IMAR 2019 — Solutions

Problem 1. Let ABC be an acute triangle, let D, E, F be the feet of the altitudes from A, B, C, respectively, and let M, N, P be the midpoints of the sides BC, CA, AB, respectively. The circles BDP and CDN cross again at X, the circles CEM and AEP cross again at Y, and the circles AFN and BFM cross again at Z. Prove that the lines AX, BY, CZ are concurrent.

Solution 1. We show that the lines AX, BY, CZ are the symmedians of the triangle ABC from A, B, C, respectively, and hence concur at the Lemoine point of the triangle.

Clearly, it is sufficient to prove that AX is the A-symmedian of the triangle ABC. To this end, we show that the triangles XAB and XCA are similar. It then follows that the line AX bisects the angle BXC, and $XB/XC = AB^2/AC^2$, so AX is indeed the A-symmedian of the triangle ABC.

To prove similarity, we first show that the quadrangle $ANXP$ is cyclic. By the preceding, $\angle DXP = 180^\circ - \angle PBD$, and $\angle NXD = 180^\circ - \angle DCN$, so

$$\begin{aligned}\angle PXN &= 360^\circ - \angle DXP - \angle NXD = 360^\circ - (180^\circ - \angle PBD) - (180^\circ - \angle DCN)\\ &= \angle PBD + \angle DCN = \angle ABC + \angle BCA = 180^\circ - \angle CAB = 180^\circ - \angle NAP,\end{aligned}$$

showing that the quadrangle $ANXP$ is indeed cyclic.

We are now in a position to show that $\angle XAB = \angle XCA$. Refer to the previous paragraph to write $\angle XAB = \angle XAP = \angle XNP$. On the other hand, the midline NP and the circle CDN are clearly tangent at N, so $\angle XNP = \angle XCN = \angle XCA$. Consequently, $\angle XAB = \angle XCA$.

A similar argument shows that $\angle ABX = \angle CAX$, so the triangles XAB and XCA are indeed similar. This ends the proof.

Solution 2. As in the previous solution, we show that AX is the A-symmedian of the triangle ABC.

To this end, invert from A with power $AB \cdot AC$. The images of B, C, D, N, P and X under this inversion are located as follows:

(1) The image of B is the point B' on the ray AB emanating from A such that $AB' = AC$; similarly, the image of C is the point C' on the ray AC emanating from A such that $AC' = AB$, so the triangles ABC and $AC'B'$ are reflexions of one another in their common internal A-bisectrix;

(2) The image of D is the antipode D' of A in the circle $AB'C'$, since

$$AD' = \frac{AB \cdot AC}{AD} = \frac{AB \cdot AC \cdot BC}{2 \cdot \text{area}\, ABC} = \frac{AC' \cdot AB' \cdot B'C'}{2 \cdot \text{area}\, AC'B'}$$

which is the diameter of the circle $AC'B'$;

(3) The images N' and P' of N and P, respectively, are the reflexions of A across C' and B', respectively; and

(4) Since the angles $AB'D'$ and $AC'D'$ are both right, by **(2)**, and B' and C' are the midpoints of the segments AP' and AN', respectively, by **(3)**, D' is the centre of the circle $AN'P'$, so the image of X is the midpoint X' of the segment $N'P'$.

Finally, since $B'C'$ and $N'P'$ are parallel, the line AXX' also bisects the segment $B'C'$, and since the triangles ABC and $AC'B'$ are reflexions of one another in their common internal A-bisectrix, it is indeed the A-symmedian of the triangle ABC.

Problem 2. Let f_0, f_1, f_2 and f_3 be polynomials in $\mathbb{R}[X]$ such that $f_k(1) = f_{k+1}(0)$, $k = 0, 1, 2, 3$ (indices are reduced modulo 4). Show that there exists a polynomial f in $\mathbb{R}[X, Y]$ such that $f(X, 0) = f_0(X)$, $f(1, Y) = f_1(Y)$, $f(1 - X, 1) = f_2(X)$, and $f(0, 1 - Y) = f_3(Y)$.

Solution 1. The idea is to consider a suitable $\mathbb{R}[Y]$-linear combination of $f_0(X)$ and $f_2(1-X)$, namely, $(1 - Y)f_0(X) + Yf_2(1 - X)$, and a suitable $\mathbb{R}[X]$-linear combination of $f_1(Y)$ and $f_3(1-Y)$, namely, $Xf_1(Y)+(1-X)f_3(1-Y)$, along with a suitable degree 2 corrective term, $a_{11}XY + a_{10}X + a_{01}Y + a_{00}$, collect them together to form

$$f(X,Y) = (1-Y)f_0(X)+Xf_1(Y)+Yf_2(1-X)+(1-X)f_3(1-Y)+a_{11}XY+a_{10}X+a_{01}Y+a_{00},$$

and require the latter to satisfy the conditions in the statement.

Thus, the condition $f(X, 0) = f_0(X)$ requires $a_{10} = f_3(1) - f_1(0)$ and $a_{00} = -f_3(1)$. Refer to the hypothesis on the f_k to write $a_{10} = f_0(0) - f_1(0)$ and $a_{00} = -f_0(0)$

Next, the condition $f(0, 1 - Y) = f_3(Y)$ requires $a_{01} = f_0(0) - f_2(1)$ and $f_2(1) + a_{01} + a_{00} = 0$. Notice that the latter holds automatically. With reference again to the hypothesis on the f_k, $a_{01} = f_0(0) - f_3(0)$.

Similarly, the condition $f(1, Y) = f_1(Y)$ requires $a_{11} = f_0(1) - f_2(0) - a_{01}$ and $f_0(1) + a_{10} + a_{00} = 0$. The latter holds again automatically, while the former yields $a_{11} = f_1(0) - f_2(0) - f_0(0) + f_3(0)$.

Finally, the condition $f(1 - X, 1) = f_2(X)$ requires $f_3(0) - f_1(1) - a_{11} - a_{10} = 0$ and $f_1(1) + a_{11} + a_{10} + a_{01} + a_{00} = 0$, both of which hold by the preceding and the hypothesis on the f_k.

In terms of the data, the desired polynomial is

$$\begin{aligned} f(X,Y) = {} & (1-Y)f_0(X) + Xf_1(Y) + Yf_2(1-X) + (1-X)f_3(1-Y) - \\ & \big(f_0(0) - f_1(0) + f_2(0) - f_3(0)\big)XY + \big(f_0(0) - f_1(0)\big)X + \big(f_0(0) - f_3(0)\big)Y - f_0(0). \end{aligned}$$

Solution 2. The polynomial in the previous solution may equally well be obtained as follows: Begin by seeking a two-variable polynomial f of the form

$$f(X,Y) = g(X,Y) + Xh(Y) + Yk(X) + aXY,$$

where g is a two-variable polynomial satisfying $g(X, 0) = f_0(X)$ and $g(0, 1 - Y) = f_3(Y)$, h and k are one-variable polynomials subject to $h(0) = k(0) = 0$ and $h(1) = k(1)$, and a is a real number to be determined from the requirements. The common value $h(1) = k(1)$ will come out of the choice of h and k.

Notice that such a choice of g, h and k yields $f(X, 0) = f_0(X)$ and $f(0, 1 - Y) = f_3(Y)$.

Now, the two-variable polynomial $g(X, Y) = f_0(X) + f_3(1-Y) - f_0(0)$ clearly fits the bill; with reference to the hypothesis on the f_k, the verification is routine and hence omitted.

Next, letting $h(Y) = f_1(Y) - g(1, Y)$ and $k(X) = f_2(1 - X) - g(X, 1)$, it is again readily checked that $h(0) = k(0) = 0$, and $h(1) = f_1(1) - g(1, 1) = f_2(0) - g(1, 1) = k(1)$. In terms of the data, $h(Y) = f_1(Y) - f_3(1-Y) + f_0(0) - f_1(0)$, $k(X) = -f_0(X) + f_2(1-X) + f_0(0) - f_3(0)$, and $h(1) = k(1) = f_0(0) - f_1(0) + f_2(0) - f_3(0)$.

At this stage, requiring $f(1,Y) = f_1(Y)$ yields $a = -f_0(0) + f_1(0) - f_2(0) + f_3(0)$; incidentally, yet not accidentally at all, this is precisely the value of a_{11} in the previous solution.

Finally, notice that $h(1) = -a$, to check the remaining requirement:

$$f(1-X,1) = g(1-X,1)+h(1)(1-X)+k(1-X)+a(1-X) = g(1-X,1)+k(1-X) = f_2(X).$$

Expressing f in terms of the data yields the polynomial in the previous solution, just as mentioned in the beginning.

Remarks. **(1)** If F is an arbitrary polynomial solution of the problem, then $F - f$ is a polynomial that vanishes identically along each of the lines $X = 0$, $X = 1$, $Y = 0$ and $Y = 1$; hence it must have the factor $X(X-1)Y(Y-1)$. Consequently, all polynomial solutions have the form $f + X(X-1)Y(Y-1)\varphi$, where φ is an arbitrary two-variable polynomial.

(2) The result generalises as follows: Given n lines $\ell_1, \ell_2, \ldots, \ell_n$ in the plane, no three of which are concurrent, and n one-variable polynomials f_k such that f_i and f_j assume the same value at $\ell_i \cap \ell_j$ for all i and all j, there exists a two-variable polynomial f whose restriction to ℓ_k is f_k for each k.

Problem 3. Is it possible to express a positive integer n congruent to 9 modulo 25 in the form $n = a(a+1)/2 + b(b+1)/2 + c(c+1)/2$, where a, b, c are non-negative integers that do not share parity?

Solution. The answer is in the affirmative. Alternatively, but equivalently, we show that, if n is a positive integer congruent to 9 modulo 25, then $N = 8n+3 = (2a+1)^2+(2b+1)^2+(2c+1)^2$ for some non-negative integers a, b, c that do not share parity; that is, N is a sum of three odd squares whose positive square roots are not congruent modulo 4. This is a special case of the following fact:

(∗) *Let (p,q,r) be a Pythagorean triple of positive integers, $p^2 + q^2 = r^2$, such that $p \equiv -1 \pmod 4$, $q \equiv 0 \pmod 4$, $r \equiv 1 \pmod 4$, and $p < q$. Then every positive integer $N \equiv 3 \pmod 8$, that is divisible by r^2, is the sum of three odd squares whose positive square roots are not congruent modulo 4.*

Consequently, a positive integer $n \equiv 3(r^2-1)/8 \pmod{r^2}$ is expressible in the form $n = a(a+1)/2 + b(b+1)/2 + c(c+1)/2$ for some non-negative integers a, b, c that do not share parity.

The problem at hand is the special case where $(p,q,r) = (3,4,5)$.

To prove (∗), notice that $N/r^2 \equiv 3 \pmod 8$, so it is not of the form $4^k(8\ell+7)$, and is therefore a sum of three odd squares (Gauss-Legendre).

Write $N = (ru)^2 + (rv)^2 + (rw)^2$ for some positive odd integers u, v, w, and assume, without loss of generality, that $u \geq v$, to write $(ru)^2 + (rv)^2 = (pu+qv)^2 + (qu-pv)^2$.

Since $(ru - rv) + \big((pu+qv) - (qu-pv)\big) \equiv (u-v) + (u+v) \equiv 2u \equiv 2 \pmod 4$, the entries of one of the pairs of positive odd integers (ru, rv), $(pu+qv, qu-pv)$ are not congruent modulo 4. This ends the proof.

Remark. There are, of course, infinitely many primitive Pythagorean triples satisfying the conditions in (∗). For instance, $p = |4m+1|$, $q = 4m(2m+1)$, $r = 4m(2m+1)+1$, where m runs through the non-zero integers. Thus, every positive integer $n \equiv 3m(2m+1)(4m^2+2m+1)$

modulo $(8m^2+4m+1)^2$ is expressible in the form $n = a(a+1)/2 + b(b+1)/2 + c(c+1)/2$ for some non-negative integers a, b, c that do not share parity. The problem at hand is the special case where $m = 1$. Similarly, let $m = -1$, to infer that every positive integer $n \equiv 63 \pmod{169}$ is expressible in the form $n = a(a+1)/2 + b(b+1)/2 + c(c+1)/2$ for some non-negative integers a, b, c that do not share parity.

Problem 4. Given a positive integer k, a *loop of length* k in a graph is a list v_1, e_1, v_2, e_2, …, v_k, e_k, where the v_i are (not necessarily distinct) vertices, the e_i are (not necessarily distinct) edges, and each e_i joins v_i and v_{i+1} (indices are reduced modulo k); the loop *traces* an edge e if $e = e_i$ for some index i. Show that a connected graph with vertex set V and edge set E has a loop of length at most $|V| + |E| - 1$ tracing every edge of the graph.

Radu Bumbăcea

Solution. Let G be a connected graph with vertex set V and edge set E; G may have loops and/or multiple edges. The idea is to expand at most $|V| - 1$ suitable edges to pairs of edges to make G into a graph $\hat{G}$ on V each vertex of which has an even degree. A maximal loop in $\hat{G}$ tracing each edge at most once is then Eulerian, i. e., traces each edge exactly once (Euler). Read back in G, i. e., identifying each cloned edge back to the original, this is the desired loop: it traces every cloned edge twice, every other edge once, and its length does not exceed $|V| - 1 + |E|$.

To obtain $\hat{G}$, consider a spanning tree T of G, i. e., a minimal connected subgraph on V. We will show that T has a subgraph S on V such that $\deg_S$ and $\deg_G$ agree modulo 2 at all vertices; clearly, the number of edges of S does not exceed the number of edges of T which is $|V| - 1$. Cloning each edge of S once, while keeping its end points fixed, of course, yields the desired additional edges in $\hat{G}$.

To obtain S from T, notice that the sums $\sum_{v\in V} \deg_T v = 2|E_T|$ and $\sum_{v\in V} \deg_G v = 2|E|$ have like parities, to infer that $\deg_T$ and $\deg_G$ disagree modulo 2 at an even number of vertices (possibly zero), say, v_1, …, v_n, v_{n+1}, …, v_{2n}. For each index i in the range 1 through n, let α_i be the unique path in T joining v_i and v_{i+n}, and collect together all edges of T lying along an even number of the α_i (zero, inclusive) to form the edge set of S.

Finally, we show that $\deg_S$ and $\deg_G$ agree modulo 2 at all vertices. To this end, fix a vertex v, and let E' be the set of all edges of T having an end point at v. For each edge e in E' and each path α_i, consider their edge-path incidence number

$$\langle e, \alpha_i\rangle = \begin{cases} 1 & \text{if } \alpha_i \text{ traces } e \\ 0 & \text{otherwise,} \end{cases}$$

and let $\equiv$ denote congruence modulo 2 to write

$$\begin{aligned}\deg_T v - \deg_S v &\equiv \sum_{e\in E'} \sum_{i=1}^{n} \langle e, \alpha_i\rangle = \sum_{i=1}^{n} \sum_{e\in E'} \langle e, \alpha_i\rangle \equiv \begin{cases} 1 & \text{if } v \text{ is a } v_i \\ 0 & \text{otherwise} \end{cases} \\ &\equiv \deg_G v - \deg_T v,\end{aligned}$$

on account of $\sum_{e\in E'} \langle e, \alpha_i\rangle$ being congruent to 1 modulo 2 if and only if α_i has an end point at v, which is the case if and only if v is one of v_i, v_{i+n}. Consequently, $\deg_S v \equiv \deg_G v$ and the conclusion follows.

Remark. In the above notation, S may alternatively, but equivalently, be described as the last subgraph in an $(n+1)$-term sequence of subgraphs of T, $S_i = (V, E_i)$, $i = 0, \ldots, n$, whose edge sets are recursively defined by $E_0 = E_T$, so $S_0 = T$, and $E_i = (E_{i-1} \smallsetminus A_i) \cup (A_i \cap (E_T \smallsetminus E_{i-1}))$, where A_i is the set of edges along α_i, $i = 1, \ldots, n$. Since $\deg_{S_{i-1}}$ and $\deg_{S_i}$ disagree modulo 2 only at v_i and v_{i+n}, it follows that $\deg_T = \deg_{S_0}$ and $\deg_{S_n} = \deg_S$ disagree modulo 2 only at v_1, …, v_n, v_{n+1}, …, v_{2n}. Consequently, $\deg_S$ and $\deg_G$ agree modulo 2 throughout.

IMAR 16 NOIEMBRIE 2019

Nr.crt.	Numele si prenumele	Unitatea de provenienta	P1	P2	P3	P4	Total	Premiu
1	MEMIS EDIS	LICEUL INTERNAŢIONAL DE INFORMATICĂ CONSTANŢA	7	7	7	7	28	I
2	MARGINEAN ANDREI	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	7	7	7	2	23	II
3	SIMON SEBASTIAN	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	7	7	7	2	23	II
4	TRAN BACH NGUYEN	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	7	7	0	7	21	III
5	ANGHEL DAVID ANDREI	SC GIMN 56 BUCURESTI	7	7	3	0	17	M
6	PARFENI ANDREI ALEXANDRU	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	7	6	3	0	16	M
7	GIRBAN ALEXANDRU	LICEUL INTERNAŢIONAL DE INFORMATICĂ CONSTANŢA	7	6	1	1	15	M
8	CARDAS TUDOR DARIUS	COLEGIUL NAŢIONAL A.T. LAURIAN BOTOSANI	7	7	0	0	14	M
9	IACOB RADU ALEXANDRU	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	7	7	0	0	14	M
10	LECOIU RADU ANDREI	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	7	0	7	0	14	M
11	BALAN TRIBUS LEON ROLAND	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	3	7	0	1	11	
12	NICOLCEA HORIA PAUL	COLEGIUL NAŢIONAL DIMITRIE CANTEMIR ONESTI	4	7	0	0	11	
13	ROBU VLAD	COLEGIUL NAŢIONAL VASILE LUCACIU	7	3	1	0	11	
14	CRACIUNESCU ION EMANUEL	C.N. FRATII BUZESTI CRAIOVA	2	7	1	0	10	
15	GASAN CAROL LUCA	COLEGIUL NAŢIONAL SFANTUL SAVA	0	7	0	2	9	
16	IOSIFESCU DINU ALEXANDRU	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	7	0	2	0	9	
17	MOLDOVAN ANDREI	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	7	0	2	0	9	
18	PANA LUCA	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	7	0	2	0	9	
19	VERGELEA VLAD STEFAN	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	7	1	0	1	9	
20	BABAN BOGDAN	COLEGIUL NAŢIONAL DE INFORMATICĂ TUDOR VIANU	7	0	1	0	8	
21	CONSTANTINESCU IUSTINIAN	COLEGIUL NAŢIONAL DE INFORMATICĂ TUDOR VIANU	7	1	0	0	8	
22	DRAGOMIRESCU ROBERT	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	7	0	1	0	8	
23	TEODORESCU ANTONIO	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	7	0	1	0	8	
24	VASILE MARIAN DANIEL	COLEGIUL N STEFAN ODOBLEJA DROBETA TR. SEVERIN	7	0	1	0	8	
25	MERCAN HORIA	COLEGIUL NAŢIONAL DE INFORMATICĂ TUDOR VIANU	7	0	0	0	7	
26	OZGULUK DENIZ	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	7	0	0	0	7	
27	PANTEA ANDREI TIBERIU	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	7	0	0	0	7	
28	PICU GEORGE	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	7	0	0	0	7	
29	ION ANREI ROBERT	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	0	3	0	2	5	
30	CABA TUDOR IOAN	COLEGIUL NAŢIONAL DE INFORMATICĂ TUDOR VIANU	1	3	0	0	4	
31	CRACIUN MIHAI	COLEGIUL NAŢIONAL DE INFORMATICĂ TUDOR VIANU	0	3	1	0	4	
32	IGNATESCU LUCA	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	3	0	1	0	4	
33	BERBECE RIAN ALEXANDRU	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	1	1	1	0	3	
34	COMAN MIHNEA-GEORGE	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	3	0	0	0	3	
35	GHEORGHE LUCA	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	2	0	0	0	2	
36	MITRACHE IONUT	LICEUL INTERNAŢIONAL DE INFORMATICĂ BUCUREŞTI	1	1	0	0	2	
37	BORNEA ANDREI	LICEUL TEORETIC MIHAI EMINESCU CALARASI	1	0	0	0	1	
38	TICUS RARES ANDREI	COLEGIUL NAŢIONAL SFANTUL SAVA	1	0	0	0	1	