Călăraşi 2012 — Solutions

Problem 1. Given a positive integer n, determine the maximum number of lattice points in the plane a square of side length $n + 1/(2n+1)$ may cover.

Solution. The required maximum is $(n+1)^2$. Clearly, the square $[-\epsilon/2, n+\epsilon/2] \times [-\epsilon/2, n+\epsilon/2]$, $0 \le \epsilon < 1$, covers exactly $(n+1)^2$ lattice points.

We now proceed to show that any (closed) square of side length $n + \epsilon$, $0 \le \epsilon \le 1/(2n+1)$, covers at most $(n+1)^2$ lattice points.

The case $n = 1$ is settled by a metric argument: the diameter of a square of side length $1+\epsilon$ is $(1+\epsilon)\sqrt{2}$, whereas the diameter of any configuration of five lattice points is at least $\sqrt{5} > (1+\epsilon)\sqrt{2}$ in the slightly wider range $0 \le \epsilon < \sqrt{10}/2 - 1$.

Henceforth assume $n \ge 2$ and consider the convex hull K of the lattice points covered by a square of side length $n+\epsilon$, $0 \le \epsilon \le 1/(2n+1)$. Clearly, $\operatorname{area} K \le (n+\epsilon)^2$, the area of the square. On the other hand, by Pick's theorem, $\operatorname{area} K = m - k/2 - 1$, where m is the number of lattice points covered by K, and k is the numer of lattice points on the boundary of K. Therefore,

$$m = \operatorname{area} K + k/2 + 1 \le (n+\epsilon)^2 + k/2 + 1.$$

To find an upper bound for k, notice that the perimeter of K does not exceed the perimeter of the square which is $4(n+\epsilon) \le 4n + 4/(2n+1) < 4n+1$, for $n \ge 2$. Since the distance between two lattice points is at least 1, it follows that $k \le 4n$. Consequently,

$$m \le (n+\epsilon)^2 + 2n + 1 = (n+1)^2 + 2n\epsilon + \epsilon^2 < (n+1)^2 + 1$$

in the slightly wider range $0 \le \epsilon < \sqrt{n^2+1} - n$. The conclusion follows.

Problem 2. Let ABC be an acute triangle and let A_1, B_1, C_1 be points on the sides BC, CA and AB, respectively. Show that the triangles ABC and $A_1B_1C_1$ are similar ($\angle A = \angle A_1$, $\angle B = \angle B_1$, $\angle C = \angle C_1$) if and only if the orthocentre of the triangle $A_1B_1C_1$ and the circumcentre of the triangle ABC coincide.

Solution. Let triangles ABC and $A_1B_1C_1$ be similar, $\angle A = \angle A_1 = \alpha$, $\angle B = \angle B_1 = \beta$, $\angle C = \angle C_1 = \gamma$, and let O be the orthocentre of the triangle $A_1B_1C_1$. Then $\angle OB_1C_1 = 90^\circ - \gamma$, $\angle OC_1B_1 = 90^\circ - \beta$, so $\angle B_1OC_1 = 180^\circ - (90^\circ - \gamma) - (90^\circ - \beta) = \beta + \gamma$. Since $\angle B_1AC_1 + \angle B_1OC_1 = \alpha + \beta + \gamma$, the quadrangle AC_1OB_1 is cyclic, so $\angle OAB_1 = 90^\circ - \beta$ and $\angle OAC_1 = \angle OB_1C_1 = 90^\circ - \gamma$. Similarly, the quadrangles BA_1OC_1 and CB_1OA_1 are cyclic, so $\angle OBC_1 = 90^\circ - \gamma$, $\angle OBA_1 = 90^\circ - \alpha$ and $\angle OCA_1 = 90^\circ - \alpha$, $\angle OCB_1 = 90^\circ - \beta$. Consequently, O is the circumcentre of the triangle ABC.

Conversely, let the circumcentre O of the triangle ABC be the orthocentre of the triangle $A_1B_1C_1$. Let $\angle A = \alpha$, $\angle B = \beta$, $\angle C = \gamma$ and $\angle A_1 = \alpha_1$, $\angle B_1 = \beta_1$, $\angle C_1 = \gamma_1$. Let the points B' on the side CA and C' on the side AB be such that the quadrangles $CB'OA_1$ and BA_1OC' are cyclic. Then so is the quadrangle $AC'OB'$. Hence $\angle OC'B' = \angle OAB' = \angle OAC = 90^\circ - \beta$. Since the supplementary angle of the angle $A_1OC'$is β, the lines A_1O and $B'C'$ are perpendicular, so the lines $B'C'$ and B_1C_1 are parallel.

Since O is the orthocentre of the triangle $A_1B_1C_1$, the line B_1C_1 separates A and O, and $\angle B_1OC_1 = 180^\circ - \alpha_1$.

Since the quadrangle $A_1OB'C$ is cyclic, $\angle A_1OB' = 180^\circ - \gamma$ and, similarly, $\angle A_1OC' = 180^\circ - \beta$. The sum of these two angles is $180^\circ + \alpha$, so the points A and O lie on opposite sides of the line $B'C'$.

Without loss of generality, we may (and will) assume that the line $B'C'$ is closer to the point A than the line B_1C_1. Then $\angle B'OC' \le \angle B_1OC_1$ and $\angle B'A_1C' \le \angle B_1A_1C_1$, so

$$\angle B'OC' + \angle B'A_1C' \le \angle B_1OC_1 + \angle B_1A_1C_1. \qquad (*)$$

Since $\angle B'A_1C' = \angle B'A_1O + \angle OA_1C' = \angle B'CO + \angle OBC' = \angle ACO + \angle OBA = 90^\circ - \beta + 90^\circ - \gamma = \alpha$, it follows that $\angle B'OC' + \angle B'A_1C' = 180^\circ - \alpha + \alpha = 180^\circ$.

Also, $\angle B_1OC_1 + \angle B_1A_1C_1 = 180^\circ - \alpha_1 + \alpha_1 = 180^\circ$.

Thus equality holds in $(*)$, and this is the case only if $\angle B'OC' = \angle B_1OC_1$ and $\angle B'A_1C' = \angle B_1A_1C_1$; that is, $\alpha_1 = \alpha$ and the lines $B'C'$ and B_1C_1 coincide. Then $\beta_1 = \beta$ and $\gamma_1 = \gamma$, so the triangles $A_1B_1C_1$ and ABC are indeed similar.

Problem 3. Let p and q, $p < q$, be two primes such that $1 + p + p^2 + \cdots + p^m$ is a power of q for some positive integer m, and $1 + q + q^2 + \cdots + q^n$ is a power of p for some positive integer n. Show that $p = 2$ and $q = 2^t - 1$, where t is prime.

Solution. Let m be the smallest positive integer such that $1 + p + p^2 + \cdots + p^m$ is a power of q, say q^s. Then $m + 1$ must be prime, for if $m + 1 = kl$, then

$$1 + p + p^2 + \cdots + p^m = \left(1 + p^l + p^{2l} + \cdots + p^{(k-1)l}\right)(1 + p + p^2 + \cdots + p^{l-1}),$$

so $1 + p + p^2 + \cdots + p^{l-1}$ is again a power of q, and minimality of m forces $l = 1$ or $k = 1$. Similarly, if n is the smallest positive integer such that $1 + q + q^2 + \cdots + q^n$ is a power of p, say p^r, then $n + 1$ must be prime.

Clearly, $p^{m+1} \equiv 1 \pmod q$ and $p^r \equiv 1 \pmod q$. Since $p \not\equiv 1 \pmod q$ and $m + 1$ is prime, $m + 1$ must divide r.

If $q \not\equiv 1 \pmod p$, a similar argument shows that $n + 1$ must divide s, so

$$(p^{m+1} - 1)(q^{n+1} - 1) = p^r q^s (p - 1)(q - 1) \ge p^{m+1} q^{n+1}$$

which is impossible.

Hence $q \equiv 1 \pmod p$, so $n + 1 \equiv 0 \pmod p$ which forces $n + 1 = p$ by primality of $n + 1$.

Recall that r is divisible by $m + 1$, say $r = r'(m + 1)$, to write

$$1 + q + q^2 + \cdots + q^n = p^r = \left(p^{m+1}\right)^{r'} = (q^s(p - 1) + 1)^{r'}$$

and deduce thereby that q^s divides $q + q^2 + \cdots + q^n$. This forces $s = 1$, so $q = 1 + p + p^2 + \cdots + p^m$.

Now suppose, if possible, that $p \ne 2$. Since p^r divides

$$q^{n+1} - 1 = q^p - 1 = (1 + p + p^2 + \cdots + p^m)^p - 1 = p^2 + p^3 N,$$

it follows that $r = 2$, so $m = 1$. Hence $q = p + 1$ which is even — a contradiction.

Consequently, $p = 2$, so $n = 1$, $q = 1 + 2 + 2^2 + \cdots 2^m = 2^{m+1} - 1$, where $m + 1$ is prime, and $r = m + 1$.

Problem 4. Given a positive integer n, show that the set $\{1, 2, \cdots, n\}$ can be partitioned into m sets, each with the same sum, if and only if m is a divisor of $n(n + 1)/2$ which does not exceed $(n + 1)/2$.

Solution. The necessity of the two conditions is easy to establish. If each block in the partition has the sum s, then $ms = 1+2+\cdots+n = n(n+1)/2$, which gives the divisibility condition. Also, $n \geq 2m-1$, for there can be at most one block with a single element.

To prove sufficiency, call a set of parameters n, m, s admissible if $ms = n(n+1)/2$ and $n \geq 2m-1$. If $n = 2m-1$, then $s = 2m-1$, and the partition is unique:

$$\{2m-1\}, \quad \{2m-2, 1\}, \quad \{2m-3, 2\}, \quad \cdots, \quad \{m, m-1\}.$$

Similarly, if $n = 2m$, then $s = 2m+1$, and the partition is again unique:

$$\{2m, 1\}, \quad \{2m-1, 2\}, \quad \cdots, \quad \{m+1, m\}.$$

If $n > 2m$ induct on n. Given an admissible set of parameters n, m, s, construct a new partition from an old partition corresponding to some admissible set of parameters n', m', s', where $n' < n$. The proof will be divided into cases. In each case, the condition $m's' = n'(n'+1)/2$ will be clear from the construction, but we must check that $n' \geq 2m'-1$.

If $2m < n < 4m-1$, then $n+1 < s < 2n$, so if we set $n' = s-n-1$, then $0 < n' < n-1$. We consider two subcases.

If s is odd, let $m' = m-n+(s-1)/2$ and $s' = s$. Here $n'-2m' = n-2m > 0$, so $n' > 2m'$. As new blocks, use the old ones and the $n-(s-1)/2$ pairs

$$\{n, s-n\}, \quad \{n-1, s-n+1\}, \quad \cdots, \quad \{(s+1)/2, (s-1)/2\}.$$

If s is even, let $m' = 2m-2n+s-1$ and $s' = s/2$. A straightforward calculation shows that $2m(n'-2m') = (n-2m)(4m-1-n) > 0$, so $n' > 2m'$ again. The old blocks and the singleton $\{s'\}$ combine in pairs to form $m-n+s'$ new blocks. The other new blocks are the $n-s'$ pairs

$$\{n, s-n\}, \quad \{n-1, s-n+1\}, \quad \cdots, \quad \{s'+1, s'-1\}.$$

Finally, if $n \geq 4m-1$, let $n' = n-2m$, $m' = m$, and $s' = s-2n+2m-1$. Clearly, $n' \geq 2m-1 = 2m'-1$. The new blocks are obtained from the old blocks by adjoining the m pairs

$$\{n, n-2m+1\}, \quad \{n-1, n-2m+2\}, \quad \cdots, \quad \{n-m+1, n-m\}$$

in any order. This completes the proof.