Test 4 — Solutions

Problem 1. Let ABC and ABD be coplanar triangles with equal perimeters. The lines of support of the internal bisectrices of the angles CAD and CBD meet at P. Show that the angles APC and BPD are congruent.

Solution. Extend the segment AC beyond C by a segment CE congruent to the segment CB, so C lies on the perpendicular bisectrix of the segment BE. Similarly, extend the segment BD beyond D by a segment DF congruent to the segment AD, so D lies on the perpendicular bisectrix of the segment AF. Since the triangles ABC and ABD have equal perimeters, the segments AE and BF are congruent.

Now let the perpendicular bisectrices of the segments AF and BE meet at Q. Clearly, the segments QA and QF, respectively QE and QB, are congruent, and since so are the segments AE and BF by the preceding, the triangles QAE and QFB are congruent. Therefore, the angles AQF and BQE are congruent, and hence so are their halves; that is, the angles AQD and BQC are congruent, and consequently so are the angles AQC and BQD.

We now show that, in fact, the points P and Q coincide, whence the conclusion. With reference again to the congruence of the triangles QAE and QFB, the angles QAE and QFB are congruent, and since the latter is the reflection of the angle QAD in the line QD, it follows that the angles $QAC = QAE$ and QAD are congruent, so the line AQ bisects the angle CAD. Similarly, the line BQ bisects the angle CBD, and consequently the points P and Q coincide.

Problem 2. Given an integer $k \geq 2$, determine the largest number of divisors the binomial coefficient $\binom{n}{k}$ may have in the range $n-k+1$, …, n, as n runs through the integers greater than or equal to k.

Solution. The required maximum is $k-1$ and is achieved, for instance, at $n = k!$. To complete the proof, we now show that at least on of the k numbers $\frac{1}{n-j}\binom{n}{k}$, $j = 0, 1, \ldots, k-1$, is not an integer. To this end, we exhibit a $\mathbb{Z}$-linear combination of these numbers which is not an integer. For instance,

$$\sum_{j=0}^{k-1} (-1)^{k-j-1} \binom{k-1}{j} \cdot \frac{1}{n-j} \binom{n}{k} = \frac{1}{k} \sum_{j=0}^{k-1} \prod_{i \neq j} \frac{n-i}{j-i} = \frac{1}{k}$$

is not an integer, since $k \geq 2$. The leftmost equality above is easily proved by noticing that the polynomial

$$\sum_{j=0}^{k-1} \prod_{i \neq j} \frac{X-i}{j-i}$$

has degree at most $k-1$, and takes on the value 1 at k distinct points, namely, 0, 1, …, $k-1$, so it is identically 1.

Problem 3. Let n be a positive integer. If σ is a permutation of the first n positive integers, let $S(\sigma)$ be the set of all distinct sums of the form $\sum_{i=k}^{\ell} \sigma(i)$, where $1 \leq k \leq \ell \leq n$.

(a) Exhibit a permutation σ of the first n positive integers such that $|S(\sigma)| \geq \lfloor (n+1)^2/4 \rfloor$.

(b) Show that $|S(\sigma)| > n\sqrt{n}/4\sqrt{2}$ for all permutations σ of the first n positive integers.

Solution. **(a)** We show that the permutation σ of the first n positive integers, defined by $\sigma(i) = (i+1)/2$ for each positive odd $i \leq n$, and $\sigma(i) = n - i/2 + 1$ for each positive even $i \leq n$, satisfies the required condition.

More precisely, we show that the $\lfloor (n+1)^2/4 \rfloor$ sums of the form $\sum_{i=k}^{\ell} \sigma(i)$, where $1 \leq k \leq \ell \leq n$ and $k \equiv \ell \pmod 2$, are pairwise distinct, so $|S(\sigma)| \geq \lfloor (n+1)^2/4 \rfloor$.

Let $1 \le k \le \ell \le n$ and $k \equiv \ell \,(\mathrm{mod}\, 2)$, and notice that $\sigma(i)+\sigma(i+1) = n+1$ for every positive odd $i \le n$, so

$$\sum_{i=k}^{\ell} \sigma(i) = \begin{cases} (\ell-k)(n+1)/2 + \sigma(\ell) & \text{if } k \text{ is odd,} \\ \sigma(k) + (\ell-k)(n+1)/2 & \text{if } k \text{ is even.} \end{cases}$$

Since the absolute value of an integer of the form $\sigma(i) - \sigma(j)$ is less than n, the above formula shows that the assignment $(k, \ell) \mapsto \sum_{i=k}^{\ell} \sigma(i)$ is indeed injective on the pairs in question.

(b) Let σ be a permutation of the first n positive integers, and split $S(\sigma)$ into $S_m(\sigma) = S(\sigma) \cap [mn+1, mn+n]$, where m runs through the integers; of course, $S_m(\sigma)$ is empty if m is negative or $m > (n-1)/2$, and $|S_0(\sigma)| \ge n$.

Leaving aside the trivial cases $n = 1$ and $n = 2$, we assume $n \ge 3$ and prove that

$$|S_m(\sigma)| + |S_{m-1}(\sigma)| > \sqrt{2n}, \tag{$*$}$$

for every non-negative integer $m \le (n+1)/4$. The conclusion then follows by summing over this range:

$$|S(\sigma)| \ge \frac{1}{2} \sum_{m=0}^{\lfloor (n+1)/4 \rfloor} (|S_m(\sigma)| + |S_{m-1}(\sigma)|) > \frac{1}{2} \left\lfloor \frac{n+5}{4} \right\rfloor \sqrt{2n} > \frac{n\sqrt{n}}{4\sqrt{2}}.$$

To prove $(*)$, fix a non-negative integer $m \le (n+1)/4$. We will show that the Minkowski difference $S_m(\sigma) - (S_m(\sigma) \cup S_{m-1}(\sigma))$ contains every positive integer less than or equal to n; alternatively, but equivalently, that it contains every $\sigma(k)$. Then so does $(S_m(\sigma) \cup S_{m-1}(\sigma)) - (S_m(\sigma) \cup S_{m-1}(\sigma))$, so $|S_m(\sigma)| + |S_{m-1}(\sigma)| = |S_m(\sigma) \cup S_{m-1}(\sigma)| > \sqrt{2n}$, for if X is a finite set of numbers such that $\{1, 2, \ldots, n\} \subseteq X - X$, then $|X| \cdot (|X| - 1) + 1 \ge |X - X| \ge 2n + 1$, so $|X| \ge (1 + \sqrt{8n+1})/2 > \sqrt{2n}$.

Finally, we show that every $\sigma(k)$ is a member of $S_m(\sigma) - (S_m(\sigma) \cup S_{m-1}(\sigma))$. To this end, fix a positive integer $k \le n$, and notice that at least one of the sums $\sum_{i=1}^{k} \sigma(i)$, $\sum_{i=k}^{n} \sigma(i)$ exceeds $n(n+1)/4 \ge mn$. Let $\sum_{i=k}^{n} \sigma(i) > mn$ — the other case is easily dealt with dually —, and consider the smallest integer $\ell \ge k$ such that $\sum_{i=k}^{\ell} \sigma(i) > mn$. With reference to this minimality, it is readily checked that the sums $\sum_{i=k}^{\ell} \sigma(i)$ and $\sum_{i=k+1}^{\ell} \sigma(i)$ belong to $S_m(\sigma)$ and $S_m(\sigma) \cup S_{m-1}(\sigma)$, respectively, so $\sigma(k)$ is indeed a member of $S_m(\sigma) - (S_m(\sigma) \cup S_{m-1}(\sigma))$.

Remark. The author shows by probabilistic methods that generically there are at least cn^2 such sums, where c is a positive absolute constant.