The 2018 Danube Competition in Mathematics, October 27^{th}

Problema 1. Un colier are n mărgele. Pe fiecare mărgea este scris un număr întreg, astfel încât suma acestor numere este $n-1$. Demonstraţi că putem tăia colierul astfel încât să obţinem un şirag de mărgele ale căror numere $x_1, x_2, \ldots, x_n$ (luate în această ordine) satisfac

$$\sum_{i=1}^{k} x_i \leq k-1$$

pentru orice $k = 1, \ldots, n$.

Problema 2. Arătaţi că există o infinitate de perechi de numere naturale nenule (m, n) astfel încât m divide n^2+1 şi n divide m^2+1.

Problema 3. Fie ABC un triunghi ascuţitunghic neisoscel. Bisectoarea unghiului A intersectează din nou cercul circumscris triunghiului ABC în punctul D. Fie O centrul cercului circumscris triunghiului ABC. Bisectoarele unghiurilor $\angle AOB$ şi $\angle AOC$ intersectează cercul γ de diametru AD în punctele P şi respectiv Q. Dreapta PQ intersectează mediatoarea segmentului AD în punctul R. Demonstraţi că $AR \parallel BC$.

Problema 4. Fie n un număr natural impar ($n \geq 3$). Fiecare pătrat unitate al unei reţele $n \times n$ se colorează cu roşu sau albastru. Două astfel de pătrate se numesc *adiacente* dacă au aceeaşi culoare şi dacă au cel puţin un vârf comun. Două pătrate unitate a, b se numesc *legate* dacă există un şir finit de pătrate unitate $c_1, c_2, \ldots, c_k$ cu $c_1 = a$ şi $c_k = b$ astfel încât c_i şi c_{i+1} sunt adiacente pentru orice $i = 1, \ldots, k-1$; altfel ele se numesc *nelegate* (de exemplu, orice două pătrate unitate de culori diferite sunt nelegate). Găsiţi numărul maxim M pentru care există o colorare care admite M pătrate unitate nelegate două câte două.

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Problem 1. Suppose we have a necklace of n beads. Each bead is labeled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, ..., x_n$ satisfy

$$\sum_{i=1}^{k} x_i \leq k-1$$

for any $k = 1, \ldots, n$.

Solution. Number the beads $1, 2, \ldots, n$ starting from some arbitrary position, and let z_i be the label of bead i, with the convention that $z_{n+i} = z_i$.

Let $S_j = z_1 + z_2 + \ldots + z_j - \frac{j(n-1)}{n}$. We have $S_j = S_{n+j}$. Then the sum of the labels on beads $m+1, m+2, \ldots, m+k$ is $S_{m+k} - S_m + \frac{k(n-1)}{n}$.

Now we choose m such that S_m is maximal and cut between m and $m+1$. If we do so, we have

$$S_{m+k} - S_m + \frac{k(n-1)}{n} \leq k - \frac{k}{n},$$

but the left side is an integer, so we can replace the right side by $k-1$.

Problem 2. Prove that there are infinitely many pairs of positive integers (m, n) such that simultaneously m divides n^2+1 and n divides m^2+1.

Solution. Denote by F_n the nth Fibonacci number. We claim that $(n, m) = (F_{2k-1}, F_{2k+1})$ is a solution for all positive integers k.

First we show that for all positive integers k,

$$F_{2k+1}^2 + 1 = F_{2k-1} \cdot F_{2k+3}.$$

This will be proved by induction on k. For $k = 1$, this is true since

$$F_3^2 + 1 = 2^2 + 1 = 5 = 1 \cdot 5 = F_1 \cdot F_5.$$

Now we suppose that $F_{2k-1}^2 + 1 = F_{2k-3} \cdot F_{2k+1}$. Note first that $F_{2k+3} = 3F_{2k+1} - F_{2k-1}$, by repeatedly applying the relation $F_{m+2} = F_{m+1} + F_m$. Then $F_{2k-1} \cdot F_{2k+3} = F_{2k-1}(3F_{2k+1} - F_{2k-1}) = 3F_{2k-1} \cdot F_{2k+1} - F_{2k-1}^2 = 3F_{2k-1} \cdot F_{2k+1} - (F_{2k-3} \cdot F_{2k+1} - 1) = F_{2k+1}(3F_{2k-1} - F_{2k-3}) + 1 = F_{2k+1} \cdot F_{2k+1} + 1 = F_{2k+1}^2 + 1$.

Now it follows immediately that F_{2k-1} divides $F_{2k+1}^2 + 1$ and F_{2k+1} divides $F_{2k-1}^2 + 1$.

Alternative solution. As the Pell equation $x^2 - 5y^2 = -1$ has the solution $(2, 1)$, then it has infinity many solutions $(x, y) \in \mathbb{N}^* \times \mathbb{N}^*$. For $n = x + 3y$ and $m = 2y$ we have:

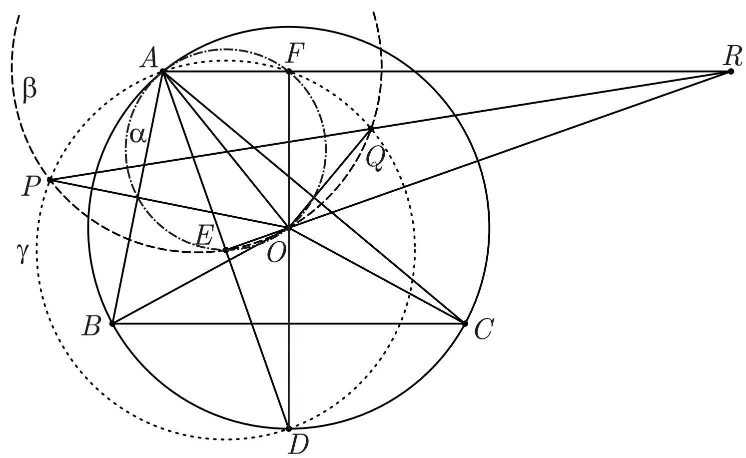
$$n^2 + 1 = x^2 + 6xy + 9y^2 + 1 = 6xy + 14y^2 = 2y(3x + 7y) = m \cdot k$$

and

$$m^2 + 1 = 4y^2 + 1 = 9y^2 - (5y^2 - 1) = 9y^2 - x^2 = (3y - x)(3y + x) = n \cdot q.$$

It is clear that $3y - x > 0$ as $m^2 + 1 > 0$.

Problem 3. Let ABC be an acute non isosceles triangle. The angle bisector of angle A meets again the circumcircle of the triangle ABC in D. Let O be the circumcenter of the triangle ABC. The angle bisectors of $\angle AOB$, and $\angle AOC$ meet the circle γ of diameter AD in P and Q respectively. The line PQ meets the perpendicular bisector of AD in R. Prove that $AR \parallel BC$.



Solution. Let E be the midpoint of AD. Since $OA = OD$ and $O \neq E$, the pependicular bisector of AD is EO.

Let F be the second intersection of OD with γ. Then $DF \perp AF$ and $DO \perp BC$, hence $BC \parallel AF$. Therefore, in order to prove the required conclusion, it suffices to prove that R is on the line AF.

The points A, E, O and F are on the circle α of diameter OA. Denote β the circumcircle of the triangle POQ. Then PQ is the radical axis of β and γ, AF is the radical axis of α and γ. So, what remains to be proven is that EO is the radical axis of α and β, that is E is on β.

Since OP is the pependicular bisector of AB, the circle α contains also the midpoint of the segment AB; the same stands for the midpoint of the segment AC. Therefore, $\angle QOR = \angle CAE$ and $\angle POE = \angle BAE$, so $\angle QOR = \angle POE$, which shows that OE is perpendicular on the bisector of $\angle POQ$. Notice also that $EP = EQ$. Denote now E' the second intersection of the bisector of $\angle POQ$ with β and E'' the antipode of E' on β. Then E'' is the common point of the perpendicular bisector of PQ and the perpendicular from O on the bisector of $\angle POQ$, hence $E'' = E$, qed.

Problem 4. Let n be an odd natural number ($n \geq 3$). Each unit square of a $n \times n$ grid it is colored either red or blue. Two unit squares are said to be *adjacent* if they are of the same color and have at least one common vertex. Two unit squares a, b are said to be *connected*, if there exists a sequence of unit squares $c_1, c_2, \ldots, c_k$ with $c_1 = a$ and $c_k = b$ such that c_i and c_{i+1} are adjacent for every $i = 1, \ldots, k-1$; otherwise they are called *disconnected* (for instance, two unit squares of different colors are disconnected). Find the maximal number M for which there exists a coloring admitting M pairwise disconnected unit squares.

Solution. The answer is $M = \frac{1}{4}(n+1)^2 + 1$.

Consider the generalized problem on a $m \times n$ grid, where $m, n \geq 3$ are both odd numbers. Suppose that the mn squares can be divided into K connected components, such that any two squares are connected if and only if they belong to the same connected component. We will prove by induction on (m, n) that (i) $K \leq \frac{1}{4}(m+1)(n+1) + 1$; (ii) If $K = \frac{1}{4}(m+1)(n+1) + 1$, then each square at the four corners of the grid is connected to none of the other squares.

When $m = n = 3$, the 8 border squares belong to at most 4 connected components. Hence $K \leq 5$, and $K = 5$ if and only if the 4 squares at the corner are of the same color, and the other 5 squares are of the other color.

Next assume $m \geq 5$. Suppose the second row of the grid can be divided into k parts, each part consisting of consecutive squares of the same color. Let x_i be the number of squares in the i^{th} part, $1 \leq i \leq k$. Let P be the number of connected components, which contain at least one square in the first row, but no square in the second row. If $k \geq 2$ then

$$P \leq \left\lceil \frac{x_1 - 1}{2} \right\rceil + \left\lceil \frac{x_k - 1}{2} \right\rceil + \sum_{i=2}^{k-1} \left\lceil \frac{x_i - 2}{2} \right\rceil \leq \frac{n - k + 2}{2}.$$

If $k = 1$, we also have $P \leq \lceil \frac{n}{2} \rceil \leq \frac{n-k+2}{2}$. Let Q be the number of connected components, which contain at least one square in the second row, but no square in the third row. We have $Q \leq \lceil \frac{k}{2} \rceil \leq \frac{k+1}{2}$. Let R be the number of connected components, which contain at least one square in the 3^{rd} to m^{th} rows. By induction hypothesis (i), we have $R \leq \frac{1}{4}(m-1)(n+1) + 1$.

If $Q = \frac{k+1}{2} = 1$ then all the squares in the second row are of the same color and thus all the squares in the 3^{rd} row are also of the same color. If $Q = \frac{k+1}{2} \geq 2$, then the first square in the 3^{rd} row is connected to the last square in the 3^{rd} row via the $2i^{th}$ part of the second row. By induction hypothesis (ii) $R \leq \frac{1}{4}(m-1)(n+1)$. Hence $Q = \frac{k+1}{2}$ and $R = \frac{1}{4}(m-1)(n+1)+1$ can't hold simultaneously. Therefore, we have

$$K = P + Q + R \leq \frac{n-k+2}{2} + \frac{k+1}{2} + \frac{1}{4}(m-1)(n+1) = \frac{1}{4}(m+1)(n+1) + 1.$$

If the equality in the above inequality holds, then we have $P = \dfrac{n-k+2}{2}$. Thus, the square at the upper-left corner is surrounded by three squares of opposite color. By symmetry, each square at the four corners of the grid is connected to none of the other squares.

If the square at position (i, j) is colored red/blue if ij is even/odd, respectively, it is easy to verify that $K = \frac{1}{4}(m+1)(n+1) + 1$.