IMAR 2017

Problema 1. Fie P un punct situat în interiorul unui triunghi ABC. Dreapta AP intersectează latura BC în punctul D; dreapta BP intersectează latura CA în punctul E; iar dreapta CP intersectează latura AB în punctul F. Cercul de diametru BC intersectează cercul de diametru AD în punctele a şi a'; cercul de diametru CA intersectează cercul de diametru BE în punctele b şi b'; iar cercul de diametru AB intersectează cercul de diametru CF în punctele c şi c'. Arătaţi că punctele a, a', b, b', c, c' sunt conciclice.

Problema 2. Fie n un număr natural nenul. Pentru fiecare număr natural nenul $k \leq n$, notăm cu r_k restul împărţirii lui 2^n la k. Arătaţi că

$$\sum_{k=1}^{n} r_k > \frac{n}{2}\left(\log_2 \frac{n}{3} - 2\right).$$

Problema 3. O mulţime *specială* este o mulţime de numere naturale impare, astfel încât niciun element al său nu divide un alt element şi orice submulţime de cardinal 3 conţine un element care divide suma celorlalte două. O mulţime specială este *maximală*, dacă ea nu este conţinută în nicio altă mulţime specială. Determinaţi numărul de elemente pe care îl poate avea o mulţime specială maximală.

Problema 4. Fie n un număr întreg mai mare sau egal cu 3 şi fie $\mathcal{P}_n$ mulţimea tuturor n-goanelor planare (simple), care nu au nicio pereche de laturi distincte paralele sau de-a lungul unei aceleiaşi drepte. Pentru fiecare poligon P din $\mathcal{P}_n$, fie $f_n(P)$ numărul minim de triunghiuri formate de drepte-suport ale laturilor lui P, a căror reuniune acoperă poligonul P. Determinaţi valoarea maximă pe care o poate lua $f_n(P)$, când P parcurge mulţimea $\mathcal{P}_n$.

IMAR 2017 — Solutions

Problem 1. Let P be a point in the interior of the triangle ABC, and let the lines AP, BP, CP meet the sides BC, CA, AB respectively at the points D, E, F. Let the circles on diameters BC and AD meet at points a and a'; the circles on diameters CA and BE meet at points b and b'; and the circles on diameters AB and CF meet at points c and c'. Show that the points a, a', b, b', c, c' lie on a circle.

Solution. Let A, B, C and P have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and

$$\mathbf{p} = \frac{\alpha}{\alpha+\beta+\gamma}\,\mathbf{a} + \frac{\beta}{\alpha+\beta+\gamma}\,\mathbf{b} + \frac{\gamma}{\alpha+\beta+\gamma}\,\mathbf{c},$$

where α, β, γ are the barycentric coordinates of P. Then D has position vector

$$\frac{\beta}{\beta+\gamma}\,\mathbf{b} + \frac{\gamma}{\beta+\gamma}\,\mathbf{c},$$

so the circles on diameters BC and AD have equations

$$(\mathbf{r}-\mathbf{b})\cdot(\mathbf{r}-\mathbf{c}) = 0 \quad \text{and} \quad (\mathbf{r}-\mathbf{a})\cdot\left(\mathbf{r} - \frac{\beta}{\beta+\gamma}\,\mathbf{b} - \frac{\gamma}{\beta+\gamma}\,\mathbf{c}\right) = 0,$$

respectively; alternatively, but equivalently, the latter reads

$$\beta(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{b}) + \gamma(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{c}) = 0.$$

Any linear combination of the two is the equation of a circle or straight line through a and a'. In particular, the linear combination formed by multiplying the first equation by $\beta\gamma$ and the second by α, and adding, is

$$\alpha\beta(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{b}) + \beta\gamma(\mathbf{r}-\mathbf{b})\cdot(\mathbf{r}-\mathbf{c}) + \gamma\alpha(\mathbf{r}-\mathbf{c})\cdot(\mathbf{r}-\mathbf{a}) = 0.$$

The symmetry of this equation shows that this circle (or, possibly, straight line) also passes through b, b' and c, c'. Finally, notice that $\alpha\beta + \beta\gamma + \gamma\alpha$ is positive when P lies inside the triangle ABC, so the locus is indeed a circle centred at the point whose barycentric coordinates are $\alpha(\beta+\gamma)$, $\beta(\gamma+\alpha)$, $\gamma(\alpha+\beta)$, respectively.

Remark. The condition for the locus to be a straight line is $\alpha\beta+\beta\gamma+\gamma\alpha = 0$, which implies that P lies on an ellipse through A, B and C that has tangents at A, B and C parallel to BC, CA and AB, respectively.

Problem 2. Let n be a positive integer. For each positive integer $k \le n$, let r_k denote the remainder 2^n leaves upon division by k. Prove that

$$\sum_{k=1}^{n} r_k > \frac{n}{2}\left(\log_2 \frac{n}{3} - 2\right).$$

Solution. Let $m = \lfloor \log_2 n \rfloor$, and split the set of integers 1 through n into $m+1$ pairwise disjoint subsets J_0, J_1, …, J_m, where J_k consists of all numbers of the form $2^k(2\ell+1)$, $\ell = 0, \ldots, \lfloor n/2^{k+1} - 1/2 \rfloor$.

If j is a member of J_k, then $r_j \geq 2^k$, unless $j = 2^k$ in which case $r_j = 0$.

If $k > \lfloor \log_2 \frac{n}{3} \rfloor = m'$, then J_k consists of 2^k alone, so it contributes nothing to sum in question.

Consequently,

$$\sum_{k=1}^{n} r_k = \sum_{k=0}^{m'} \sum_{j \in J_k} r_j \geq \sum_{k=0}^{m'} 2^k(|J_k| - 1) = \sum_{k=0}^{m'} 2^k \left\lfloor \frac{n}{2^{k+1}} - \frac{1}{2} \right\rfloor > \sum_{k=0}^{m'} 2^k \left(\frac{n}{2^{k+1}} - \frac{3}{2} \right)$$
$$= \frac{n}{2}(m' + 1) - \frac{3}{2}(2^{m'+1} - 1) > \frac{n}{2} \log_2 \frac{n}{3} - n + \frac{3}{2} > \frac{n}{2} \left(\log_2 \frac{n}{3} - 2 \right).$$

Remarks. The lower bound in the statement is rough for the sum $S_n = \sum_{k=1}^{n} r_k$ under consideration. It has been conjectured that $S_n \geq cn^2$ for some suitable positive constant c, but it is not even known whether this is the case for infinitely many values of n. It has, however, been shown that $S_n > n^2/(132 \log_2 n)$ for infinitely many values of n.

Clearly, $S_n \leq 1+2+\cdots+(n-1) = n(n-1)/2$. It has also been conjectured that $S_n \leq n^2/4$ for all but finitely many values of n, but this question is still open as well.

Problem 3. A *special* set is a set of positive odd integers no element of which divides another, and each 3-element subset of which has an member dividing the sum of the other two. A special set is *maximal* if it is contained in no other special set. Determine the number of elements a maximal special set may have.

Solution. Leaving aside the trivial case $\{1\}$, a maximal special set may have only 3, 4 or 5 elements. Begin by noticing that if $a < b$ are positive odd integers, and a does not divide b, then a, b and $2b - a$ form a special set, so a maximal special set has at least three elements.

At the other extreme, a special set — in particular, one that is maximal — has at most five elements. The proof relies on the three facts below:

(1) If $a > b > c$ form a special set, then $b + c$ is not divisible by a. This is because a is odd, and $b + c$ is a positive even integer less than $2a$.

(2) If $a > b$ are members of a special set S, then at most one of the members of S less than b does not divide the sum $a + b$. Suppose, if possible, c and d are distinct members of S less then b, neither of which divides the sum $a+b$. By **(1)**, a divides neither $b+c$ nor $b+d$, so $a + c$ and $a + d$ are both divisible by b. Then $|c - d| = |(a + c) - (a + d)|$ is a positive integer less than b and divisible by b — a contradiction.

(3) If a, b, c, d form a special set, and $a + b$ and $a + c$ are both divisible by d, then $b + c$ is not divisible by d. Otherwise, d would divide $(a + b) + (a + c) - (b + c) = 2a$, which is impossible, since d is odd and does not divide a.

We are now in a position to prove that a special set has at most five elements. Suppose, if possible, a_1, a_2, a_3, b_1, b_2, b_3 are pairwise distinct members of a special set. We may and will assume $a_1 > a_2 > a_3 > \max(b_1, b_2, b_3)$.

Fix a pair of distinct indices i and j, and write $\{i, j, k\} = \{1, 2, 3\}$. By **(2)**, some b divides both $a_i + a_k$ and $a_j + a_k$, by **(3)**, that b does not divide $a_i + a_j$, so, with reference again to **(2)**, it is the unique b not dividing $a_i + a_j$.

Consequently, the three b's may be labelled so that b_i and b_j both divide $a_i + a_j$, while b_k does not, $\{i, j, k\} = \{1, 2, 3\}$.

By **(1)**, a_1 does not divide a_3+b_2, and since b_2 does not divide a_1+a_3, it follows that a_1+b_2 is divisible by a_3. Similarly, a_2+b_1 is divisible by a_3, and hence so is $(a_1+b_2)+(a_2+b_1)=(a_1+a_2)+(b_1+b_2)$. Finally, since a_1+a_2 is divisible by a_3, by **(2)**, so is b_1+b_2, in contradiction with **(1)**.

Consequently, a special set — in particular, one that is maximal — has at most five elements.

Next, we show that 5-element special sets actually exist. Clearly, the numbers 3, 5, 7 form a special set. To enlarge this set to a 4-element special set by adjoining a positive odd integer k, notice that k divides no 2-term sum from $\{3,5,7\}$, to infer that k satisfies one of the two systems of linear congruences below:

$$\begin{cases} k+1\equiv 0 \pmod 2\\ k+5\equiv 0 \pmod 3\\ k+3\equiv 0 \pmod 7\\ k+7\equiv 0 \pmod 5\end{cases} \quad\text{or}\quad \begin{cases} k+1\equiv 0 \pmod 2\\ k+3\equiv 0 \pmod 5\\ k+5\equiv 0 \pmod 7\\ k+7\equiv 0 \pmod 3\end{cases}$$

By the Chinese remainder theorem, each of these systems has infinitely many solutions; in each case, two solutions differ by a multiple of $2\cdot 3\cdot 5\cdot 7=210$. The least positive solution of the former is 193, and the least positive solution of the latter is 107.

To enlarge the set $\{3,5,7,193\}$ to a 5-element special set by adjoining a positive odd integer k, notice again that k divides no 2-term sum from $\{3,5,7,193\}$, to infer that k satisfies the system of linear congruences

$$\begin{cases} k+1\equiv 0 \pmod 2\\ k+3\equiv 0 \pmod 5\\ k+5\equiv 0 \pmod 7\\ k+7\equiv 0 \pmod 3\\ k+7\equiv 0 \pmod{193}\end{cases};$$

clearly, 3 and 5 both divide $k+193$. As before, the Chinese remainder theorem settles the case; incidentally, the least positive solution is 3467, and all five numbers are prime.

Similarly, the set $\{3,5,7,107\}$ extends to a 5-element special set by adjoining any positive odd integer k satisfying the system of linear congruences

$$\begin{cases} k+1\equiv 0 \pmod 2\\ k+5\equiv 0 \pmod 3\\ k+3\equiv 0 \pmod 7\\ k+7\equiv 0 \pmod 5\\ k+7\equiv 0 \pmod{107}\end{cases};$$

clearly, 3 and 5 both divide $k+107$. In this case, the least positive solution is $10693=17^2\cdot 37$.

Remark. The 5-element special sets below are obtained in the same way:

$$\{3,5,13,127,17267=31\cdot 557\},\quad \{3,5,17,97,14353=31\cdot 463\},$$

$$\{3,7,11,235=5\cdot 47,26309\},\qquad \{3,7,11,437=19\cdot 23,60295=5\cdot 31\cdot 389\}.$$

We now show that the special set consisting of 3, 5, 13, 17 is maximal. Suppose, if possible, that k is a positive odd integer such that 3, 5, 13, 17, k form a special set. It is easily seen that k divides no 2-term sum from $\{3,5,13,17\}$.

We first show that $k+5$ is divisible by 3 if and only if $k+3$ is divisible by 5; since one holds, so does the other.

If $k+5$ is divisible by 3, then $k+13$ is not, so $k+3$ is divisible by 13. It follows that $k+5$ is not divisible by 13, so $k+13$ is divisible by 5, showing that $k+3$ is indeed divisible by 5.

Conversely, if $k+3$ is divisible by 5, then $k+17$ is not, so $k+5$ is divisible by 17. It follows that $k+3$ is not divisible by 17, so $k+17$ is divisible by 3, showing that $k+5$ is indeed divisible by 3.

By the preceding, $k+3$ is divisible by 13, and $k+5$ is divisible by 17. The former implies that $k+17$ is not divisible by 13, and the latter implies that $k+13$ is not divisible by 17. To derive a contradiction, recall that k divides no 2-term sum from $\{3, 5, 13, 17\}$; in particular, it does not divide the sum $13+17$.

Finally, we show that the special set consisting of 11, 15, 51 is maximal. Suppose, if possible, that k is a positive odd integer such that 11, 15, 51, k form a special set, and notice that k divides no 2-term sum from $\{11, 15, 51\}$. Consideration of 15, 51 and k shows that k is divisible by 3. It then follows that of the two numbers 15 and 51, none divides $k+11$, so $k+15$ and $k+51$ are both divisible by 11, which is clearly impossible.

Problem 4. Let n be an integer greater than or equal to 3, and let $\mathcal{P}_n$ be the collection of all planar (simple) n-gons no two distinct sides of which are parallel or lie along some line. For each member P of $\mathcal{P}_n$, let $f_n(P)$ be the least cardinal a cover of P by triangles formed by lines of support of sides of P may have. Determine the largest value $f_n(P)$ may achieve, as P runs through $\mathcal{P}_n$.

Solution. The required maximum is $n-2$. This follows from the fact that the image of f_n consists of the first $n-2$ positive integers.

Induct on n to show that $f_n(P) \le n-2$ for all P in $\mathcal{P}_n$. The base case $n=3$ is clear. If P is convex, then $f_n(P)=1$, since P has three sides whose lines of support form a triangle that covers P — simply extend two non-adjacent sides till they meet to obtain a polygon with fewer sides and proceed inductively.

If P is not convex, consider a vertex v interior to the convex hull of P. Extend one of the sides through v beyond v till it first meets again the boundary, to split P into an n_1-gon P_1 and an n_2-gon P_2, where $n_1+n_2 \le n+2$. Hence $f_n(P) \le f_{n_1}(P_1)+f_{n_2}(P_2) \le n_1-2+n_2-2 \le n-2$.

Given a positive integer $m \le n-2$, we now describe a polygon $P = a_0a_1\ldots a_{n-1}$ in $\mathcal{P}_n$ such that $f_n(P)=m$. Consider a parallelogram $aa_0a_1a_{m+1}$, let a_2, …, a_m be interior points of the triangle $a_0a_1a_{m+1}$ forming, along with a_1 and a_{m+1}, an $(m+1)$-point configuration in strictly convex position, and let a_{m+2}, …, a_{n-1} be interior points of the triangle aa_0a_{m+1} forming, along with a_0 and a_{m+1}, an $(n-m)$-point configuration in strictly convex position; clearly, it is always possible to choose the latter so that P has no parallel sides.

For convenience, a triangle formed by the lines of support of three sides of P will be called a *peritriangle*.

To show that $f_n(P) \ge m$, notice that, for each index i in the range 1 through m, covering the midpoint of the side a_ia_{i+1} of P requires a peritriangle Δ_i lying in the same half-plane as a_0 relative to the line a_ia_{i+1}, one side of which contains the line-segment a_ia_{i+1}. Since, for distinct indices i and j in the range 1 through m, the line a_ia_{i+1} separates a_0 and the midpoint of the line-segment a_ja_{j+1}, the Δ_k are pairwise distinct, so $f_n(P) \ge m$.

To conclude that $f_n(P)=m$, notice that P is covered by the m peritriangles formed by the lines a_0a_1, a_ia_{i+1} and $a_{i+1}a_{i+2}$, where i runs from 1 through m.