First Test — Solutions

Problem 1. Let ABC be a triangle, let A', B', C' be the orthogonal projections of the vertices A, B, C on the lines BC, CA and AB, respectively, and let X be a point on the line AA'. Let γ_B be the circle through B and X, centred on the line BC, and let γ_C be the circle through C and X, centred on the line BC. The circle γ_B meets the lines AB and BB' again at M and M', respectively, and the circle γ_C meets the lines AC and CC' again at N and N', respectively. Show that the points M, M', N and N' are collinear.

Solution. Let H be the orthocentre of the triangle ABC. The line AH is the radical axis of the circles γ_B and γ_C, hence $HM' \cdot HB = HN' \cdot HC$ and $AM \cdot AB = AN \cdot AC$, so the lines $M'N'$ and MN are both antiparallel to BC.

The circle γ_B meets the line BC again at B_1. Then the lines MM' and BC are antiparallel, since $BMM'B_1$ is a cyclic quadrangle. The conclusion follows.

Problem 2. Given an integer $n \geq 2$, show that there exist $n+1$ numbers x_1, x_2, …, x_n, x_{n+1} in $\mathbb{Q} \setminus \mathbb{Z}$ such that $\{x_1^3\} + \{x_2^3\} + \cdots + \{x_n^3\} = \{x_{n+1}^3\}$, where $\{x\}$ is the fractional part of the real number x.

Solution. Notice that, if $w_1 < w_2 < \cdots < w_{n+1} < w_{n+2}$ are positive integers such that

$$w_1^3 + w_2^3 + \cdots + w_{n+1}^3 = w_{n+2}^3, \qquad (*)$$

then the numbers $x_k = w_k/w_{n+2}$, $k = 1, 2, \ldots, n$, and $x_{n+1} = -w_{n+1}/w_{n+2}$ meet the required conditions.

We now show by induction on $n \geq 2$ that there exist integers $3 = w_1 < w_2 < \cdots < w_{n+1} < w_{n+2}$ satisfying $(*)$.

The equalities $3^3 + 4^3 + 5^3 = 6^3$ and $3^3 + 15^3 + 21^3 + 36^3 = 39^3$ settle the cases $n = 2$ and $n = 3$, respectively.

Finally, for the induction step $n \mapsto n+2$, notice that if $3 < w_2 < \cdots < w_{n+1} < w_{n+2}$ are $n+2$ integers satisfying $(*)$, then $3 < 4 < 5 < 2w_2 < \cdots < 2w_{n+1} < 2w_{n+2}$ are $n+4$ integers satisfying the corresponding condition.

Problem 3. Given a triangle $A_0A_1A_2$, determine the locus of the centres of the equilateral triangles $X_0X_1X_2$ satisfying the condition that each of the lines X_kX_{k+1} passes through A_k (all indices are reduced modulo 3).

Solution. Erect the three outer Napoleon triangles associated with the triangle $A_0A_1A_2$, and let γ_k, $k = 0, 1, 2$, be the corresponding circumcircles.

From any point X_0 on γ_0 draw lines $X_0A_2X_1$ and $X_0A_1X_2$, where X_1 lies on γ_1 and X_2 lies on γ_2. The points X_1, A_0, X_2 are collinear, and the triangle $X_0X_1X_2$ is an equilateral triangle satisfying the conditions in the statement.

Let M_k be the midpoint of the minor arc $A_{k+1}A_{k+2}$ of the circle γ_k, and notice that the triangle $M_0M_1M_2$ is equilateral, since the M_k are the centres of the inner Napoleon triangles associated with the triangle $A_0A_1A_2$. The centre X of the triangle $X_0X_1X_2$ is the intersection of X_0M_0 and X_1M_1, which must intersect at $60°$. Since the locus of X includes the three points M_k, it turns out that the locus of X is the circle $M_0M_1M_2$.

Similarly, another circle is obtained by starting with the inner Napoleon triangles. The required locus is a pair of circles.

Problem 4. Let k be a positive integer and let m be a positive odd integer. Show that there exists a positive integer n such that $m^n + n^m$ has at least k distinct prime factors.

Solution. Design a set of k primes $p_1 < p_2 < \cdots < p_k$ as follows. Begin by choosing $p_1 > 2m$. Having selected p_j, use Dirichlet's theorem to choose a prime

$$p_{j+1} \equiv -1 \pmod{p_1(p_1-1)p_2(p_2-1)\cdots p_j(p_j-1)}.$$

If $i < j$, then $p_i < p_j$, so p_j does not divide $p_i - 1$; further, $p_j - 1 \equiv -2 \pmod{p_i(p_i-1)}$, so p_i does not divide $p_j - 1$.

Next, use the Chinese Remainder Theorem, to choose a positive integer n such that $n \equiv -1 \pmod{p_1p_2\cdots p_k}$ and $n \equiv 0 \pmod{(p_1-1)\cdots(p_k-1)}$.

Finally, since $p_1p_2\cdots p_k$ and m are coprime, and $(p_1-1)(p_2-1)\cdots(p_k-1)$ divides n, and m is odd, Euler's Theorem applies to show that $m^n + n^m \equiv 1 + (-1)^m \equiv 0 \pmod{p_1p_2\cdots p_k}$. The conclusion follows.

Problem 5. Let n be an integer greater than 1 and let S be a finite set containing more than $n+1$ elements. Consider the collection of all sets $\mathcal{A}$ of susets of S satisfying the following two conditions:

(a) Each member of $\mathcal{A}$ contains at least n elements of S; and

(b) Each element of S is contained in at least n members of $\mathcal{A}$.

Determine $\max_{\mathcal{A}} \min_{\mathcal{B}} |\mathcal{B}|$, as $\mathcal{B}$ runs through all subsets of $\mathcal{A}$ whose members cover S, and $\mathcal{A}$ runs through the above collection.

Solution. The required number is $m = |S| - n$. We begin by showing that any set $\mathcal{A}$ of subsets of S satifying the two conditions in the statement has a subcover of cardinality at most m.

This is clear if S is a member of $\mathcal{A}$.

Assume henceforth that $\mathcal{A}$ does not contain S. If some member A of $\mathcal{A}$ has more than n elements, for each element of $S \setminus A$ choose a containing member of $\mathcal{A}$. The latter along with A form a subcover of $\mathcal{A}$ of cardinality $|S \setminus A| + 1 \le |S| - (n+1) + 1 = m$.

Assume henceforth that each member of $\mathcal{A}$ has exactly n elements. Fix a member A of $\mathcal{A}$.

If some member B of $\mathcal{A}$ contains more than one element of $S \setminus A$, for each element of $S \setminus (A \cup B)$ choose a containing member of $\mathcal{A}$. The latter along with A and B form a subcover of $\mathcal{A}$ of cardinality $|S \setminus (A \cup B)| + 2 = |S \setminus A| - |(S \setminus A) \cap B| + 2 \le |S \setminus A| = m$.

Finally, if no member of $\mathcal{A}$ contains more than one element of $S \setminus A$, write $S \setminus A = \{x_1, x_2, \ldots, x_m\}$, choose a member A_1 of $\mathcal{A}$ containing x_1 and notice that $A \setminus A_1$ is a singleton set, say $A \setminus A_1 = \{x\}$. Since x_2 is contained in at least n members of $\mathcal{A}$, each of which contains (exactly) $n-1$ elements of A, we may choose a member A_2 of $\mathcal{A}$ containing both x and x_2 (recall that $n \ge 2$). If $n \ge 3$, continue choosing members A_i of $\mathcal{A}$ containing x_i, $i = 3, \ldots, n$, to form an m-element subcover of $\mathcal{A}$ consisting of $A_1, A_2, \ldots, A_m$.

To complete the proof, we produce a set of subsets of S satisfying the two conditions in the statement, no subcover of which has less than m members. To this end, write $S = \{1, 2, \ldots, m+n\}$, $m \ge 2$, and let $S_1, S_2, \ldots, S_n$ be the $(n-1)$-element subsets of the upper part $\{m+1, m+2, \ldots, m+n\}$ of S. The sets $S_{i,j} = S_j \cup \{i\}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, satisfy both conditions in the statement and at least m of them are needed to cover S. (The condition $m \ge 2$ is required for an element in the upper part to lie in at least n of these sets.)