Stelele Matematicii 2021 — Juniori

Problema 1. Pentru fiecare număr întreg $n \geq 3$, notăm cu s_n suma tuturor numerelor prime strict mai mici decât n. Arătaţi că există o infinitate de numere întregi $n \geq 3$, astfel încât n şi s_n să fie relativ prime.

Problema 2. Fie m şi n două numere întregi, $m \geq 3$ şi $n \geq 3$. Fiecare celulă a unui tablou cu m linii şi n coloane este colorată cu una din două culori, astfel încât:

(1) Pe fiecare coloană să apară ambele culori; şi

(2) Pe oricare două linii, celulele situate la intersecţia cu o coloană să aibă o aceeaşi culoare, pentru exact k coloane.

Arătaţi că, dacă m este impar, atunci

$$\frac{n(m-1)}{2m} \leq k \leq \frac{n(m-2)}{m}.$$

Exemplu. Pentru $m = 5$ şi $n = 10$, tabelele de mai jos ilustrează cazurile $k = 4$ (stânga) şi $k = 6$ (dreapta); a = albastru şi r = roşu:

a	a	a	a	r	r	r	r	r	r
a	r	r	r	a	a	a	r	r	r
r	a	r	r	a	r	r	a	a	r
r	r	a	r	r	a	r	a	r	a
r	r	r	a	r	r	a	r	a	a

a	r	r	r	r	r	a	a	a	a
r	a	r	r	r	a	r	a	a	a
r	r	a	r	r	a	a	r	a	a
r	r	r	a	r	a	a	a	r	a
r	r	r	r	a	a	a	a	a	r

Problema 3. Fie ABC un triunghi şi fie M mijlocul laturii BC. Simetrica dreptei AM în raport cu bisectoarea interioară a unghiului $\angle BAC$, intersectează a doua oară cercul circumscris triunghiului ABC în punctul D. Fie Q, respectiv R, piciorul perpendicularei din D pe dreapta AC, respectiv AB, şi fie X un punct pe dreapta QR, diferit de Q şi R. Perpendiculara în X pe dreapta DX intersectează dreapta AC, respectiv AB, în punctul V, respectiv W. Arătaţi că mijlocul segmentului VW este situat pe dreapta BC.

Problema 4. Fie k un număr natural nenul şi fie a, b şi c numere reale strict pozitive. Arătaţi că

$$a\big(1 - a^k\big) + b\big(1 - (a+b)^k\big) + c\big(1 - (a+b+c)^k\big) < \frac{k}{k+1}.$$

2021 Stars of Mahtematics, Junior Grade — Solution to Problem 1

Problem 1. For every integer $n \geq 3$, let s_n be the sum of all primes (strictly) less than n. Show that there are infinitely many integers $n \geq 3$ such that s_n is coprime to n.

Russian Competition

Solution. It is clearly sufficient to show that at least one of the entries of every pair of consecutive odd primes satisfies the condition in the statement. (Inidentally, notice that $s_5 = 2 + 3 = 5$, so 5 and s_5 are not relatively prime.)

For a prime p, the condition is that s_p be not divisible by p. Let (p, q) be a pair of consecutive odd primes; say, $p < q$, so $s_q = s_p + p$. Clearly, it is sufficient to show that, if $s_p = kp$ for some positive integer k, then $s_q = (k+1)p$ is not divisible by q. Alternatively, but equivalently, that $k + 1$ is not divisible by q.

To prove the latter, notice that $kp = s_p < 1+2+\cdots+(p-1) = \frac{1}{2}p(p-1)$, so $k < \frac{1}{2}(p-1)$. Consequently, $k + 1 < \frac{1}{2}(p + 1) < p < q$, showing that $k + 1$ is indeed not divisible by q.

2021 Stars of Mahtematics, Junior Grade — Solution to Problem 2

Problem 2. Fix integers $m \geq 3$ and $n \geq 3$. Each cell of an array with m rows and n columns is coloured one of two colours such that:

(1) Both colours occur on every column; and

(2) On every two rows the cells on the same column share colour on exactly k columns.

Show that, if m is odd, then

$$\frac{n(m-1)}{2m} \leq k \leq \frac{n(m-2)}{m}.$$

The Problem Selection Committee

Convention. The pairs considered in both solutions in the sequel are all *unordered*. For convenience, two cells on the same column form a *vertical pair*. A *bicolour* vertical pair is one whose cells bear distinct colours; otherwise the vertical pair is *monochromatic*.

Solution 1. Count the total number of bicolour vertical pairs. There are $\frac{1}{2}m(m-1)$ pairs of rows, each of which contains exactly $n-k$ bicolour vertical pairs, so the array contains exactly $\frac{1}{2}m(m-1)(n-k)$ such pairs.

We will show that the total number of monochromatic vertical pairs in the array is at least $n(m-1)$ and, if m is odd, at most $\frac{1}{4}n(m^2-1)$; if m is even, it is at most $\frac{1}{4}nm^2$. The conclusion then follows at once, by the count in the preceding paragraph — the obvious manipulations and calculations are omitted.

To establish the bounds, it is sufficient to show that the number of bicolour pairs along any column is at least $m-1$ and, if m is odd, at most $\frac{1}{4}(m^2-1)$; if m is even, it is at most $\frac{1}{4}m^2 > \frac{1}{4}(m^2-1)$.

Fix a column and let p be the number of cells of one colour on that column. Clearly, there are $p(m-p)$ bicolour pairs along the column.

Since $(p-1)(m-p-1) \geq 0$, it follows that $p(m-p) \geq m-1$, showing that the number of bicolour pairs along the column is at least $m-1$.

On the other hand, $p(m-p) \leq \frac{1}{4}m^2$, so $p(m-p) \leq \left\lfloor \frac{1}{4}m^2 \right\rfloor$. If m is odd, then $\left\lfloor \frac{1}{4}m^2 \right\rfloor = \frac{1}{4}(m^2-1)$, so the number of bicolour pairs along the column is at most $\frac{1}{4}(m^2-1)$.

This completes the argument and concludes the proof.

Solution 2. Count the total number of monochromatic vertical pairs. There are $\frac{1}{2}m(m-1)$ pairs of rows, each of which contains exactly k monochromatic vertical pairs, so the array contains exactly $\frac{1}{2}km(m-1)$ such pairs.

We will show that the total number of monochromatic vertical pairs in the array is at most $\frac{1}{2}n(m-1)(m-2)$ and, if m is odd, at least $\frac{1}{4}n(m-1)^2$; if m is even, it is at least $\frac{1}{4}nm(m-2)$. The conclusion then follows at once, by the count in the preceding paragraph.

To establish the bounds, it is sufficient to show that the number of monochromatic pairs along any column is at most $\frac{1}{2}(m-1)(m-2)$ and, if m is odd, at least $\frac{1}{4}(m-1)^2$; if m is even, it is at least $\frac{1}{4}m(m-2) < \frac{1}{4}(m-1)^2$.

Fix a column and let p be the number of cells of one colour along that column, and let $q = m-p$ be the number of cells of the other colour. The cells along the column form $\frac{1}{2}p(p-1)$

pairs of one colour, and $\frac{1}{2}q(q-1)$ pairs of the other colour. Hence there are $\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1)$ monochromatic pairs along the column. It is easily seen that this number falls between the two bounds mentioned in the preceding paragraph.

To check that $\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1) \le \frac{1}{2}(m-1)(m-2)$, write $q = m-p$ and carry out calculations to obtain the equivalent inequality $(p-1)(p-m+1) \le 0$. This holds, since p lies precisely in the range 1 through $m-1$.

Similarly, checking that $\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1) \ge \frac{1}{4}(m-1)^2$ if m is odd, amounts to $(2p-m)^2 \ge 1$, which is clearly the case by an obvious parity argument.

The weaker inequality, $\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1) = \frac{1}{2}(p^2+q^2) - \frac{1}{2}(p+q) = \frac{1}{2}(p^2+q^2) - \frac{1}{2}m \ge \frac{1}{4}(p+q)^2 - \frac{1}{2}m = \frac{1}{4}m(m-2)$, holds whatever the parity of m; in particular, if m is even.

This completes the argument and concludes the proof.

Remark. The bounds in the statement can both be achieved. For instance, let $m = 5$, $n = 10$, and write b for 'blue' and r for 'red'. The two arrays below achieve the lower bound $k = 4$ (left) and the upper bound $k = 6$ (right), respectively:

b	b	b	b	r	r	r	r	r	r
b	r	r	r	b	b	b	r	r	r
r	b	r	r	b	r	r	b	b	r
r	r	b	r	r	b	r	b	r	b
r	r	r	b	r	r	b	r	b	b

b	r	r	r	r	r	b	b	b	b
r	b	r	r	r	b	r	b	b	b
r	r	b	r	r	b	b	r	b	b
r	r	r	b	r	b	b	b	r	b
r	r	r	r	b	b	b	b	b	r

2021 Stars of Mahtematics, Junior Grade — Solution to Problem 3

Problem 3. Let ABC be a triangle and let M be the midpoint of the side BC. The reflexion of the line AM in the internal bisectrix of the angle $\angle BAC$ crosses the circumcircle of the triangle ABC again at D. Let Q and R be the feet of the perpendiculars from D on the lines AC and AB, respectively, and let X be a point on the line QR, different from both Q and R. The line through X and perpendicular to DX crosses the lines AC and AB at V and W, respectively. Show that the midpoint of the segment VW lies on the line BC.

Adapted from The Amer. Math. Monthly

Solution. (*by The Problem Selection Committee*) Let P be the foot of the perpendicular from D on the line BC, and recall that P, Q and R are collinear — the line ℓ through these points is the Simson line of D with respect to the triangle ABC. If $X = P$, the conclusion is clear, since $V = C$ and $W = B$.

For a generic X on ℓ, that is, $X \neq P$, Q, R, the line VW crosses the line BC at some point U. We will show that U is the midpoint of the segment VW.

Apply Menelaus' theorem to triangle AVW and transversal UBC to write

$$\frac{UV}{UW} \cdot \frac{BW}{BA} \cdot \frac{CA}{CV} = 1.$$

It is therefore sufficient to show that $CV/BW = CA/BA$. The latter is a consequence of the fact that (CDV, BDW), (ACD, AMB) and (ABD, AMC) are pairs of similar triangles. Indeed, assuming these similarities and recalling that $MB = MC$, the desired equality follows from the corresponding similarity ratios below:

$$\frac{CV}{BW} = \frac{CD}{BD}, \qquad \frac{CD}{MB} = \frac{AC}{AM}, \qquad \frac{BD}{MC} = \frac{AB}{AM}.$$

We now turn to prove the three similarities above. The last two offer no difficulty — they both follow by isogonality at A and standard angle chase in the circle through A, B, C, D. For instance, $\angle(AD, AC) = \angle(AB, AM)$, by isogonality at A, and $\angle(DC, DA) = \angle(BC, BA) = \angle(BM, BA)$, on account of A, B, C, D being concyclic. The triangles ACD and AMB are therefore similar. The pair of triangles (ABD, AMC) is dealt with similarly.

To deal with the pair (CDV, BDW), proceed by angle chase in the different cyclic quadrangles that form in the configuration. Thus, $\angle(CD, CV) = \angle(CD, CA) = \angle(BA, BD) = \angle(BW, BD)$, on account of A, B, C, D being concyclic; and $\angle(VC, VD) = \angle(VQ, VD) = \angle(XQ, XD) = \angle(XR, XD) = \angle(WR, WD) = \angle(WB, WD)$, where the third equality holds on account of D, Q, V, X being concyclic (Q and X both lie on the circle on diameter DV), and the fifth — on account of D, R, W, X being concyclic (X and R both lie on the circle on diameter DW). The triangles CDV and BDW are therefore similar. This completes the argument and concludes the proof.

Remark. The particular case where X is the orthogonal projection of D on ℓ shows that P is the midpoint of the segment QR.

2021 Stars of Mahtematics, Junior Grade — Solution to Problem 4

Problem 4. Let k be a positive integer, and let a, b and c be positive real numbers. Show that

$$a\bigl(1-a^k\bigr)+b\bigl(1-(a+b)^k\bigr)+c\bigl(1-(a+b+c)^k\bigr)<\frac{k}{k+1}.$$

* * *

Solution. Let S denote the sum in the left-hand member of the required inequality, and let $x=a$, $y=a+b$ and $z=a+b+c$. Then $b=y-x$, $c=z-y$, so $0<x<y<z$, and $S=x\bigl(1-x^k\bigr)+(y-x)\bigl(1-y^k\bigr)+(z-y)\bigl(1-z^k\bigr)$.

The special case of the AM-GM inequality, $\frac{1}{k+1}\bigl(u^{k+1}+kv^{k+1}\bigr)\geq uv^k$, $u\geq 0$, $v\geq 0$, is used in the sequel; the inequality is strict unless $u=v$.

The particular case, $\frac{1}{k+1}u^{k+1}+\frac{k}{k+1}\geq u$, $u\geq 0$, is used in the last relation below.

The required inequality now follows from the chain of relations below:

$$\begin{aligned}
S &= x\bigl(1-x^k\bigr)+(y-x)\bigl(1-y^k\bigr)+(z-y)\bigl(1-z^k\bigr)\\
&= x+(y-x)+(z-y)-x^{k+1}-(y-x)y^k-(z-y)z^k\\
&= z-x^{k+1}+xy^k-y^{k+1}+yz^k-z^{k+1}\\
&< z-x^{k+1}+\tfrac{1}{k+1}\bigl(x^{k+1}+ky^{k+1}\bigr)-y^{k+1}+\tfrac{1}{k+1}\bigl(y^{k+1}+kz^{k+1}\bigr)-z^{k+1}\\
&= z+\Bigl(-x^{k+1}+\tfrac{1}{k+1}x^{k+1}\Bigr)+\Bigl(\tfrac{k}{k+1}y^{k+1}-y^{k+1}+\tfrac{1}{k+1}y^{k+1}\Bigr)+\Bigl(\tfrac{k}{k+1}z^{k+1}-z^{k+1}\Bigr)\\
&= z-\tfrac{k}{k+1}x^{k+1}-\tfrac{1}{k+1}z^{k+1}=\tfrac{k}{k+1}-\tfrac{k}{k+1}x^{k+1}-\Bigl(\tfrac{1}{k+1}z^{k+1}+\tfrac{k}{k+1}-z\Bigr)\leq\tfrac{k}{k+1}-\tfrac{k}{k+1}x^{k+1}\\
&< \tfrac{k}{k+1}.
\end{aligned}$$

This ends the proof.

Stelele Matematicii 2021 — Seniori

Problema 1. Pentru fiecare număr întreg $n \geq 3$, notăm cu s_n suma tuturor numerelor prime strict mai mici decât n. Există o infinitate de numere întregi $n \geq 3$, astfel încât n şi s_n să fie relativ prime?

Problema 2. Fie n un număr natural nenul. Arătaţi că există un polinom f de grad n cu coeficienţi întregi, astfel încât $f^2 = (X^2 - 1)g^2 + 1$, unde g este un polinom cu coeficienţi întregi.

Problema 3. Fie ABC un triunghi, fie D punctul în care simediana din A a triunghiului ABC intersectează a doua oară cercul ABC şi fie Q, respectiv R, piciorul perpendicularei din D pe dreapta AC, respectiv AB. Considerăm un punct variabil X pe dreapta QR, diferit de Q şi R. Perpendiculara în X pe dreapta DX intersectează dreapta AC, respectiv AB, în punctul V, respectiv W. Determinaţi locul geometric al mijlocului segmentului VW.

Problema 4. Fixăm un număr întreg $n \geq 4$. Fie $\mathcal{C}_n$ mulţimea tuturor configuraţiilor C de n puncte în plan, astfel încât triunghiul format de oricare trei puncte din C să aibă aria strict mai mare decât 1. Pentru fiecare configuraţie C din $\mathcal{C}_n$, notăm cu $f(n, C)$ numărul maxim de puncte pe care le poate conţine o subconfiguraţie a lui C, supusă condiţiei ca distanţa dintre oricare două puncte distincte ale sale să fie strict mai mare decât 2. Determinaţi valoarea minimă $f(n)$ pe care poate să o ia $f(n, C)$, când C parcurge mulţimea $\mathcal{C}_n$.

2021 Stars of Mahtematics, Senior Grade — Solution to Problem 1

Problem 1. For every integer $n \geq 3$, let s_n be the sum of all primes (strictly) less than n. Are there infinitely many integers $n \geq 3$ such that s_n is coprime to n?

Russian Competition

Solution. The answer is in the affirmative. To prove this, we show that at least one of the entries of every pair of consecutive odd primes satisfies the condition in the statement. (Inidentally, notice that $s_5 = 2 + 3 = 5$, so 5 and s_5 are not relatively prime.)

For a prime p, the condition is that s_p be not divisible by p. Let (p, q) be a pair of consecutive odd primes; say, $p < q$, so $s_q = s_p + p$. Clearly, it is sufficient to show that, if $s_p = kp$ for some positive integer k, then $s_q = (k+1)p$ is not divisible by q. Alternatively, but equivalently, that $k + 1$ is not divisible by q.

To prove the latter, notice that $kp = s_p < 1+2+\cdots+(p-1) = \frac{1}{2}p(p-1)$, so $k < \frac{1}{2}(p-1)$. Consequently, $k + 1 < \frac{1}{2}(p + 1) < p < q$, showing that $k + 1$ is indeed not divisible by q.

2021 Stars of Mahtematics, Senior Grade — Solution to Problem 2

Problem 2. Let n be a positive integer. Show that there exists a degree n polynomial f with integral coefficients such that $f^2 = (X^2-1)g^2+1$, where g is a polynomial with integral coefficients.

* * *

Solution 1. Use the binomial expansion to write $\left(a+\sqrt{a^2-1}\right)^n = f(a)+g(a)\sqrt{a^2-1}$, for any real number $a \geq 1$, where f and g are polynomials with integral coefficients, of degree n and $n-1$, respectively. Then $\left(a-\sqrt{a^2-1}\right)^n = f(a)-g(a)\sqrt{a^2-1}$, so

$$\begin{aligned}1 &= \left(a+\sqrt{a^2-1}\right)^n\left(a-\sqrt{a^2-1}\right)^n = \left(f(a)+g(a)\sqrt{a^2-1}\right)\left(f(a)-g(a)\sqrt{a^2-1}\right)\\ &= f(a)^2-g(a)^2(a^2-1).\end{aligned}$$

Since the latter holds for infinitely many values of $a \geq 1$, the conclusion follows.

Solution 2. Read the required condition as a Pell equation in $\mathbb{Z}[X]$, $f^2-(X^2-1)g^2=1$, whose starting solution for $n=1$ is $f_1 = X$ and $g_1 = 1$, to infer that, if f_n and g_n both have a positive leading coefficient and solve the problem for n, then $f_{n+1} = Xf_n+(X^2-1)g_n$ and $g_{n+1} = f_n + Xg_n$ both have a positive leading coefficient and solve the problem for $n+1$. The degree condition is clearly satisfied, the coefficients are all integral, and it is readily checked that $f_{n+1}^2-(X^2-1)g_{n+1}^2 = f_n^2-(X^2-1)g_n^2 = \cdots = f_1^2-(X^2-1)g_1^2 = 1$.

Solution 3. For every real number a, write $\cos na = f(\cos a)$ and $\sin na = g(\cos a)\cdot\sin a$, where f and g are uniquely defined polynomials with integral coefficients, of degree n and $n-1$, respectively. These two relations are easily established together by induction on n. Then

$$1 = (\cos na)^2+(\sin na)^2 = f(\cos a)^2-\Big((\cos a)^2-1\Big)\,g(\cos a)^2,$$

for every real number a. As a runs through the real numbers, $t=\cos a$ takes on infinitely many real values (in fact, t hits every value in the closed interval $[-1,1]$) at each of which $f(t)^2 = (t^2-1)g(t)^2+1$. Consequently, $f^2=(X^2-1)g^2+1$.

Remark. The three solutions are, of course, linked to one another. Consider the approach in Solution 1 to write

$$\begin{aligned}f_{n+1}(a)+g_{n+1}(a)\sqrt{a^2-1} &= \left(a+\sqrt{a^2-1}\right)^{n+1} = \left(a+\sqrt{a^2-1}\right)\left(a+\sqrt{a^2-1}\right)^n\\ &= \left(a+\sqrt{a^2-1}\right)\left(f_n(a)+g_n(a)\sqrt{a^2-1}\right)\\ &= \left(af_n(a)+(a^2-1)g_n(a)\right)+\left(f_n(a)+ag_n(a)\right)\sqrt{a^2-1}.\end{aligned}$$

Since f_n, g_n, f_{n+1} and g_{n+1} all have integral coefficients, letting a run through the positive integers, rational and irrational part identification then provides the recurrence formulae in Solution 2.

Solutions 1 and 3 are also linked to one another. For instance, let $a=\cos\alpha$ in Solution 1, and use de Moivre's formula, on one hand, and the standard binomial expansion, on the other. Real and imaginary part identification then yields the polynomials referred to in Solution 3.

The recurrence formulae in Solution 2 mimic those for the standard Pell equation:

$$\begin{pmatrix} f_{n+1} \\ g_{n+1} \end{pmatrix} = \begin{pmatrix} X & X^2-1 \\ 1 & X \end{pmatrix} \begin{pmatrix} f_n \\ g_n \end{pmatrix} = \begin{pmatrix} f_1 & X^2-1 \\ g_1 & f_1 \end{pmatrix} \begin{pmatrix} f_n \\ g_n \end{pmatrix}.$$

On the other hand, they also mimic the well-known formulae expressing $\cos(n+1)a$ and $\sin(n+1)a$ in terms of $\cos a$, $\sin a$, $\cos na$ and $\sin na$, written formally in the form

$$\begin{aligned} \cos(n+1)a &= \cos a \cdot \cos na + \big((\cos a)^2 - 1\big) \cdot \frac{\sin na}{\sin a}, \\ \frac{\sin(n+1)a}{\sin a} &= \cos na + \cos a \cdot \frac{\sin na}{\sin a}. \end{aligned}$$

This is essentially what links Solutions 2 and 3.

2021 Stars of Mahtematics, Senior Grade — Solution to Problem 3

Problem 3. Let ABC be a triangle, let its A-symmedian cross the circle ABC again at D, and let Q and R be the feet of the perpendiculars from D on the lines AC and AB, respectively. Consider a variable point X on the line QR, different from both Q and R. The line through X and perpendicular to DX crosses the lines AC and AB at V and W, respectively. Determine the geometric locus of the midpoint of the segment VW.

Adapted from The Amer. Math. Monthly

Solution. (*by The Problem Selection Committee*) The required locus is the line BC with the points B and C removed. The latter correspond, in fact, to the removed positions of $X:$ If $X = R$, then $V = A$ and W is (apparently) not defined; and if $X = Q$, then $W = A$ and V is (apparently) not defined. The reason why W should be the reflexion of A across B, in the former case, and V should be the reflexion of A across C, in the latter, will become apparent later on.

Let P be the foot of the perpendicular from D on the line BC, and recall that P, Q and R are collinear — the line ℓ through these points is the Simson line of D with respect to the triangle ABC. If $X = P$, then $V = C$, $W = B$, and the point of the locus is the midpoint M of the segment BC.

For a generic X on ℓ, that is, $X \neq P$, Q, R, the line VW crosses the line BC at some point U. We will show that U is the midpoint of the segment VW. This is the reason why the above choices of V and W, for X at the removed positions on ℓ, are natural.

Apply Menelaus' theorem to triangle AVW and transversal UBC to write

$$\frac{UV}{UW} \cdot \frac{BW}{BA} \cdot \frac{CA}{CV} = 1.$$

It is therefore sufficient to show that $CV/BW = CA/BA$. The latter is a consequence of the fact that (CDV, BDW), (ACD, AMB) and (ABD, AMC) are pairs of similar triangles. Indeed, assuming these similarities and recalling that $MB = MC$, the desired equality follows from the corresponding similarity ratios below:

$$\frac{CV}{BW} = \frac{CD}{BD}, \qquad \frac{CD}{MB} = \frac{AC}{AM}, \qquad \frac{BD}{MC} = \frac{AB}{AM}.$$

We now turn to prove the three similarities above. The last two offer no difficulty — they both follow by isogonality at A and standard angle chase in the circle through A, B, C, D. For instance, $\angle(AD, AC) = \angle(AB, AM)$, by isogonality at A, and $\angle(DC, DA) = \angle(BC, BA) = \angle(BM, BA)$, on account of A, B, C, D being concyclic. The triangles ACD and AMB are therefore similar. The pair of triangles (ABD, AMC) is dealt with similarly.

To deal with the pair (CDV, BDW), proceed by angle chase in the different cyclic quadrangles that form in the configuration. Thus, $\angle(CD, CV) = \angle(CD, CA) = \angle(BA, BD) = \angle(BW, BD)$, on account of A, B, C, D being concyclic; and $\angle(VC, VD) = \angle(VQ, VD) = \angle(XQ, XD) = \angle(XR, XD) = \angle(WR, WD) = \angle(WB, WD)$, where the third equality holds on account of D, Q, V, X being concyclic (Q and X both lie on the circle on diameter DV), and the fifth — on account of D, R, W, X being concyclic (X and R both lie on the circle on diameter DW). The triangles CDV and BDW are therefore similar.

Consequently, the required locus is a subset of the line BC with the points B and C removed. (Incidentally, the particular case where X is the orthogonal projection of D on ℓ shows that P is the midpoint of the segment QR.)

Conversely, let U be a point on the line BC, different from both B and C. The circle ω on diameter DU and the Simson line ℓ meet at P and some point X, not necessarily distinct; the two are tangent at P, in which case $X = P$ for $U = M$ and no other position of U. The line through X and perpendicular to DX is then UX. Letting the line UX cross the lines AC and AB at V and W, respectively, it follows, by the preceding, that U is the midpoint of the segment VW.

Consequently, the required locus is indeed the line BC with the points B and C removed.

It should now be clear that letting W be the reflexion of A across B if $X = R$, and letting V be the reflexion of A across C if $X = Q$, fills in the two gaps in the range of X and the locus. In this setting, as X traces the Simson line, the locus is the line BC.

For completeness, we show that ω and ℓ are tangent at P for $U = M$; since the circle through D and tangent at P to ℓ is unique, for no other position of U are ω and ℓ tangent.

The desired tangency is equivalent to $\angle(MB, MD) = \angle(PR, PD)$. Let AM cross the circle ABC again at E. Isogonality at A shows that D and E are reflexions of one another in the perpendicular bisectrix of the segment BC. Then so are the triangles BDM and CEM. Then

$$\begin{aligned}
\angle(MB, MD) &= \angle(ME, MC) && \text{by symmetry,}\\
&= \angle(MA, MB) && \text{since } AR \text{ and } BC \text{ cross at } M,\\
&= \angle(CA, CD) && \text{since } \triangle ADC \sim \triangle ABM,\\
&= \angle(BA, BD) && \text{since } A,\ B,\ C,\ D \text{ are concyclic,}\\
&= \angle(PR, PD) && \text{since } B,\ D\ P,\ R \text{ are concyclic.}
\end{aligned}$$

This completes the argument and concludes the proof.

Remarks. (1) The circle ω establishes a bicontinuous bijection $X \leftrightarrow U$ between the lines ℓ and BC. This shows again why it is natural to let V be the reflexion of A across C if $X = Q$, and let W be the reflexion of A across B if $X = R$. Another reason is provided in the first paragraph of the next remark.

(2) Since $CV/BW = CA/BA =$ constant, the assignment $V \leftrightarrow W$ links the lines CA and AB projectively, so the lines VW envelop a parabola γ. The required locus can then be derived from the focal and projective properties of γ.

Alternatively, but equivalently, γ can be described as follows: Let ℓ' be the image of ℓ under the homothety χ of ratio 2 centred at D; that is, ℓ' is the line through the reflexions of D in the lines BC, CA, AB. For each X on ℓ, let X' on ℓ' be the image of X under χ, and consider the point T where the line through X' and perpendicular to ℓ' crosses the perpendicular bisectrix of the segment DX' (which is VW, the line through X and perpendicular to DX, as described in the statement). Since X' is the reflexion of D across X, it follows that $TD = TX' = \operatorname{dist}(T, \ell')$; that is, as X traces ℓ, T traces the parabola γ with focus D and directrix ℓ', and the tangent of γ at T is the line VW. In particular, the line BC is the tangent of γ at M, the line CA is the tangent of γ at the reflexion of A across C, the line AB is the tangent of γ at the reflexion of A across B, and ℓ is the tangent of γ at the vertex (the orthogonal projection of D on ℓ).

(3) As incidentally mentioned in the solution, P is the midpoint of the segment QR. Since AD is a diameter of the circle AQR, the reflexion D' of D across P is the orthocentre of the triangle AQR, and isogonality at A implies that the lines AM and $QR = \ell$ are perpendicular. Hence D' lies on the line AM. These facts can all be established, more or less directly, in several other ways as well.

(4) Assume it has somehow been proved or guessed that the locus lies along some line k. Letting $X = P$ shows that M lies on k. Letting X be the orthogonal projection of D on ℓ, shows that P lies on k as well. The points M and P are distinct, unless the triangle ABC is isosceles with apex at A. Leaving this trivial case aside, it follows that k is the line MP, which is no other than BC.

2021 Stars of Mahtematics, Senior Grade — Solution to Problem 4

Problem 4. Fix an integer $n \geq 4$. Let $\mathcal{C}_n$ be the collection of all n-point configurations in the plane every three points of which span a triangle of area (strictly) greater than 1. For each configuration C in $\mathcal{C}_n$, let $f(n, C)$ be the maximal size a subconfiguration of C may have subject to the condition that every pair of distinct points be (strictly) more than 2 distance apart. Determine the minimum value $f(n)$ the size $f(n, C)$ achieves as C runs through $\mathcal{C}_n$.

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Solution. The required minimum is $f(n) = \lceil n/3 \rceil$ and is achieved, for instance, by the configuration described in the next block of four paragraphs.

Let $m = \lfloor n/3 \rfloor$. Consider the vertex set of $m + 1$ equilateral triangles, $\Delta_1, \ldots, \Delta_{m+1}$, of side length 2 each, e. g., translated copies of one another, widely and suitably spread in the plane. Here by 'widely' it is understood that $\operatorname{dist}(\Delta_i, \Delta_j) > 2$ for all $i \neq j$; and by 'suitably', that each Δ_i lies outside every forbidden area strip determined by a segment with one end point at a vertex of a Δ_j, $j \neq i$, and the other at a vertex of a Δ_k, $k \neq i$ (possibly, $j = k$) — this is possible, since admissible regions (rooms) opening wider and wider to infinity always form in the complement of a finite number of strips.

Removal of $3m - n + 3$ vertices of Δ_{m+1} leaves a vertex configuration C in $\mathcal{C}_n$.

Choose one vertex from each Δ_i (Δ_{m+1} inclusive if $m < n/3$) to obtain a subconfiguration of C of size $\lceil n/3 \rceil$ every two points of which are (strictly) more than 2 distance apart.

Clearly, every subconfiguration of C of larger size contains points 2 distance apart. Consequently, $f(n, C) = \lceil n/3 \rceil$, so $f(n) \leq \lceil n/3 \rceil$.

To show that $f(n) \geq \lceil n/3 \rceil$, it is sufficient to prove that $f(n, C) \geq n/3$ for any configuration C in $\mathcal{C}_n$. Fix such a configuration C.

For convenience, a segment of length at most 2 will be referred to as a *short* segment; and a segment with both end points in C will be referred to as a segment *in* C.

Consider the short segment graph G associated with C: the points of C form the vertex set, and the short segments in C form the edge set.

We will show that G is (vertex) 3-colourable, i. e. the points of C can be assigned one of three colours so that the end points of each short segment in C are assigned distinct colours. The most frequently coloured points then form a subconfiguration of at least $n/3$ points every two of which are (strictly) more than 2 distance apart; that is, $f(n, C) \geq n/3$.

To prove 3-colourability, induct on the number of vertices. For convenience, start with the trivial base case $n = 3$. For the induction step, let a be a vertex of G.

If $\deg a \leq 2$, then a colour is always available for a to extend a valid 3-colouring of $G - a$ to one of G.

If $\deg a \geq 3$, then a has a neighbour b such that the line ℓ through a and b leaves neighbours of a, and hence points of C, on either side. Let H be the short segment graph associated with the subconfiguration consisting of a, b and all points of C on one side of ℓ, and let K be the short segment graph associated with the subconfiguration consisting of a, b and all points of C on the other side of ℓ. Clearly, H and K are both subgraphs of G.

Since the segment ab is short, the area condition forces all points in $C \setminus \{a, b\}$ to be (strictly) more than 1 distance away from ℓ. Consequently, every edge of G is either one of H or one of K, and the two share one single edge, namely, ab. (No edge crosses ℓ.)

By the induction hypothesis, H and K are both 3-colourable. The previous paragraph shows that the restriction of any 3-colouring of H to $H - a - b$ and the restriction of any

3-colouring of K to $K - a - b$ always combine to provide a 3-colouring of $G - a - b$. To obtain an overall 3-colouring of G from one of H and one of K, it is therefore sufficient to make the two agree at both a and b. To this end, perform at most two colour swaps in one of them. This completes the argument and concludes the proof.

Remarks. (1) The area condition implies that, if ab and ac are edges of G, then their lines of support form an angle (strictly) greater than $30°$ and (strictly) less than $150°$. Consequently, the degree of every vertex of G is at most 5. Without further work, however, this fact alone does not even imply 5-colourability.

In fact, the area condition implies that the line of support of an edge of G hits no other edge, unless the two edges hinge at a common vertex. In particular, no two edges cross, so G is geometrically plane. At this stage, reference alone to the highly non-trivial four colour theorem merely proves 4-colourability.

Using ideas in the last three paragraphs of the solution, it can be shown that G has at most $2n - 3$ edges. It then follows that the average vertex degree does not exceed $4 - 6/n$, so G has a vertex of degree at most 3. Again, without further work, this merely proves 4-colourability without reference to the four colour theorem.

The authors suspect and are about to prove that G has in fact at most $7n/5$ edges. If this is indeed the case, then the average vertex degree does not exceed $2 \cdot 7/5 < 3$, so G has a vertex of degree at most 2, and 3-colourability now follows at once.

The short segment graph is not necessarily 2-colourable, as the configuration described in the solution clearly shows.

(2) We exhibit two configurations, one in $\mathcal{C}_4$, and the other in $\mathcal{C}_5$, every three points of which span a triangle with at least one short side. Consequently, they both achieve the corresponding $f(n)$. Moreover, the short segment graph associated with each of these configurations is connected and has the largest possible number of edges. For any $n \geq 5$, using widely and suitably spread (translated) copies of the configuration in $\mathcal{C}_5$ produces a configuration in $\mathcal{C}_n$ whose short segment graph has at most $7n/5$ edges.

For convenience, in what follows, a segment of length (strictly) greater than 2 will be referred to as a *long segment*.

The configuration in $\mathcal{C}_4$ consists of the vertices of a lozenge of side length 2 with an internal angular span of $60°$ at some vertex. There are five short segments and one long segment. This configuration achieves $f(4) = 2$. The short segment graph is connected and has the largest possible number of edges; it contains (two) triangles, so it is not 2-colourable.

The configuration in $\mathcal{C}_5$ consists of the vertices of a convex pentagon $ABCDE$ such that $AB = AC = AD = AE = CD = 2$, BE and CD are parallel, and

$$1 < \operatorname{dist}(BE, CD) < \sqrt{3} - \tfrac{1}{2}(\sqrt{6} - \sqrt{2}). \qquad (*)$$

Before checking the area condition, we show that $BC < 2$; by symmetry, $DE = BC < 2$, accounting thus for another two short segments. The inequality $BC < 2$ is equivalent to $\angle BAC < 60°$. Letting M be the midpoint of BE, the desired angular inequality follows from the fact that $\angle BAM$ is acute and $\angle CAM = 30°$.

We now check that every three the vertices of the pentagon span a triangle whose area is (strictly) greater than 1. By symmetry, the area condition is to be checked out only for the triangles ABC, ABD, ABE, ACD, BCD and BCE.

The first four triangles above span an area (stricltly) greater than 1, since their internal angular spans at A all lie (strictly) between 30° and 150°. Indeed, the smallest of these is $\angle BAC = \angle BAM - 30^\circ$, and the largest is $\angle BAE = 2\angle BAM$, so it is sufficient to show that $60^\circ < \angle BAM < 75^\circ$. Alternatively, but equivalently, $\frac{1}{2}\sqrt{3} > \cos\angle BAM > \frac{1}{4}(\sqrt{6} - \sqrt{2})$. To establish the latter, write $\cos\angle BAM = AM/AB = AM/2$, and $AM = \operatorname{dist}(A, CD) - \operatorname{dist}(BE, CD) = \sqrt{3} - \operatorname{dist}(BE, CD)$, and refer to $(*)$.

Continuing, area $BCD > 1$, since $CD = 2$, and the B-altitude is (strictly) greater than 1, by $(*)$; and area $BCE > 1$, since the C-altitude is (strictly) greater than 1, by $(*)$, and $BE = 2 \cdot BM = 2 \cdot AB \cdot \sin\angle BAM = 4\sin\angle BAM > 4\sin 60^\circ = 2\sqrt{3} > 2$.

Consequently, the vertices of the pentagon form a configuration in $\mathcal{C}_5$. There are seven short segments, three long segments, and the triangle spanned by every three vertices has at least one short side. This configuration achieves $f(5) = 2$. The short segment graph is connected and has the largest possible number of edges; it is not 2-colourable, since it contains triangles, such as ABC, ACD and ADE.