Solutions Călăraşi 2013

Problem 1. Given six points on a circle, A, a, B, b, C, c, show that the Pascal lines of the hexagrams $AaBbCc$, $AbBcCa$, $AcBaCb$ are concurrent.

Solution. The lines Aa and bC meet at D, and the lines Bb and cA meet at D' to determine the Pascal line of the hexagram $AaBbCc$; similarly, the lines Bc and aA meet at E, and the lines Ca and bB meet at E' to determine the Pascal line of the hexagram $AbBcCa$; finally, the lines Cb and cB meet at F, and the lines Ac and aC meet at F' to determine the Pascal line of the hexagram $AcBaCb$. By Desargues' theorem, the lines DD', EE', FF' are concurrent if and only if the pairs of lines DE and $D'E'$, EF and $E'F'$, FD and $F'D'$ meet at three collinear points. Since the latter lie on the Pascal line of the hexagram $AcBbCa$, the conclusion follows.

Problem 2. Let a, b, c, n be four integers, where $n \geq 2$, and let p be a prime dividing both a^2+ab+b^2 and $a^n+b^n+c^n$, but not $a+b+c$; for instance, $a \equiv b \equiv -1 \pmod 3$, $c \equiv 1 \pmod 3$, n a positive even integer, and $p = 3$ or $a = 4$, $b = 7$, $c = -13$, $n = 5$, and $p = 31$ satisfy these conditions. Show that n and $p-1$ are not coprime.

Solution. Throughout the proof congruences are taken modulo p. Begin by ruling out the case $p = 2$. If $p = 2$, then $a^2 + ab + b^2$ and $a^n + b^n + c^n$ are both even, and $a + b + c$ is odd. The first condition forces both a and b even, so c is also even by the second, contradicting the third. Consequently, p must be odd, and the conclusion follows unless n is odd.

Henceforth assume n odd. It is easily seen from the conditions in the statement that $a \not\equiv 0$, so a^{-1} exists modulo p, and the hypotheses yield $B^2 + B + 1 \equiv 0$, $B^n + C^n + 1 \equiv 0$ and $B + C + 1 \not\equiv 0$, where $B = a^{-1}b$ and $C = a^{-1}c$. The first congruence yields $B^3 \equiv 1$, where $B \not\equiv 1$. (Otherwise, $3 \equiv 0$, so $p = 3$, $C^n \equiv 1$ and $C \not\equiv 1$ which is impossible since n is odd.) Hence 3 is a factor of $p-1$; in particular, $p \geq 7$ and $p - 1$ is divisible by 6, so the conclusion follows unless $n = 6m \pm 1$.

Let $n = 6m \pm 1$ and recall that $B^3 \equiv 1$ to deduce that $B^{2n} + B^n + 1 \equiv B^{\pm 2} + B^{\pm 1} + 1 \equiv 0$, so $C^n \equiv B^{2n} \equiv (-B-1)^n$; that is, $\left(-C(B+1)^{-1}\right)^n \equiv 1$, since $B + 1 \not\equiv 0$. The condition $B + C + 1 \not\equiv 0$ shows that $-C(B+1)^{-1} \not\equiv 1$, so the multiplicative order of $-C(B+1)^{-1}$ in $\mathbb{Z}_p^*$ is a divisor d of n, greater than 1. Since d is also a divisor of $p-1$, the conclusion follows.

Remark. The argument shows that if n is a prime greater than 3, then $p-1$ is divisible by $6n$. This is precisely the case in the second example in the statement.

Problem 3. Show that, for every integer $r \geq 2$, there exists an r-chromatic simple graph (no loops, nor multiple edges) which has no cycle of less than 6 edges.

Solution. The case $r = 2$ is clear: Any cycle of even length works. In the other cases, define a sequence of graphs G_r, $r \geq 3$, as follows. The graph G_3 is a cycle of just 7 edges. (Any larger odd number would do.) When G_r is defined, with n_r vertices say, construct G_{r+1} as follows. Consider

$$\binom{rn_r - r + 1}{n_r}$$

disjoint copies of G_r. Adjoin $rn_r - r + 1$ extra vertices. Set up a one-to-one correspondence between the copies of G_r and the n_r-element sets of extra vertices. Join each copy of G_r to the members of the corresponding n_r-element set of extra vertices by n_r disjoint new edges (no two have a common end). The resulting graph is G_{r+1}.

The construction ensures that no graph G_r has a cycle of less than 6 edges.

Clearly, G_3 is 3-chromatic. If $r \geq 3$ and G_{r+1} has a colouring C in r or fewer colours, then some n_r of the extra vertices in G_{r+1} must share the same colour in C, so the corresponding copy of G_r must be coloured in $r-1$ or fewer colours. It follows, by descending induction, that G_3 must be coloured in 2 or fewer colours which is, of course, impossible. Consequently, no G_r can be coloured in less than r colours.

This does not prove that G_r is r-chromatic, but if it is not, deletion of some monochromatic classes of vertices together with their incident edges yields one such.

Problem 4. Show that there exists a proper non-empty subset S of the set of real numbers such that, for every real number x, the set $\{nx + S \colon n \in \mathbb{N}\}$ is finite, where $nx + S = \{nx + s \colon s \in S\}$.

Solution. Let H be a Hamel basis; that is, H is a set of real numbers such that every real number x can uniquely be written in the form

$$x = \sum_{h \in H} q(x,h) \cdot h, \qquad (*)$$

where the $q(x,h)$ are all rational and vanish for all but a finite number (depending on x) of h's. The existence of Hamel bases can be proved via Zorn's lemma or Zermelo's well ordering theorem or any other statement equivalent to the axiom of choice.

We are now going to prove that the set S of those real numbers x whose $q(x,h)$ in $(*)$ are all integral satisfies the required condition.

To this end, fix a real number x. Since the conclusion is clear if $x = 0$, let x be different from 0 and let $m(x)$ be the least common multiple of the denominators of the non-vanishing $q(x,h)$ in $(*)$. Finally, notice that $m(x) \cdot x$ is a member of S, to conclude that any set of the form $nx+S$, where n is a non-negative integer, must be one of the sets $rx+S$, $r = 0, 1, \ldots, m(x)-1$.