Second Test — Solutions

Problem 1. Let ABC be a triangle and let X, Y, Z be interior points on the sides BC, CA, AB, respectively. Show that the magnified image of the triangle XYZ under a homothecy of factor 4 from its centroid covers at least one of the vertices A, B, C.

Solution 1. Since the problem is of an affine nature, we may (and will) assume that the triangle XYZ is equilateral. The triangle ABC has at least one vertex angle, say at A, greater than or equal to 60°, so A is covered by the closed circumdisc OYZ, where O is the centre of the triangle XYZ. Since the latter is covered by the 4-fold blow-up of the triangle XYZ from O, the conclusion follows.

Solution 2. Suppose, if possible, that none of the vertices A, B, C is covered by the 4-fold blow-up of the triangle XYZ from its centroid. Then the distance of the point A to the line YZ is greater than the distance of the point X to this line, so the area of the triangle AYZ is greater than the area of the triangle XYZ. Similarly, the triangles BZX and CXY both have an area greater than that of the triangle XYZ, in contradiction with the well known fact that of the four triangles AYZ, BZX, CXY, XYZ, the latter has not the smallest area.

Problem 2. Let a be a real number in the open interval $(0, 1)$, let n be a positive integer and let $f_n \colon \mathbb{R} \to \mathbb{R}$, $f_n(x) = x + x^2/n$. Show that

$$\frac{a(1-a)n^2 + 2a^2n + a^3}{(1-a)^2n^2 + a(2-a)n + a^2} < (\underbrace{f_n \circ \cdots \circ f_n}_{n})(a) < \frac{an + a^2}{(1-a)n + a}.$$

Solution. Let $a_k = (\underbrace{f_n \circ \cdots \circ f_n}_{k})(a)$, $k \in \mathbb{N}$, and notice that

$$1/a_{k+1} = 1/a_k - 1/(a_k + n), \quad k \in \mathbb{N},$$

to deduce that $1/a_n = 1/a - \sum_{k=0}^{n-1} 1/(a_k + n)$, so

$$1/a - n/(a+n) < 1/a_n < 1/a - n/(a_n + n), \qquad (*)$$

since the a_k form an increasing sequence of positive real numbers. The first inequality above yields the required upper bound,

$$a_n < \frac{an + a^2}{(1-a)n + a}.$$

Plugged into the rightmost expression in $(*)$, this upper bound yields the required lower bound,

$$a_n > \frac{a(1-a)n^2 + 2a^2n + a^3}{(1-a)^2n^2 + a(2-a)n + a^2}.$$

Problem 3. Determine all positive integers n such that all positive integers less than n and coprime to n be powers of primes.

Solution. Let $p_1 = 2 < p_2 = 3 < p_3 = 5 < \cdots$ be the sequence of primes and let q and r, $q < r$, be the first two primes which do not divide n. A necessary and sufficient condition that

n be of the required type is that $n < qr$. Each of the primes less than r and different from q divides n, and so does their product. Therefore the product of all primes less than r does not exceed $nq < q^2 r$. If $r = p_m$, then $q \le p_{m-1}$, so $p_1 p_2 \cdots p_{m-2} < p_{m-1} p_m$.

Notice that 6 is the first index k such that $p_1 p_2 \cdots p_{k-2} > p_{k-1} p_k$. Now, if $p_1 p_2 \cdots p_{k-2} > p_{k-1} p_k$ for some index $k \ge 6$, then (by Bertrand-Tchebysheff) $p_1 p_2 \cdots p_{k-1} > p_{k-1}^2 p_k > 2p_{k-1} \cdot 2p_k > p_k p_{k+1}$, so $p_1 p_2 \cdots p_{k-2} > p_{k-1} p_k$ for all indices $k \ge 6$.

Consequently, $m \le 5$, $r = p_m \le p_5 = 11$, $q \le p_4 = 7$, and $n < qr \le p_4 p_5 = 7 \cdot 11 = 77$. Examination of the integers less than 77 quickly yields the required numbers: 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 18, 20, 24, 30, 42, 60.

Problem 4. Let f be the function of the set of positive integers into itself, defined by $f(1) = 1$, $f(2n) = f(n)$ and $f(2n+1) = f(n) + f(n+1)$. Show that, for any positive integer n, the number of positive odd integers m such that $f(m) = n$ is equal to the number of positive integers less than and coprime to n.

Solution. With reference to the recurrence for f, notice that if n is a positive even, respectively odd, integer, then $f(n) < f(n+1)$, respectively $f(n) \ge f(n+1)$, so $f(n) < f(n+1)$ if and only if n is even.

With reference again to the recurrence for f, an esay induction shows $f(n)$ and $f(n+1)$ coprime for each positive integer n.

Discarding the trivial case $n = 1$, given a positive integer $n \ge 2$, it follows that if m is a positive odd integer such that $f(m) = n$, then $f(m-1)$ is a positive integer less than and coprime to n.

Next, we prove that for every pair of coprime positive integers (k, n) there exists a unique positive integer m such that $k = f(m)$ and $n = f(m+1)$. If, in addition, $k < n$, then m is even by the preceding, so $m + 1$ is a positive odd integer such that $f(m+1) = n$ and the conclusion follows.

To prove the above claim, proceed by induction on $k + n$. The base case, $k + n = 2$, i.e. $k = n = 1$, is clear. If $k + n > 2$, apply the induction hypothesis to the pair $(k, n-k)$ or $(k-n, n)$, according as to $k < n$ or $k > n$. In the former case, $k = f(m) = f(2m)$ and $n = k + f(m+1) = f(m) + f(m+1) = f(2m+1)$ for some positive integer m; in the latter, $n = f(m+1) = f(2m+2)$ and $k = f(m) + n = f(m) + f(m+1) = f(2m+1)$ for some positive integer m. This establishes the existence of the desired positive integer.

To prove uniqueness, write $k = f(m)$ and $n = f(m+1)$ for some positive integer m, and consider again the two possible cases.

If $k < n$, then m is even, say $m = 2m'$, where m' is a positive integer, so $k = f(2m') = f(m')$ and $n - k = f(2m'+1) - f(m') = f(m'+1)$. The induction hypothesis applies to the pair $(k, n-k)$ to imply uniqueness of m', hence uniqueness of m.

If $k > n$, then m is odd, say $m = 2m'+1$, where m' is a non-negative integer, so $k - n = f(2m'+1) - f(2m'+2) = f(m') + f(m'+1) - f(m'+1) = f(m')$ and $n = f(2m'+2) = f(m'+1)$. The induction hypothesis applies now to the pair $(k-n, n)$ to imply uniqueness of m', hence again uniqueness of m. This completes the induction step and ends the proof.