Third Test — Solutions

Problem 1. Let ABC be an isosceles triangle, $AB = AC$, and let M and N be points on the sides BC and CA, respectively, such that the angles BAM and CNM are equal. The lines AB and MN meet at P. Show that the internal angle bisectrices of the angles BAM and BPM meet at a point on the line BC.

Solution. By the internal angle bisectrix theorem, it is sufficient to show that $PB/PM = AB/AM$. Since the angles BAM and CNM, respectively ABM and MCN, are equal, so are the angles AMB and CMN. Hence the angles BMP and AMC are equal, and since the angles MBP and ACM are also equal, the triangles BMP and CMA are similar. Consequently, $BP/AC = MP/AM$. Finally, rewrite the latter in the form $BP/MP = AC/AM$ and replace AC by $AB = AC$.

Problem 2. For every positive integer n, let $\sigma(n)$ denote the sum of all positive divisors of n (1 and n, inclusive). Show that a positive integer n, which has at most two distinct prime factors, satisfies the condition $\sigma(n) = 2n - 2$ if and only if $n = 2^k\left(2^{k+1}+1\right)$, where k is a non-negative integer and $2^{k+1}+1$ is prime.

Solution. Sufficiency is a routine verification. To prove necessity, assume first that $n = 2^k p^l$, where k and l are non-negative integers, and p is an odd prime. Notice that l must be positive, so

$$1 + \frac{1}{p} \le \frac{\sigma(p^l)}{p^l} = \frac{p - \frac{1}{p^l}}{p-1} < \frac{p}{p-1},$$

whence (since σ is multiplicative)

$$\left(2^{k+1}-1\right)\left(1+\frac{1}{p}\right) \le \frac{\sigma(n)}{p^l} = 2^{k+1} - \frac{2}{p^l} < \left(2^{k+1}-1\right)\frac{p}{p-1}.$$

By the first inequality, $\left(2^{k+1}-1\right)(1+1/p) < 2^{k+1}$, so $p > 2^{k+1}-1$, i.e., $p \ge 2^{k+1}+1$ since p is odd. On the other hand, $p < 2^{k+1} + 2(p-1)/p^l$, by the second inequality, so $2(p-1) > p^l$, and consequently $l = 1$ and $p = 2^{k+1}+1$.

To rule out the case $n = p^k q^l$, where p and q are distinct odd primes, and k and l are positive integers, write

$$2 - \frac{2}{n} = \frac{\sigma(n)}{n} = \frac{\sigma(p^k)}{p^k}\cdot\frac{\sigma(q^l)}{q^l} < \frac{p}{p-1}\cdot\frac{q}{q-1}.$$

Alternatively, but equivalently,

$$\frac{1}{p-1} + \frac{1}{q-1} + \frac{1}{(p-1)(q-1)} + \frac{2}{n} > 1,$$

so $\min(p,q) = 3$, say $p = 3$. Then $3/(q-1) + 4/n > 1$, and it follows that $q = 5$ and $k = l = 1$, i.e., $n = 15$ which does not satisfy the condition $\sigma(n) = 2n - 2$. This completes the proof.

Remarks. A positive integer n satisfying the condition $\sigma(n) = 2n - 2$ is called a perfect-plus-two (pp2) number. The result in the problem shows that $3 = 2^0(2^{0+1}+1)$, $10 = 2\cdot 5 = 2^1(2^{1+1}+1)$, $136 = 2^3\cdot 17 = 2^3(2^{3+1}+1)$, $128\cdot 257 = 2^7(2^{7+1}+1)$ are all pp2 integers.

It can be shown that a pp2 odd integer greater than 3, if any, must have at least four distinct prime factors, and if, in addition, it is not divisible by 3, then it must have at least seven distinct prime factors.

Pp1 integers, i.e., positive integers n such that $\sigma(n) = 2n - 1$, have equally well been considered, but the powers of 2 are the only ones known.

Problem 3. Determine the smallest real constant c such that

$$\sum_{k=1}^{n} \left(\frac{1}{k} \sum_{j=1}^{k} x_j \right)^2 \le c \sum_{k=1}^{n} x_k^2,$$

for all positive integers n and all positive real numbers $x_1, \ldots, x_n$.

Solution. The best constant is $c = 4$. We first show that, if n is a positive integer and $x_1, \ldots, x_n$ are positive real numbers, then

$$\sum_{k=1}^{n} \left(\frac{1}{k} \sum_{j=1}^{k} x_j \right)^2 + \frac{2}{n} \left(\sum_{k=1}^{n} x_k \right)^2 < 4 \sum_{k=1}^{n} x_k^2,$$

so $c \le 4$. To prove the above inequality, proceed by induction on n. The base case, $n = 1$, is clear. For the induction step, let $\bar{x}_k = (x_1 + \cdots + x_k)/k$, $k \ge 1$, and notice that it is sufficient to show that $(2n+3)\bar{x}_{n+1}^2 - 2n\bar{x}_n^2 < 4x_{n+1}^2$. Since $x_{n+1} = (n+1)\bar{x}_{n+1} - n\bar{x}_n$, this is equivalent to $2n(2n+1)\bar{x}_n^2 - 8n(n+1)\bar{x}_n\bar{x}_{n+1} + (4n^2 + 6n + 1)\bar{x}_{n+1}^2 > 0$. The left-hand member is a quadratic form in $\bar{x}_n$ and $\bar{x}_{n+1}$ whose discriminant is $-2n$ and the inequality follows.

To show $c \ge 4$, we prove that

$$\sum_{k=1}^{n} \left(\frac{1}{k} \sum_{j=1}^{k} \frac{1}{\sqrt{j}} \right)^2 > 4 \sum_{k=1}^{n} \frac{1}{k} - 24.$$

Divergence of the harmonic series settles the case. Write $1/\sqrt{j} > 2(\sqrt{j+1} - \sqrt{j})$, to obtain

$$\left(\frac{1}{k} \sum_{j=1}^{k} \frac{1}{\sqrt{j}} \right)^2 > \frac{4}{k^2} \left(\sqrt{k+1} - 1 \right)^2 > \frac{4}{k} \left(1 - \frac{2}{\sqrt{k}} \right) = \frac{4}{k} - \frac{8}{k\sqrt{k}}.$$

Finally, notice that $1/(2k\sqrt{k}) < 1/\sqrt{k-1} - 1/\sqrt{k}$, $k \ge 2$, to get

$$\sum_{k=1}^{n} \frac{1}{k\sqrt{k}} \le 3 - \frac{2}{\sqrt{n}} < 3,$$

and deduce thereby the desired inequality.

Problem 4. Let n be a positive integer, and let A_n, respectively B_n, be the set of non-negative integers $k < n$ such that the number of distinct prime factors of $\gcd(k, n)$ is even, respectively odd. Show that $|A_n| = |B_n|$ if n is even, and $|A_n| > |B_n|$ if n is odd.

Solution. Since $\gcd(k, n)$ depends only upon the residue class of k modulo n, $|A_n| - |B_n| = \sum_k (-1)^{s(k,n)}$, where $s(k, n)$ is the number of distinct prime factors of $\gcd(k, n)$, and k ranges over any complete residue system modulo n.

We shall prove that the above sum equals $n\prod_{p|n}(1-2/p)$, p prime, whence the conclusion; the latter is precisely the number of positive integers $k < n$ such that k and $k+1$ are both coprime to n.

In the above notation, let $e(k,n) = (-1)^{s(k,n)}$ and let $f(n) = \sum_k e(k,n)$. We show that f is a numerical multiplicative function — that is, if n_1 and n_2 are coprime positive integers, then $f(n_1 n_1) = f(n_1)f(n_2)$.

If n_1 and n_2 are coprime positive integers, then $e(k, n_1n_2) = e(k,n_1)e(k,n_2)$. Further, if k_i ranges once over a complete residue system modulo n_i, $i = 1, 2$, then $k = k_1n_2 + k_2n_1$ ranges once over a complete residue system modulo n_1n_2, and $e(k,n_i) = e(k_i,n_i)$, $i = 1, 2$. Hence $e(k,n_1n_2) = e(k,n_1)e(k,n_2) = e(k_1,n_1)e(k_2,n_2)$, and $f(n_1n_2) = \sum_k e(k,n_1n_2) = \sum_{k_1}\sum_{k_2} e(k_1,n_1)e(k_2,n_2) = f(n_1)f(n_2)$.

Finally, if p is a prime, and m is a positive integer, then $f(p^m)$ equals the number of k's coprime to p, which is $p^m - p^{m-1}$, minus the number of k's divisible by p, which is p^{m-1}, so $f(p^m) = p^m(1-2/p)$. This ends the proof.