8$, $m = 1$ works for $k = 1, 2, 3$, so let $k \geq 3$ and let m be a positive integer such that $a^m + a + 2$ is divisible by 2^k.

If $(a^m + a + 2)/2^k$ is even, then $a^m + a + 2$ is clearly divisible by 2^{k+1}.

If $(a^m + a + 2)/2^k$ is odd, we will show that $a^{m+2^{k-2}} + a + 2$ is divisible by 2^{k+1}. To this end, write $a^{m+2^{k-2}} + a + 2 = a^{2^{k-2}}(a^m + a + 2) - (a+2)\left(a^{2^{k-2}} - 1\right)$. The first term is an odd multiple of 2^k, and it is sufficient to prove that so is the second.

Induct on $k \geq 3$ to show that $a^{2^{k-2}} - 1$ is an odd multiple of 2^k. Since $a \equiv 3 \pmod 8$, this is clearly the case if $k = 3$, and the induction step follows from the identity $a^{2^{k-1}} - 1 = \left(a^{2^{k-2}} - 1\right)\left(a^{2^{k-2}} + 1\right) = \left(a^{2^{k-2}} - 1\right)\left(\left(a^{2^{k-2}} - 1\right) + 2\right)$. This completes the proof.

Problem 3. Given a positive integer n, show that for no set of integers modulo n, whose size exceeds $1 + \sqrt{n+4}$, is it possible that the pairwise sums of unordered pairs be all distinct.

Solution. Let $S \subseteq \mathbb{Z}/n\mathbb{Z}$ be a set whose pairwise sums of unordered pairs are distinct, and consider the pairwise differences of *ordered* pairs.

The crucial observation is that if a difference $d \neq 0$ occurs twice in S, then these two occurrences must be adjacent in a 3-term arithmetic progression in S (with difference d). Indeed, if a, $a+d$, a', $a'+d$ are all in S, where $a \neq a'$, then $a + (a'+d) = a' + (a+d)$ so, by assumption on S, $a' = a \pm d$, i.e. the ordered pairs $(a, a+d)$ and $(a', a'+d)$ are adjacent in a 3-term arithmetic progression.

We also observe that, excepting the arithmetic progression of common difference $n/2$, in case n is even, no two 3-term arithmetic progressions can have the same central term, since if a, $a \pm d$,

$a \pm d'$ are all in S, where $d, d' \neq 0, n/2$ are distinct, then $(a + d') + (a - d') = (a + d) + (a - d)$ would violate the assumption on S.

It follows that the number of 3-term arithmetic progressions in S is at most $|S| + 2$. Now any non-zero difference not occurring in a 3-term arithmetic progression can appear at most once in S, and those which do appear in 3-term arithmetic progressions can appear at most twice, except $\pm n/3$ in case n is divisible by 3, which can appear three times.

Consequently, the total number of non-zero differences appearing in S is at least $|S|(|S| - 1) - (|S| + 2) - 2 = |S|^2 - 2|S| - 4$. If $|S| > 1 + \sqrt{n+4}$, this quantity is greater than $n - 1$ — a contradiction.

Problem 4. Let $ABCD$ be a convex quadrangle, and let P, Q, R and S be points on the sides AB, BC, CD and DA, respectively. The line segments PR and QS cross at O. Suppose that each of the quadrangles $APOS$, $BQOP$, $CROQ$ and $DSOR$ has an incircle. Prove that the lines AC, PQ and RS are concurrent or parallel to each other.

Solution. Application of Menelaus' theorem to triangle ABC and line PQ, and triangle ACD and line RS shows the conclusion equivalent to

$$\frac{AP}{BP} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = 1.$$

To prove the latter, usage is made of the lemma below.

Lemma. *If M is the incentre of a circumscribed quadrangle $EFGH$, then*

$$\frac{EF \cdot FG}{GH \cdot HE} = \frac{MF^2}{MH^2}.$$

Proof. Notice that $\angle EMH + \angle FMG = \angle EMF + \angle GMH = 180^\circ$, $\angle FGM = \angle HGM$, and $\angle HEM = \angle FEM$, to get, by the law of sines

$$\frac{EF}{MF} \cdot \frac{FG}{MF} = \frac{\sin\angle EMF \cdot \sin\angle FMG}{\sin\angle FEM \cdot \sin\angle FGM} = \frac{\sin\angle GMH \cdot \sin\angle EMH}{\sin\angle HGM \cdot \sin\angle HEM} = \frac{GH}{MH} \cdot \frac{HE}{MH}.$$

The lemma follows.

Next, let I, J, K and L be the incentres of the quadrangles $APOS$, $BQOP$, $CROQ$ and $DSOR$, respectively, and apply the lemma to these four quadrangles to get

$$\frac{AP}{BP} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = \frac{IP^2}{JP^2} \cdot \frac{JQ^2}{KQ^2} \cdot \frac{KR^2}{RL^2} \cdot \frac{LS^2}{IS^2}.$$

Notice further that O and P lie on the circle on diameter IJ, on opposite sides of this diameter. Similarly, O and Q lie on the circle on diameter JK, on opposite sides of this diameter. It follows that $\angle JIP = \angle JOP = \angle JOQ = \angle JKQ$, so the right triangles IPJ and KQJ are similar, and $IP/JP = KQ/JQ$. Similarly, $KR/LR = IS/LS$, and the conclusion follows.