Second Selection Test — Solutions

Problem 1. Let a and b be distinct positive real numbers such that $\lfloor na \rfloor$ divides $\lfloor nb \rfloor$ for every positive integer n. Show that a and b are both integer.

Solution. Since the $\lfloor nb \rfloor / \lfloor na \rfloor$ form a sequence of positive integers converging to b/a, it follows that $b = ma$ for some integer $m \geq 2$, and $\lfloor nb \rfloor = m \lfloor na \rfloor$ for all n large enough. Consequently, if n is large enough, then $\lfloor nma \rfloor = m \lfloor na \rfloor$, so $mna < m \lfloor na \rfloor + 1$; that is, $na < \lfloor na \rfloor + 1/m \leq \lfloor na \rfloor + 1/2$. Hence $\{na\} = na - \lfloor na \rfloor < 1/2$, so the set $\{\{na\} \colon n \in \mathbb{Z}_+\}$ is not dense in the closed unit interval $[0,1]$, and a must be rational, say $a = p/q$, where p and q are coprime positive integers. If $q \geq 2$, choose n large enough such that $np \equiv -1 \pmod{q}$, to reach a contradiction: $1/2 > \{na\} = \{np/q\} = \{(q-1)/q\} = 1 - 1/q \geq 1/2$. Consequently, $q = 1$ and the conclusion follows.

Problem 2. The vertices of two acute triangles all lie on a same circle. The midpoints of two sides of one triangle both lie on the nine-point circle of the other triangle. Show that the two triangles share the same nine-point circle.

Solution. We shall start stating a well-known result.

Lemma. Let ABC be a triangle with circumcenter O and nine point center N. If O' is the reflection of O across BC then the points A, N, O' are collinear and $NA = NO'$.

Proof. This is a simple consequence of the fact that O' is the circumcenter of BHC (H being the ortocenter of ABC) and the remark that the nine point circle of ABC is the homothetic image of the circumcenter of BHC under $\mathcal{H}(A, 1/2)$.

Returning to the problem, consider a triangle ABC inscribed in Γ of center O and nine point center γ of center N. The second triangle is XYZ such that the midpoints of XY and XZ lie on γ. This can be reformulated by saing that Y, Z are the intersection points of the image of γ under $\mathcal{H}(X, 2)$ (say γ') with Γ.

We have two cases: $\gamma' = \Gamma$ and $\gamma' \neq \Gamma$. If $\gamma' \neq \Gamma$ they must be symetric with respect to YZ. By construction, the center O' of γ' and the circumcenter O of ABC are symetric with respect to YZ. By the Lemma, the nie point center N' of XYZ coincides with the midpoint of XO'. But N is the midpoint of XO', so $N = N'$, and we are done.

If $\gamma' = \Gamma$, it follows that $\mathcal{H}(X, 2)$ maps γ into Γ. But the two homotheties that map γ into Γ are $\mathcal{H}(G, -2)$ and $\mathcal{H}(H, 2)$. Thus we must have $X \equiv H$, meaning that the ortocenter of ABC lies on Γ. This happens iff triangle ABC is right-angled. Since, by hypothesis ABC is aucte, this case cannot hold.

Remark. We point out, that by taking ABC to be right angled, say in A, $X \equiv A$, Y and Z arbitrary on Γ, then theh Euler circle of AYZ is symetric of γ across $Y'Z'$, ehere Y' and Z' are the midpoints of XY and XZ (IN this case γ is tangent in A at Γ having half of its radius).

Problem 3. Let S be the set of rational numbers of the form

$$\frac{(a_1^2 + a_1 - 1)(a_2^2 + a_2 - 1) \cdots (a_n^2 + a_n - 1)}{(b_1^2 + b_1 - 1)(b_2^2 + b_2 - 1) \cdots (b_n^2 + b_n - 1)},$$

where n, a_1, a_2, …, a_n, b_1, b_2, …, b_n run through the positive integers. Show that S contains infinitely many primes.

Solution. Clearly, S is closed under multiplication and division: if r and s are members of S, so are rs and r/s.

If a is a positive integer, and $p \neq 5$ is a prime factor of $a^2 + a - 1$, then $p \equiv \pm 1 \pmod 5$. To prove this, notice that $(2a+1)^2 \equiv 5 \pmod p$, so 5 is a quadratic residue modulo p. By quadratic reciprocity, p is a quadratic residue modulo 5, so $p \equiv \pm 1 \pmod 5$. Notice also that S contains 5, for $5 = 2^2 + 2 - 1$.

We now show by induction that S contains all primes congruent to ± 1 modulo 5. Since there are infinitely many such, the conclusion follows. To begin, notice that 11 and 19 both are in S: $11 = 3^2 + 3 - 1$, and $19 = 4^2 + 4 - 1$.

Consider now a prime $q \equiv \pm 1 \pmod 5$, and assume that S contains all primes $p < q$, $p \equiv \pm 1 \pmod 5$. Since q is a quadratic residue modulo 5, quadratic reciprocity shows that 5 is a quadratic residue modulo q, so there exists a in $\{1, 2, \ldots, q-1\}$ such that $a^2 + a - 1 = mq$ for some positive integer m. Notice that $a^2 + a - 1 \leq (q-1)^2 + (q-1) - 1 = q^2 - q - 1 < q^2$, to deduce that $m < q$. If $m = 1$, then $q = a^2 + a - 1$ which is a member of S. If $m > 1$, and p is a prime factor of m, then p is also a prime factor of $a^2 + a - 1$, so $p = 5$ or $p \equiv \pm 1 \pmod 5$. In either case, p is a member of S, so m is a member, for S is closed under multiplication. Since $q = (a^2 + a - 1)/m$, and S is closed under division, it follows that q is indeed a member of S. This completes the proof.

Remark. Since S contains all primes congruent to ± 1 modulo 5, it must contain 31. Although there is no integer a such that $a^2 + a - 1 = 31$, the latter may be written in the form $(12^2 + 12 - 1)/(2^2 + 2 - 1)$ which explicitly exhibits 31 as a member of S.

Problem 4. Given an integer $k \geq 2$, exhibit an infinite set $\mathcal{A}$ of sets of positive integers satisfying the two conditions below:

(a) The intersection of the members of every k-element subset of $\mathcal{A}$ is a singleton set; and
(b) The intersection of the members of every $(k+1)$-element subset of $\mathcal{A}$ is empty.

Solution. Biject the set of k-element sets of positive integers with the set of positive integers to label the former $S_1, S_2, \ldots, S_n, \ldots$. For every positive integer m, set $A_m = \{n \colon m \in S_n\}$.

If m and m' are distinct positive integers, there exist distinct positive integers n and n' such that $m \in S_n$ and $m' \in S_{n'}$. Consequently, $n \in A_m \setminus A_{m'}$ and $n' \in A_{m'} \setminus A_m$; in particular, $A_m \neq A_{m'}$, so the A's form an infinite set $\mathcal{A}$.

Next, if $m_1, m_2, \ldots, m_k$ are distinct positive integers, then $A_{m_1} \cap A_{m_2} \cap \ldots \cap A_{m_k} = \{n\}$, where n is the index of the label of the set $\{m_1, m_2, \ldots, m_k\}$ in the list $S_1, S_2, \ldots$. Consequently, $\mathcal{A}$ satisfies **(a)**.

Finally, if $m_1, m_2, \ldots, m_k, m_{k+1}$ are distinct positive integers, then $\{m_1, m_2, \ldots, m_k\}$ and $\{m_2, \ldots, m_k, m_{k+1}\}$ have different labels in the list $S_1, S_2, \ldots$, so $A_{m_1} \cap A_{m_2} \cap \ldots \cap A_{m_k} \cap A_{m_{k+1}}$ is empty. Consequently, $\mathcal{A}$ satisfies **(b)**.