Language: **English**

Saturday, April 28, 2012

Problem 1. Let A, B and C be points lying on a circle Γ with centre O. Assume that $\angle ABC > 90^\circ$. Let D be the point of intersection of the line AB with the line perpendicular to AC at C. Let ℓ be the line through D which is perpendicular to AO. Let E be the point of intersection of ℓ with the line AC, and let F be the point of intersection of Γ with ℓ that lies between D and E.

Prove that the circumcircles of triangles BFE and CFD are tangent at F.

Problem 2. Prove that

$$\sum_{\text{cyc}}(x+y)\sqrt{(z+x)(z+y)} \geq 4(xy+yz+zx),$$

for all positive real numbers x, y, and z.

The notation above means that the left-hand side is

$$(x+y)\sqrt{(z+x)(z+y)}+(y+z)\sqrt{(x+y)(x+z)}+(z+x)\sqrt{(y+z)(y+x)}.$$

Problem 3. Let n be a positive integer. Let $P_n = \{2^n, 2^{n-1}\cdot 3, 2^{n-2}\cdot 3^2, \dots, 3^n\}$. For each subset X of P_n, we write S_X for the sum of all elements of X, with the convention that $S_\emptyset = 0$ where $\emptyset$ is the empty set. Suppose that y is a real number with $0 \leq y \leq 3^{n+1} - 2^{n+1}$.

Prove that there is a subset Y of P_n such that $0 \leq y - S_Y < 2^n$.

Problem 4. Let $\mathbb{Z}^+$ be the set of positive integers. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that the following conditions both hold:

(i) $f(n!) = f(n)!$ for every positive integer n,

(ii) $m - n$ divides $f(m) - f(n)$ whenever m and n are different positive integers.

Each problem is worth 10 points.

Time allowed: 4 hours and 30 minutes.