The 7th Romanian Master of Mathematics Competition

Solutions for the Day 2

Problem 4. Let ABC be a triangle, let D be the touchpoint of the side BC and the incircle of the triangle ABC, and let J_b and J_c be the incentres of the triangles ABD and ACD, respectively. Prove that the circumcentre of the triangle AJ_bJ_c lies on the bisectrix of the angle BAC.

(Russia) Fedor Ivlev

Solution. Let the incircle of the triangle ABC meet CA and AB at points E and F, respectively. Let the incircles of the triangles ABD and ACD meet AD at points X and Y, respectively. Then $2DX = DA+DB-AB = DA+DB-BF-AF = DA-AF$; similarly, $2DY = DA-AE = 2DX$. Hence the points X and Y coincide, so $J_bJ_c \perp AD$.

Now let O be the circumcentre of the triangle AJ_bJ_c. Then $\angle J_bAO = \pi/2 - \angle AOJ_b/2 = \pi/2 - \angle AJ_cJ_b = \angle XAJ_c = \frac{1}{2}\angle DAC$. Therefore, $\angle BAO = \angle BAJ_b + \angle J_bAO = \frac{1}{2}\angle BAD + \frac{1}{2}\angle DAC = \frac{1}{2}\angle BAC$, and the conclusion follows.

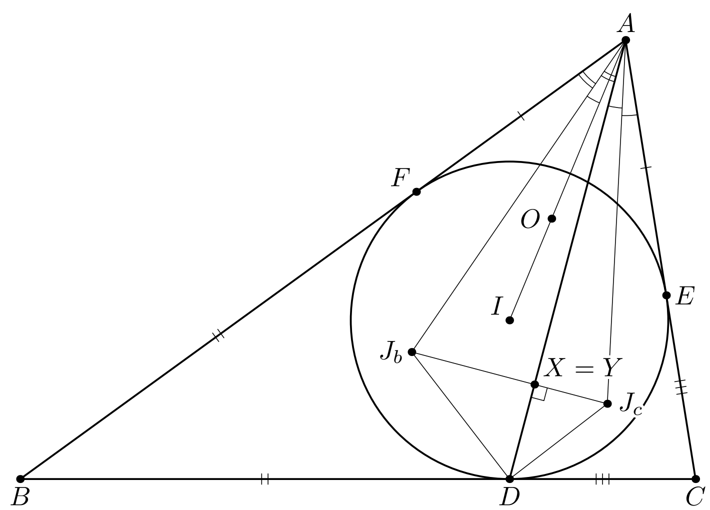

Fig. 1

Problem 5. Let $p \geq 5$ be a prime number. For a positive integer k we denote by $R(k)$ the remainder of k when divided by p. Determine all positive integers $a < p$ such that

$$m + R(ma) > a$$

for every $m = 1, 2, \ldots, p-1$.

(BULGARIA) ALEXANDER IVANOV

Solution. The required integers are $p-1$ along with all the numbers of the form $\lfloor p/q \rfloor$, $q = 2, \ldots, p-1$. In other words, these are $p-1$, along with the numbers $1, 2, \ldots, \lfloor \sqrt{p} \rfloor$, and also the (distinct) numbers $\lfloor p/q \rfloor$, $q = 2, \ldots, \left\lfloor \sqrt{p} - \frac{1}{2} \right\rfloor$.

We begin by showing that these numbers satisfy the conditions in the statement. It is readily checked that $p-1$ satisfies the required inequalities, since $m + R(m(p-1)) = m + (p-m) = p > p-1$ for all $m = 1, \ldots, p-1$.

Now, consider any number a of the form $a = \lfloor p/q \rfloor$, where q is an integer greater than 1 but less than p; then $p = aq + r$ with $0 < r < q$. Choose any integer $m \in (0, p)$ and write $m = xq + y$ with $x, y \in \mathbb{Z}$, $0 < y \leq q$ (notice that x is nonnegative). Then

$$R(ma) = R(ay + xaq) = R(ay + xp - xr) = R(ay - xr).$$

Since $ay - xr \leq ay \leq aq < p$, we obtain $R(ay - xr) \geq ay - xr$ and hence

$$m + R(ma) \geq (xq + y) + (ay - xr) = x(q - r) + y(a + 1) \geq a + 1$$

by $q > r$ and $y \geq 1$. Thus a satisfies the required condition.

Finally, we show that if an integer $a \in (0, p-1)$ satisfies the required condition then a is indeed of the form $a = \lfloor p/q \rfloor$ for some integer $q \in (0, p)$. This is clear for $a = 1$, so we may (and will) assume that $a \geq 2$.

Write $p = aq + r$ with $q, r \in \mathbb{Z}$ and $0 < r < a$; since $a \geq 2$ we have $q < p/2$. Choose $m = q + 1 < p$; we have $R(ma) = R(aq + a) = R(p + (a - r)) = a - r$, so

$$a < m + R(ma) = q + 1 + a - r,$$

which yields $r < q + 1$. Moreover, if $r = q$, then $p = q(a+1)$ which is impossible by $1 < a + 1 < p$. Thus $r < q$, and we have

$$0 \leq \frac{p}{q} - a = \frac{r}{q} < 1,$$

which proves $a = \lfloor p/q \rfloor$.

Problem 6. Given a positive integer n, determine the largest real number μ satisfying the following condition: for every $4n$-point configuration C in an open unit square U, there exists an open rectangle in U, whose sides are parallel to those of U, which contains exactly one point of C, and has an area greater than or equal to μ.

(Bulgaria) Nikolai Beluhov

Solution. The required maximum is $\frac{1}{2n+2}$. To show that the condition in the statement is not met if $\mu > \frac{1}{2n+2}$, let $U = (0,1) \times (0,1)$, choose a small enough positive ϵ, and consider the configuration C consisting of the n four-element clusters of points $\left(\frac{i}{n+1} \pm \epsilon\right) \times \left(\frac{1}{2} \pm \epsilon\right)$, $i = 1, \ldots, n$, the four possible sign combinations being considered for each i. Clearly, every open rectangle in U, whose sides are parallel to those of U, which contains exactly one point of C, has area at most $\left(\frac{1}{n+1} + \epsilon\right) \cdot \left(\frac{1}{2} + \epsilon\right) < \mu$ if ϵ is small enough.

We now show that, given a finite configuration C of points in an open unit square U, there always exists an open rectangle in U, whose sides are parallel to those of U, which contains exactly one point of C, and has an area greater than or equal to $\mu_0 = \dfrac{2}{|C| + 4}$.

To prove this, usage will be made of the following two lemmas whose proofs are left at the end of the solution.

Lemma 1. *Let k be a positive integer, and let $\lambda < \dfrac{1}{\lfloor k/2 \rfloor + 1}$ be a positive real number. If $t_1, \ldots, t_k$ are pairwise distinct points in the open unit interval $(0,1)$, then some t_i is isolated from the other t_j by an open subinterval of $(0,1)$ whose length is greater than or equal to λ.*

Lemma 2. *Given an integer $k \geq 2$ and positive integers $m_1, \ldots, m_k$,*

$$\left\lfloor \frac{m_1}{2} \right\rfloor + \sum_{i=1}^{k} \left\lfloor \frac{m_i}{2} \right\rfloor + \left\lfloor \frac{m_k}{2} \right\rfloor \leq \sum_{i=1}^{k} m_i - k + 2.$$

Back to the problem, let $U = (0,1) \times (0,1)$, project C orthogonally on the x-axis to obtain the points $x_1 < \cdots < x_k$ in the open unit interval $(0,1)$, let ℓ_i be the vertical through x_i, and let $m_i = |C \cap \ell_i|$, $i = 1, \ldots, k$.

Setting $x_0 = 0$ and $x_{k+1} = 1$, assume that $x_{i+1} - x_{i-1} > (\lfloor m_i/2 \rfloor + 1)\mu_0$ for some index i, and apply Lemma 1 to isolate one of the points in $C \cap \ell_i$ from the other ones by an open subinterval $x_i \times J$ of $x_i \times (0,1)$ whose length is greater than or equal to $\mu_0/(x_{i+1} - x_{i-1})$. Consequently, $(x_{i-1}, x_{i+1}) \times J$ is an open rectangle in U, whose sides are parallel to those of U, which contains exactly one point of C and has an area greater than or equal to μ_0.

Next, we rule out the case $x_{i+1} - x_{i-1} \leq (\lfloor m_i/2 \rfloor + 1)\mu_0$ for all indices i. If this were the case, notice that necessarily $k > 1$; also, $x_1 - x_0 < x_2 - x_0 \leq (\lfloor m_1/2 \rfloor + 1)\mu_0$ and $x_{k+1} - x_k < x_{k+1} - x_{k-1} \leq (\lfloor m_k/2 \rfloor + 1)\mu_0$. With reference to Lemma 2, write

$$\begin{aligned}
2 = 2(x_{k+1} - x_0) &= (x_1 - x_0) + \sum_{i=1}^{k} (x_{i+1} - x_{i-1}) + (x_{k+1} - x_k) \\
&< \left(\left(\left\lfloor \frac{m_1}{2} \right\rfloor + 1 \right) + \sum_{i=1}^{k} \left(\left\lfloor \frac{m_i}{2} \right\rfloor + 1 \right) + \left(\left\lfloor \frac{m_k}{2} \right\rfloor + 1 \right) \right) \cdot \mu_0 \\
&\leq \left(\sum_{i=1}^{k} m_i + 4 \right) \mu_0 = (|C| + 4)\mu_0 = 2,
\end{aligned}$$

and thereby reach a contradiction.

Finally, we prove the two lemmas.

Proof of Lemma 1. Suppose, if possible, that no t_i is isolated from the other t_j by an open subinterval of $(0,1)$ whose length is greater than or equal to λ. Without loss of generality, we may (and will) assume that $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$. Since the open interval (t_{i-1}, t_{i+1}) isolates t_i from the other t_j, its length, $t_{i+1} - t_{i-1}$, is less than λ. Consequently, if k is odd we have $1 = \sum_{i=0}^{(k-1)/2}(t_{2i+2} - t_{2i}) < \lambda\big(1 + \frac{k-1}{2}\big) < 1$; if k is even, we have $1 < 1 + t_k - t_{k-1} = \sum_{i=0}^{k/2-1}(t_{2i+2} - t_{2i}) + (t_{k+1} - t_{k-1}) < \lambda\big(1 + \frac{k}{2}\big) < 1$. A contradiction in either case.

Proof of Lemma 2. Let I_0, respectively I_1, be the set of all indices i in the range 2, …, $k-1$ such that m_i is even, respectively odd. Clearly, I_0 and I_1 form a partition of that range. Since $m_i \geq 2$ if i is in I_0, and $m_i \geq 1$ if i is in I_1 (recall that the m_i are positive integers),

$$\sum_{i=2}^{k-1} m_i = \sum_{i\in I_0} m_i + \sum_{i\in I_1} m_i \geq 2|I_0| + |I_1| = 2(k-2) - |I_1|, \quad \text{or} \quad |I_1| \geq 2(k-2) - \sum_{i=2}^{k-1} m_i.$$

Therefore,

$$\begin{aligned} \left\lfloor \frac{m_1}{2} \right\rfloor + \sum_{i=1}^{k} \left\lfloor \frac{m_i}{2} \right\rfloor + \left\lfloor \frac{m_k}{2} \right\rfloor &\leq m_1 + \left(\sum_{i=2}^{k-1} \frac{m_i}{2} - \frac{|I_1|}{2} \right) + m_k \\ &\leq m_1 + \left(\frac{1}{2}\sum_{i=2}^{k-1} m_i - (k-2) + \frac{1}{2}\sum_{i=2}^{k-1} m_i \right) + m_k \\ &= \sum_{i=1}^{k} m_i - k + 2. \end{aligned}$$

□

Remark. In case $4n$ is replaced by a positive integer k not divisible by 4, we do not yet know the maximal μ satisfying the corresponding condition.