Soluţii juniori

Problema 1

Se consideră suma $S = x_1x_2 + x_3x_4 + ... + x_{2015}x_{2016}$, unde $x_1, x_2, ..., x_{2016} \in \left\{\sqrt{3}-\sqrt{2}, \sqrt{3}+\sqrt{2}\right\}$. Este posibil să avem $S = 2016$?

Cristian Lazăr

Soluţie.
Răspuns: Da.

Termenii sumei sunt de forma $\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)=1$, $\left(\sqrt{3}+\sqrt{2}\right)^2 = 5+2\sqrt{6}$ sau $\left(\sqrt{3}-\sqrt{2}\right)^2 = 5-2\sqrt{6}$.

Să spunem că sunt a termeni de primul tip, b termeni de al doilea tip şi c de al treilea tip .
Avem $a+b+c=1008$ şi $a+b\left(5+2\sqrt{6}\right)+c\left(5-2\sqrt{6}\right)=2016$, echivalent cu $a+5b+5c-2016=\sqrt{6}(2c-2b)$. De aici deducem că $b=c$.
Obţinem $a+2b=1008$ şi $a+10b=2016$. Prin urmare $a=756$ şi $b=c=126$.

Problema 2

Determinați numerele naturale $n>1$ care au proprietatea că, pentru orice divizor $d>1$ al lui *n*, numerele d^2+d+1 și d^2-d+1 sunt prime.

Lucian Petrescu

Soluţie.
Răspuns: $n \in \{2;3;6\}$.

Arătăm mai întâi că *n* este liber de pătrate. Dacă prin absurd ar exista $d>1$, astfel încât $d \mid n$ și $d^2 \mid n$, atunci conform ipotezei ar rezulta că numărul $\left(d^2\right)^2+d^2+1=d^4+d^2+1$ este prim.

Dar $d^4+d^2+1=\left(d^2+d+1\right)\left(\underbrace{d^2-d+1}_{\geq 3}\right)$, contradicţie.

Deci $n = p_1 \cdot p_2 \cdot \ldots \cdot p_s$, cu $s \in \mathbb{N}^*$ și $p_1 < p_2 < \ldots < p_s$ numere prime.

Fie $p \geq 5$ un număr prim, divizor al lui n. Atunci $p \equiv 1 \pmod 6$ sau $p \equiv 5 \pmod 6$.

Dacă $p \equiv 1 \pmod 6$ obținem că numărul $p^2+p+1 \equiv 3 \pmod 6$ (și $p^2+p+1>3$) este compus, fals. Dacă $p \equiv 5 \pmod 6$ rezultă că numărul $p^2-p+1 \equiv 3 \pmod 6$ (și $p^2-p+1>3$) este compus, fals.

Așadar, *n* nu are factori primi $p \geq 5$ de unde rezultă că $p_i \in \{2;3\}$, adică $n \in \{2;3;6\}$, toate cele trei valori ale lui *n* verificând ipoteza problemei.

Problema 3

Se consideră un triunghi *ABC*, cu $AB < AC$, în care punctul *I* este intersecţia bisectoarelor şi punctul *M* este mijlocul laturii [*BC*]. Dacă $IA = IM$, determinaţi cea mai mică măsură posibilă a unghiului AIM.

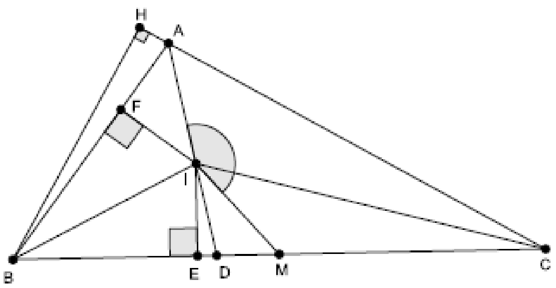
[∗∗∗]

Soluţie.

Răspuns: $150°$.

Fie $\{D\} = AI \cap BC$. Deoarece $AB < AC$, atunci $D \in (BM)$ şi $m(ACB) < m(ABC)$.

Avem $m(IDB) = m(DAC) + m(ACB) < m(DAB) + m(ABD) = m(ADC)$, deci unghiul IDB este ascuţit.



Fie *F* şi *E* proiecţiile punctului *I* pe dreptele *AB*, respectiv *BC*.

Deducem că $E \in (BD) \subset BM$.

Cum $\Delta IBF \equiv \Delta IBE$ (CU) şi $\Delta IFA \equiv \Delta IEM$ (IC), rezultă că $BA = BM = \frac{BC}{2}$ şi $\Delta IBA \equiv \Delta IBM$.

Avem $m(MID) = m(IDB) - m(IMB) = m(DAC) + m(ACD) - m(IAB) = m(ACD)$.

Ca urmare, $m(AIM) = 180^\circ - m(ACB)$.(1)

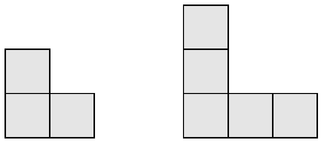
Fie *H* proiecţia punctului *B* pe dreapta *AC*. Rezultă că $BH \le AB = \frac{BC}{2}$.

Deducem că $m(ACB) \le 30°$. (2)

Din (1) şi (2) rezultă că $m(AIM) \ge 180^\circ - 30^\circ = 150^\circ$.

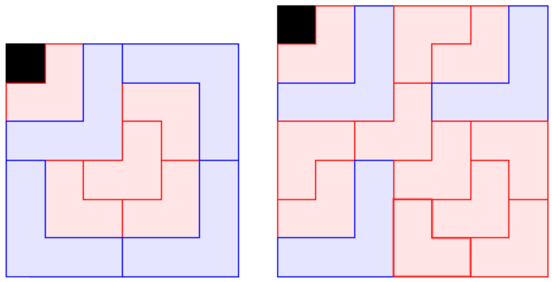
Problema 4

Un pătrat unitate este îndepărtat din colţul unui pătrat $n \times n$, unde $n \in \mathbb{N}$, $n \geq 2$.
Demonstraţi că suprafaţa rămasă poate fi pavată cu dale formate din 3 sau 5 pătrate unitate de forma celor din figurile de mai jos:

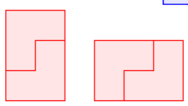


[* * *]

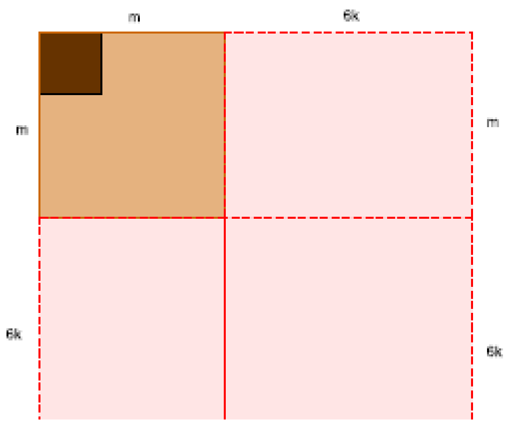
Soluţie.
Vom numi un pătrat descris de enunţ „ pătrat ciobit”.
Mai întâi constatăm că, pentru $m \in \{2,3,4,5,6,7\}$ orice pătrat ciobit $m \times m$ poate fi pavat ca mai jos.

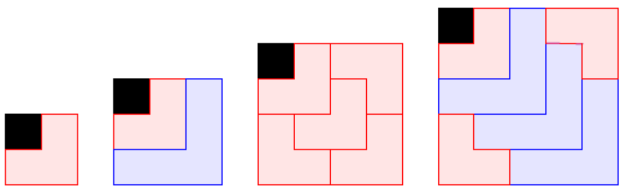


În plus, dacă $p \in \mathbb{N}^*$, orice dreptunghi $6p \times m$, cu $m \in \{2,3,4,5,6,7\}$, precum şi pătratul $6p \times 6p$ pot fi pavate cu dreptunghiuri de tipul



Pentru oricare n, $n \geq 8$, există $k \in \mathbb{N}^*$ astfel încât $n = 6k + m$, unde $m \in \{2,3,4,5,6,7\}$.
Prin urmare, orice pătrat ciobit $n \times n$ se va obţine din pătratul ciobit $m \times m$, $m \in \{2,3,4,5,6,7\}$, care se bordează cu două dreptunghiuri $6k \times m$ şi un pătrat $6k \times 6k$ astfel:

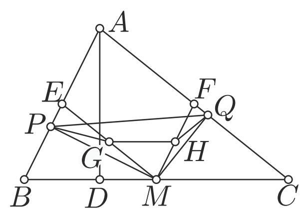




The 2016 Danube Competition in Mathematics, October 29$^{\text{th}}$

1. Let ABC be a triangle, D the foot of the altitude from A and M the midpoint of the side BC. Let S be a point on the closed segment DM and let P, Q the projections of S on the lines AB and AC respectively. Prove that the length of the segment PQ does not exceed one quarter the perimeter of the triangle ABC.

Adrian Zahariuc



Solution. From the cyclic quadrilateral $APSQ$, whose circumcircle has diameter AS, $PQ = AS \sin \widehat{PAQ}$. So, the largest value of PQ is obtained when AS is largest, that is when $S = M$.

In the case $S = M$, denote E, F the midpoints of the segments AB, respectively AC and G, H the midpoints of the segments ME, respectively MC. Then

$$PQ \leq PG + GH + HQ = \frac{1}{2}(EM + EF + FM) = \frac{1}{4}P_{\triangle ABC},$$

as desired. Equality occurs if and only if ABC is equilateral.

2. A bank has a set S of codes for its customers, in the form of sequences of 0 and 1, each sequence being of length n. Two codes are called *close* if they are different at exactly one position. It is known that each code from S has exactly k close codes in S.

a) Show that S has an even number of elements.

b) Show that S contains at least 2^k codes.

Solution. Start by noticing that we can suppose that S contains the nil code $(0, 0, \ldots, 0)$: otherwise take a code $x \in S$ and replace S with the set $S' = x + S = \{x + y \mid y \in S\}$, where addition is taken mod 2. Then S' and S have the same cardinal and S' fulfills the same condition as S.

In the sequel we will denote $w(x)$ the number of non-nil components of the code x and $S_i = \{x \in S \mid w(x) = i\}$.

a) Consider the bipartite graph $G = A \cup B$, where the vertices are the codes, $A = \{x \in S \mid w(x) = \text{even}\}$, $B = \{x \in S \mid w(x) = \text{odd}\}$ and an edge between two vertices exists if and only if the corresponding codes are close. Count the edges of this graph: from A emerge $k \cdot |A|$ edges and from B emerge $k \cdot |B|$ edges. But $k \cdot |A| = k \cdot |B|$, hence A and B have the same number of vertices, whence the conclusion.

b) Notice first that $|S_0| = 1$. Take now $x \in S_i$. The set S_{i-1} has at most i codes close to x and the set S_{i+1} has at least $k - i$ codes close to x. On the other hand, each code from S_{i+1} has at most $i + 1$ close codes in the set S_i.

Consider now the (bipartite) subgraph $S_i \cup S_{i+1}$. We count the number N of its edges twice: first we count the edges emerging from S_i to find that $N \geq (k - i)|S_i|$, then we count the edges emerging from S_{i+1} to find that $N \leq (i + 1)|S_{i+1}|$. So, $(i + 1)|S_{i+1}| \geq (k - i)|S_i|$.

The last inequality yields inductively to $|S_i| \geq C_k^i$ for every $0 \leq i \leq k$, whence $|S| \geq 2^k$.

3. Let $n > 1$ be an integer and $a_1, a_2, \ldots, a_n$ be positive integers with sum 1.
a) Show that there exists a constant $c \geq 1/2$ so that

$$\sum_{k=1}^{n} \frac{a_k}{1 + (a_0 + \cdots + a_{k-1})^2} \geq c,$$

where $a_0 = 0$.
b) Show that 'the best' value of c is at least $\pi/4$.

Solution. For the first part, notice that

$$\sum_{k=1}^{n} \frac{a_k}{1 + (a_0 + \cdots + a_{k-1})^2} \geq \sum_{k=1}^{n} \frac{a_k}{(1 + a_0 + a_1 + \cdots + a_{k-1})^2}.$$

Since

$$\begin{aligned} \frac{a_k}{(1 + a_0 + a_1 + \cdots + a_{k-1})^2} &\geq \frac{a_k}{(1 + a_0 + a_1 + \cdots + a_{k-1})(1 + a_0 + a_1 + \cdots + a_k)} \\ &= \frac{1}{1 + a_0 + a_1 + \cdots + a_{k-1}} - \frac{1}{1 + a_0 + a_1 \cdots + a_k}, \end{aligned}$$

the relation follows by summing.

For the second part we denote $x_k = a_0 + a_1 + \cdots + a_k$ and use the inequality

$$\frac{x_{k+1} - x_k}{1 + x_k^2} \geq \arctan x_{k+1} - \arctan x_k,$$

valid for $1 \geq x_{k+1} \geq x_k \geq 0$. This is equivalent to $\tan \frac{x_{k+1} - x_k}{1 + x_k^2} \geq \frac{x_{k+1} - x_k}{1 + x_k x_{k+1}}$, and results from $\tan x \geq x$ for $x \in (0, \frac{\pi}{2})$.

Remark. Taking $a_k = 1/n$, $k = 1, 2, \ldots, n$ and $n \to \infty$, a limit argument shows that the value $c = \pi/4$ cannot be improved.

4. Prove that there exist only finitely many positive integers n such that

$$\left(\frac{n}{1} + 1\right)\left(\frac{n}{2} + 2\right)\left(\frac{n}{3} + 3\right) \ldots \left(\frac{n}{n} + n\right)$$

is an integer.

Adrian Zahariuc

Solution. Assume that n is sufficiently large. We claim that there exists a prime number $p \leq n$ such that p does not divide $k^2 + n$, for any integer k.

Lemma. There exist coprime positive integers a, b of different parities such that

$$3n \leq (a^2 + b^2)^2 \leq 4n.$$

Proof. We may take $a = 1$ and b even maximal such that $a^2 + b^2 \leq 2\sqrt{n}$, etc. □

Note that $N = 4n - (a^2 + b^2)^2 \equiv 3 (\mathrm{mod} 4)$, so we can find a prime factor p of N which is of the form $4k + 3$, since $N \geq 0$. We claim that p satisfies the desired property.

Recall that for a prime number $p \equiv 3 (\mathrm{mod} 4)$, if p divides $x^2 + y^2$, then p divides both x and y. Assume that there was some k such that p divides $k^2 + n$. Then

$$p | (2k)^2 + (a^2 + b^2)^2,$$

hence $p | a^2 + b^2$, since $p \equiv 3 (\mathrm{mod} 4)$. Then, $p | a$ and $p | b$, contradicting $\gcd(a, b) = 1$.