Test 5 — Solutions

Problem 1. Let ABC be a triangle. Let P_1 and P_2 be points on the side AB such that P_2 lies on the segment BP_1 and $AP_1 = BP_2$; similarly, let Q_1 and Q_2 be points on the side BC such that Q_2 lies on the segment BQ_1 and $BQ_1 = CQ_2$. The segments P_1Q_2 and P_2Q_1 meet at R, and the circles P_1P_2R and Q_1Q_2R meet again at S, situated inside triangle P_1Q_1R. Finally, let M be the midpoint of the side AC. Prove that the angles P_1RS and Q_1RM are equal.

Solution. Throughout the solution, $[XYZ]$ and $d(X, YZ)$ denote the area of the triangle XYZ and the distance from the point X to the line YZ, respectively.

Since the quadrilaterals SRQ_2Q_1 and SRP_2P_1 are cyclic, $\angle SQ_1R = \angle SQ_2R$ and $\angle SP_1R = \angle SP_2R$ (see Fig. **??**), so the triangles SP_1Q_2 and SP_2Q_1 are similar, and

$$\frac{d(S, P_1Q_2)}{d(S, P_2Q_1)} = \frac{P_1Q_2}{P_2Q_1}. \tag{1}$$

Let K and L be the midpoints of AB and AC, respectively; then $P_1K = P_2K$ and $Q_1L = Q_2L$. Therefore, $d(Q_1, MP_1) + d(Q_2, MP_1) = 2d(L, MP_1)$, so

$$[MP_1Q_2] + [MP_1Q_1] = 2[MP_1L] = ML \cdot d(P_1, ML) = [ABC]/2.$$

Similarly, $[MP_2Q_1] + [MP_1Q_1] = [ABC]/2 = [MP_1Q_2] + [MP_1Q_1]$. Thus $[MP_1Q_2] = [MP_2Q_1]$, whence

$$\frac{d(M, P_1Q_2)}{d(M, P_2Q_1)} = \frac{P_2Q_1}{P_1Q_2}. \tag{2}$$

In the angle P_1RQ_1, the condition $d(X, P_1Q_2)/d(X, P_2Q_1) = \alpha$ determines a ray emanating from R. Moreover, rays symmetric with respect to the bisectrix of $\angle P_1RQ_1$ correspond to reciprocal values of α. Consequently, relations (1) and (2) show that the rays RS and RM are symmetric with respect to this angle bisectrix, and the conclusion follows.

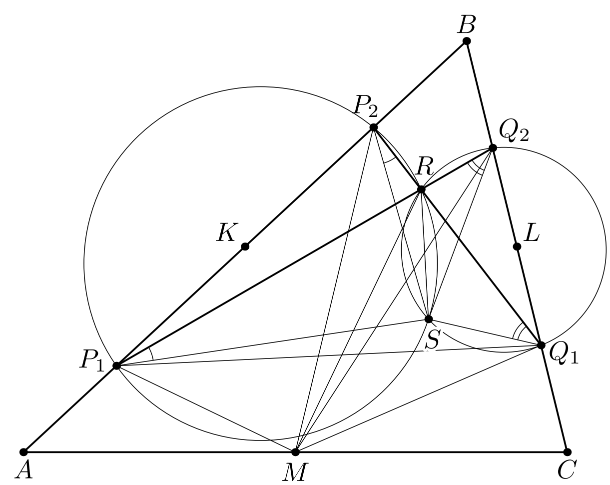

Fig. 1

Remark. Relation (2) may be obtained in several different ways. For instance, one may notice that the midpoints X, X_1 and X_2 of the segments BR, P_1Q_1 and P_2Q_2, respectively, are collinear — the Gauss-Newton line ℓ of the quadrilateral $P_1P_2Q_1Q_2$ (see Fig. **??**). Moreover, KX_1LX_2 is the Varignon parallelogram of the quadrilateral $P_1P_2Q_2Q_1$, hence the segments X_1X_2 and KL have a common midpoint N. Since XN is a midline in the triangle BMR, the line RM is parallel to ℓ, so $\overrightarrow{RM} = \alpha\overrightarrow{X_1X_2} = \frac{\alpha}{2}\big(\overrightarrow{P_1Q_2} + \overrightarrow{P_1Q_2}\big)$ which easily yields (2).

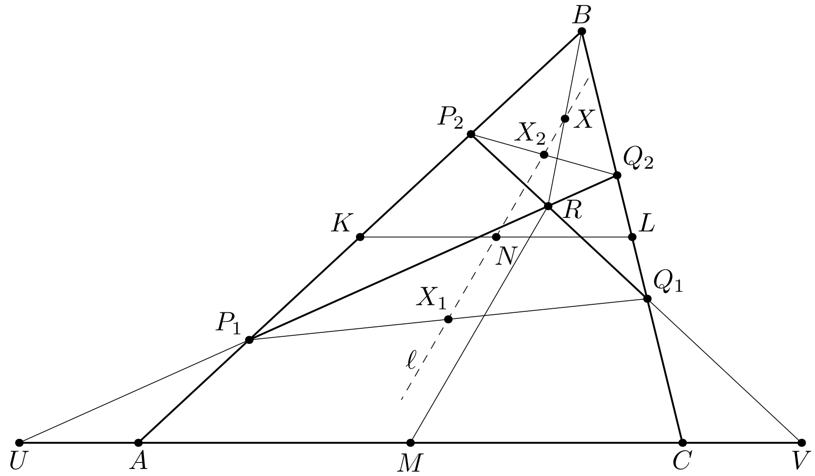

Fig. 2

Yet another way of obtaining the same relation is the following. Let the lines P_1Q_2 and P_2Q_1 meet AC at points U and V, respectively. By Menelaus' theorem,

$$\frac{AU}{UC} = \frac{AP_1}{P_1B} \cdot \frac{BQ_2}{Q_2C} = \frac{BP_2}{P_2A} \cdot \frac{CQ_1}{Q_1B} = \frac{CV}{VA},$$

so $AU = CV$. Thus RM is a median in the triangle RUV, so

$$1 = \frac{[MUR]}{[MVR]} = \frac{RU \cdot d(M, P_1Q_2)}{RV \cdot d(M, P_2Q_1)}, \qquad \text{or} \qquad \frac{d(M, P_1Q_2)}{d(M, P_2Q_1)} = \frac{RV}{RU}.$$

Finally, comparing the projections of the segments RU, RV, P_1Q_2, and P_2Q_1 onto the perpendicular to AC, we get $RU/P_1Q_2 = RV/P_2Q_1$, or $RV/RU = P_2Q_1/P_1Q_2$, as required.

Similar calculations may be carried out in terms of the sines of the corresponding angles $\angle P_1RM$, $\angle Q_2RM$, $\angle P_1RS$, and $\angle Q_2RS$, instead of distances from points to lines.

Problem 2. Let n be an integer greater than 1, and let p be a prime divisor of n. A confederation consists of p states, each of which has exactly n airports. There are p air companies operating interstate flights only such that every two airports in different states are joined by a direct (two-way) flight operated by one of these companies. Determine the maximal integer N satisfying the following condition: In every such confederation it is possible to choose one of the p air companies and N of the np airports such that one may travel (not necessarily directly) from any one of the N chosen airports to any other such only by flights operated by the chosen air company.

Solution. The required maximum is n. The following example shows that N cannot exceed n. Split the n airports in the i-th state, $i = 1, \ldots, p$, into p disjoint groups of n/p airports each, $A_{i,j}$, $j = 1, \ldots, p$. Let the k-th air company, $k = 1, \ldots, p$, operate direct flights between every airport in $A_{i,j}$ and every airport in $A_{i',j'}$ if $i \neq i'$ and $j' - j \equiv k(i' - i) \pmod{p}$, and operate no flights between the airports in $A_{i,j}$ and those in $A_{i',j'}$ otherwise.

Since p is prime, for every $i \neq i'$ and every j, j', there exists $1 \leq k \leq p$ satisfying the previous congruence modulo p, so every two airports in different states are connected by a flight. On the other hand, for any given $k = 1, \ldots, p$, the np airports are split into p disjoint n-element groups, namely, $\bigsqcup_{i=1}^{p} A_{i,j+ki}$, $j = 1, \ldots, p$ (the indices are reduced modulo p), such that there are no flights between different groups operated by the k-th air company. Consequently, $N \leq n$.

Slightly more generally, if the number of states is $m \geq 2$, and the number p of air companies is not necessarily prime, nor a divisor of n, we show that there always are at least mn/p airports connected by flights operated by one air company. Assume the contrary and let G_k, $k = 1, \ldots, p$, be the graph whose vertices are the np airports, two vertices being joined by an edge in G_k if the k-th air company operates a direct flight between the corresponding airports. Thus, each G_k is a p-partite graph on n-element classes (recall that each air company operates interstate flights only). By assumption, each component of every G_k has less than mn/p vertices.

Consider a component of G_k, and let n_i be the number of vertices in this component located in the i-th state. The number of edges in the component under consideration does not exceed

$$\sum_{1 \leq i < j \leq m} n_i n_j \leq \frac{m-1}{2m} \left(\sum_{i=1}^{m} n_i \right)^2 < \frac{m-1}{2m} \cdot \frac{mn}{p} \sum_{i=1}^{m} n_i = \frac{n(m-1)}{2p} \sum_{i=1}^{m} n_i.$$

Summing over all components of G_k, it follows that that the total number of edges of G_k is less than $mn \cdot n(m-1)/(2p) = mn^2(m-1)/(2p)$. Consequently, the total number of flights is less than $p \cdot mn^2(m-1)/(2p) = n^2 \cdot m(m-1)/2 = n^2 \binom{m}{2}$. Since the latter is the number of pairs of airports from different states, we reach a contradiction.

Problem 3. Define a sequence of integers by $a_0 = 1$, and $a_n = \sum_{k=0}^{n-1} \binom{n}{k} a_k$, $n \geq 1$. Let m be a positive integer, let p be a prime, and let q and r be non-negative integers. Prove that the difference $a_{p^m q+r} - a_{p^{m-1}q+r}$ is divisible by p^m.

Solution. Consider the $\mathbb{R}$-vector space $\mathbb{R}[X]$ of all polynomials with real coefficients and define an $\mathbb{R}$-linear functional $L\colon \mathbb{R}[X] \to \mathbb{R}$ by $LX^n = a_n$, $n = 0, 1, 2, \ldots$. Thus, if $f = \sum_k \alpha_k X^k$, then $Lf = \sum_k \alpha_k a_k$. Since $(X+1)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$, $n \geq 1$,

$$L(X+1)^n = \sum_{k=0}^{n} \binom{n}{k} LX^k = \sum_{k=0}^{n} \binom{n}{k} a_k = \sum_{k=0}^{n-1} \binom{n}{k} a_k + a_n = 2a_n = 2LX^n,$$

so $Lf(X+1) = 2Lf(X) - f(0)$ for every polynomial f in $\mathbb{R}[X]$. In particular, take $f = \binom{X}{k}$ and use the relation $\binom{X+1}{k} = \binom{X}{k} + \binom{X}{k-1}$, $k \geq 1$, to get $L\binom{X}{k} = L\binom{X}{k-1}$, $k \geq 1$, and deduce that $L\binom{X}{k} = 1$, $k = 0, 1, 2, \ldots$. Further, if a polynomial f in $\mathbb{R}[X]$ is integral valued, i.e., $f(k)$ is integral for every integral k, then $f = \sum_k \alpha_k \binom{X}{k}$ for some integers α_k, so $Lf = \sum_k \alpha_k$ is an integer. Finally, since $a^{p^m} \equiv a^{p^{m-1}} \pmod{p^m}$ for all integers a,

$$f = p^{-m}\left(X^{p^m q+r} - X^{p^{m-1}q+r}\right)$$

is an integral valued polynomial in $\mathbb{R}[X]$, so $Lf = (a_{p^m q+r} - a_{p^{m-1}q+r})/p^m$ is an integer, as required.