Test de Selecţie pentru EGMO 2014 (fete) şi MofM 2014 – Soluţii

Problem 1. Given $n+1$ distinct real numbers in the interval $[0,1]$, prove there exist two of them $a \neq b$, such that $ab|a-b| < \dfrac{1}{3n}$.

AoPS

Solution. Index the numbers $0 \leq a_0 < a_1 < \cdots < a_n \leq 1$. If $a_0 = 0$ we're done; if not, $\sum_{k=0}^{n-1} a_k a_{k+1}(a_{k+1}-a_k) = \dfrac{1}{3}\left(a_n^3 - a_0^3 - \sum_{k=0}^{n-1}(a_{k+1}-a_k)^3\right) < \dfrac{1}{3}$, so there will exist $0 \leq k \leq n-1$ such that $\boxed{a_k a_{k+1}(a_{k+1}-a_k) < \dfrac{1}{3n}}$ (by an averaging argument). ■

Problem 2. What is the minimum number $m(n)$ of edges of K_n (the complete graph on $n \geq 4$ vertices) that can be colored red, such that any K_4 subgraph contains a red K_3? For example, $m(4) = 3$.

AoPS

Solution. The answer is in fact quite easy to get. Assume the edge ab is not red. Then the fact that among any $\{a,b,x,y\}$ has to exist a red triangle forces xy to be red, and moreover, either ax, ay to be red or bx, by to be red. That means $K_n - \{a,b\} = K_{n-2}$ is red. Let A be the set of vertices x such that ax is red, and B be the set of vertices y such that by is red; it follows $A \cup B = K_n \setminus \{a,b\}$. If we could take $x \in A \setminus B$ and $y \in B \setminus A$, then $\{a,b,x,y\}$ would be a contradiction, so say $B \setminus A = \emptyset$, thus $A = K_n \setminus \{a,b\}$, therefore $K_n - \{b\} = K_{n-1}$ is red. That is enough, so $\boxed{m(n) = (n-1)(n-2)/2}$. ■

Problem 3. Let $0 < p \leq Q$ be fixed real numbers, and let a, b, x and y be positive real numbers, such that $\begin{cases} ax & \leq p \\ ay & \leq Q \\ bx & \leq Q \\ by & \leq Q \end{cases}$. Determine the maximum value of $(a+b)(x+y)$, and the cases of equality.

Sgall's Lemma

Solution. Let us normalize, by taking $\lambda = \dfrac{y}{x}$, $\mu = \dfrac{b}{a}$, $m = \min\{\lambda,\mu\}$, $M = \max\{\lambda,\mu\}$, $p' = \dfrac{p}{ax}$ and $Q' = \dfrac{Q}{ax}$, and dividing all inequations by ax, to get $\begin{cases} 1 & \leq p' \\ m & \leq Q' \\ M & \leq Q' \\ mM & \leq Q' \end{cases}$.

We thus need to maximize $(1+m)(1+M)$. We claim the maximum is $2(p'+Q')$.

- If $m < 1$, then $1 + m + M + mM < 2 + 2Q' \leq 2(p'+Q')$.
- If $1 \leq m$, then $(m-1)(M-1) \geq 0$, so $m + M \leq 1 + mM$, thus $1 + m + M + mM \leq 2(1+mM) \leq 2(p'+Q')$. Equality is reached if and only if $p' = 1$, $m = 1$, $M = Q'$.

Going back to the original variables, the above means $\boxed{(a+b)(x+y) \leq 2(p+Q)}$, with equality occuring if and only if $\boxed{p = ax}$ and $\boxed{y = x \text{ and } Q = bx}$ or $\boxed{b = a \text{ and } Q = ay}$. ■

Problem 4. Say that a (nondegenerate) triangle is *funny* if it satisfies the condition that the altitude, median, and angle bisector drawn from one of the vertices partition the triangle into 4 non-overlapping triangles whose areas form (in some order) a 4-term arithmetic sequence. (One of these 4 triangles is allowed to be degenerate.) Find, with proof, all funny triangles.

Math Prize for Girls 2013

Solution. (L. Ploscaru) Să presupunem că cele trei ceviene pleacă din A, cu $AB < AC$ ($\triangle ABC$ nu poate evident fi isoscel în A; din ipoteză se deduce şi că triunghiul *funny* nu poate fi obtuzunghic în B sau C.). Ordinea dreptelor este

$$AB - \text{înălţimea} - \text{bisectoarea} - \text{mediana} - AC$$

(se demonstrează eventual uitându-ne la picioarele lor pe BC). Ideea principală este să demonstrăm că un triunghi *funny* ABC e dreptunghic (paranteza din ipoteză face aluzie la această posibilitate; dacă nu erau triunghiuri *funny* dreptunghice, nu îşi avea rostul).

Să zicem că M este mijlocul lui BC; atunci aria$[ABM]$ = aria$[ACM]$, deci clar ACM e triunghiul cu cea mai mare arie. Fie q, $q+r$, $q+2r$, $q+3r$ ariile. Cele 3 triunghiuri mici îl partiţionează pe ABM, deci $3q+3r = q+3r$, de unde $q = 0$, iar atunci singurul fel în care 2 din cele 5 drepte de mai sus pot coincide este $AB \perp BC$.

Acum problema e aproape gata; luăm D piciorul bisectoarei, şi prin simpla formulă aria $= \dfrac{1}{2}$baza$\times$înălţimea, vom obţine că $\{BD, DM, MC\} = \{x, 2x, 3x\}$ pentru un x real pozitiv. Evident $MC = 3x$, iar atunci în fiecare dintre cele două cazuri aplicăm teorema bisectoarei ca să aflăm valoarea raportului $AB/AC = \cos A$, şi am terminat. Obţinem $\angle A \in \{\arccos(1/5), \arccos(1/2) = \pi/3\}$ (deci unul dintre triunghiuri este cel de unghiuri $30°, 60°, 90°$, dar mai există un caz). ∎

Problem 5. For positive real numbers a, b, c with $a^2+b^2+c^2 \geq 3$, prove the inequality

$$\frac{a^2}{1+bc} + \frac{b^2}{1+ca} + \frac{c^2}{1+ab} \geq \frac{3}{2}$$

and determine its case(s) of equality.

Show that if $a^2+b^2+c^2 < 3$, the inequality may hold no more.

DAN SCHWARZ, variant of Italian Test

Solution. It is enough to consider the case $a^2+b^2+c^2 = 3$. Indeed, for $k \geq 1$ we have $\dfrac{(ka)^2}{1+(kb)(kc)} \geq \dfrac{a^2}{1+bc}$ *et.al.*

We then have $1+bc \leq 1+\dfrac{b^2+c^2}{2} = \dfrac{5-a^2}{2}$, hence $\dfrac{a^2}{1+bc} \geq \dfrac{2a^2}{5-a^2}$ *et.al.* Then the function $f\colon [0,3] \to \mathbb{R}$ given by $f(t) = \dfrac{2t}{5-t} = \dfrac{10}{5-t} - 2$ is clearly convex, therefore we have (by Jensen's inequality)

$$f(a^2)+f(b^2)+f(c^2) \geq 3f\left(\frac{a^2+b^2+c^2}{3}\right) = 3f(1) = \frac{3}{2}.$$

Thus the inequality is proved, with the obvious equality case when $a^2+b^2+c^2 = 3$ and $a = b = c = 1$.

For $a^2+b^2+c^2 < 3$ the inequality will hold no more; just consider $0 < a = b = c = k < 1$, and then LHS $= \dfrac{3k^2}{1+k^2} < \dfrac{3}{2}$. ∎

Alternative Solution. Trying the Cauchy-Schwarz inequality, just for $a^2+b^2+c^2 = 3$ (seen to be enough)

$$\frac{a^2}{1+bc} + \frac{b^2}{1+ca} + \frac{c^2}{1+ab} \geq \frac{(a+b+c)^2}{3+bc+ca+ab} = \frac{3+2(ab+bc+ca)}{3+ab+bc+ca}$$

will not work this time, since the hopeful continuation towards value $\dfrac{3}{2}$ would require $6+4(ab+bc+ca) \geq 9+3(ab+bc+ca)$, *i.e.* $ab+bc+ca \geq 3$, which in fact it is precisely the other way around.

If however we try a common trick, and write

$$\frac{a^2}{1+bc}+\frac{b^2}{1+ca}+\frac{c^2}{1+ab}=\frac{a^4}{a^2+a^2bc}+\frac{b^4}{b^2+b^2ca}+\frac{c^4}{c^2+c^2ab},$$

then we can continue by Cauchy-Schwarz

$$\frac{a^4}{a^2+a^2bc}+\frac{b^4}{b^2+b^2ca}+\frac{c^4}{c^2+c^2ab}\geq\frac{(a^2+b^2+c^2)^2}{(a^2+b^2+c^2)+abc(a+b+c)}=\frac{9}{3+abc(a+b+c)}.$$

Now, in order to continue with $\geq\frac{3}{2}$, we need $abc(a+b+c)\leq 3$, which holds true, since $abc\leq\left(\frac{a^2+b^2+c^2}{3}\right)^{3/2}=1$ and $a+b+c\leq\sqrt{3(a^2+b^2+c^2)}=3$; the equality case follows as above. ■

Alternative Solution. (C. Popescu) The required inequality is a consequence of the following inequality

$$\sum\frac{a^2}{1+bc}\geq\frac{3(a^2+b^2+c^2)}{3+a^2+b^2+c^2}.$$

To prove the latter, apply Jensen's inequality to the convex function $t\mapsto(1+t)^{-1}$, $t>-1$, at $t_1=bc$, $t_2=ca$ and $t_3=ab$, with weights $\lambda_1=a^2/(a^2+b^2+c^2)$, $\lambda_2=b^2/(a^2+b^2+c^2)$ and $\lambda_3=c^2/(a^2+b^2+c^2)$, respectively, to obtain

$$\sum\frac{a^2}{a^2+b^2+c^2}\cdot\frac{1}{1+bc}\geq\frac{1}{1+\sum\frac{a^2}{a^2+b^2+c^2}\cdot bc}=\frac{a^2+b^2+c^2}{a^2+b^2+c^2+abc(a+b+c)},$$

and get thereby

$$\sum\frac{a^2}{1+bc}\geq\frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+abc(a+b+c)}.$$

Now,

$$abc(a+b+c)\leq\frac{1}{3^3}(a+b+c)^3(a+b+c)\leq\frac{1}{3^3}\cdot 3^2(a^2+b^2+c^2)^2=\frac{1}{3}(a^2+b^2+c^2)^2,$$

so

$$\sum\frac{a^2}{1+bc}\geq\frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+\frac{1}{3}(a^2+b^2+c^2)^2}=\frac{3(a^2+b^2+c^2)}{3+a^2+b^2+c^2}.$$

This ends the proof. ■

Remarks. Notice that

$$\frac{a^4}{1+bc}+\frac{b^4}{1+ca}+\frac{c^4}{1+ab}\geq\frac{(a^2+b^2+c^2)^2}{3+bc+ca+ab}=\frac{9}{3+ab+bc+ca}\geq\frac{3}{2}$$

again works immediately.

The original Italian Test problem was to prove for $a^2+b^2+c^2=3$ the inequality

$$\frac{1}{1+bc}+\frac{1}{1+ca}+\frac{1}{1+ab}\geq\frac{3}{2},$$

much easier to handle. A "brute force" solution is also possible here, but more difficult to compute for the variant asked above. In fact $a^2+b^2+c^2\leq 3$ is both needed, and enough, for the Italian problem.

Combining the two, both holding for $a^2+b^2+c^2=3$, allows us to then claim that

$$\frac{1+a^2}{1+bc}+\frac{1+b^2}{1+ca}+\frac{1+c^2}{1+ab}\geq 3.$$

In a continuation to his Alternative Solution, Călin Popescu also offers the following generalization.

It can be shown along the same lines that if n is a positive integer, α is a real number larger than 1, and $a_1, \ldots, a_n$ are positive real numbers, then

$$\sum_{k=1}^{n} \frac{a_k^{\alpha}}{1+a_1\cdots a_{k-1}a_{k+1}\cdots a_n} \geq \frac{n^{(n-1)/\alpha}\sum_{k=1}^{n} a_k^{\alpha}}{n^{(n-1)/\alpha}+\left(\sum_{k=1}^{n} a_k^{\alpha}\right)^{(n-1)/\alpha}}.$$

In particular, if $n \geq 3$ and $\alpha = n-1$, then

$$\sum_{k=1}^{n} \frac{a_k^{n-1}}{1+a_1\cdots a_{k-1}a_{k+1}\cdots a_n} \geq \frac{n\sum_{k=1}^{n} a_k^{n-1}}{n+\sum_{k=1}^{n} a_k^{n-1}},$$

so, if a is a positive real number lesser than n, and $\sum_{k=1}^{n} a_k^{n-1} \geq \dfrac{an}{n-a}$, then

$$\sum_{k=1}^{n} \frac{a_k^{n-1}}{1+a_1\cdots a_{k-1}a_{k+1}\cdots a_n} \geq a.$$

Problem 6. Find the formula of the general term of a real numbers sequence $(x_n)_{n\geq 1}$ satisfying

$$\begin{cases} x_1 = 3 \\ 3(x_{n+1}-x_n) = \sqrt{x_{n+1}^2+16}+\sqrt{x_n^2+16} \end{cases}$$

AoPS

Solution. It is clear the sequence is (strictly) increasing. Then

$$3(x_{n+1}-x_n) = \sqrt{x_{n+1}^2+16}+\sqrt{x_n^2+16} = \frac{(x_{n+1}-x_n)(x_{n+1}+x_n)}{\sqrt{x_{n+1}^2+16}-\sqrt{x_n^2+16}}$$

allows us to write $x_{n+1}+x_n = 3\left(\sqrt{x_{n+1}^2+16}-\sqrt{x_n^2+16}\right)$. So $4x_{n+1}-5x_n = 3\sqrt{x_n^2+16}$. Square it, write it for the next index, subtract the two and factorize, in order to get $8(x_{n+2}-x_n)(2x_{n+2}-5x_{n+1}+2x_n) = 0$, hence $2x_{n+2}-5x_{n+1}+2x_n = 0$. By the known methods, the general solution is $x_n = \alpha 2^n + \beta 2^{-n}$. Since the sequence can in fact be prolonged to the left, to $x_0 = 0$, the coefficients can be determined to be $\alpha = 2$, $\beta = -2$, so $\boxed{x_n = 2^{n+1}-2^{-n+1}}$.

Alternatively, if we compute the first few terms and "guess" this formula, it is a simple task to check it verifies the recurrence formula, since

$$3(x_{n+1}-x_n) = 3\left(2^{n+2}-\frac{1}{2^n}-2^{n+1}+\frac{1}{2^{n-1}}\right) = 3\left(2^{n+1}+\frac{1}{2^n}\right),$$

$$\sqrt{x_{n+1}^2+16}+\sqrt{x_n^2+16} = \left(2^{n+2}+\frac{1}{2^n}\right)+\left(2^{n+1}+\frac{1}{2^{n-1}}\right) = 3\left(2^{n+1}+\frac{1}{2^n}\right).$$

There are merits in it, especially if one has seen in the past such relations. ■