IMAR 2013 — Solutions

Problem 1. Given a prime $p \geq 5$, show that there exist at least two distinct primes q and r in the range 2, 3, ..., $p-2$ such that $q^{p-1} \not\equiv 1 \pmod{p^2}$ and $r^{p-1} \not\equiv 1 \pmod{p^2}$.

Solution 1. In what follows, all congruences are to be understood modulo p^2. An integer n coprime to p will be called *proper* if $n^{p-1} \equiv 1$, and *improper* otherwise. Our solution is based on the following two simple facts:

(1) An improper integer greater than 1 has at least one improper prime divisor; and

(2) If k is an integer coprime to p and n is a proper integer, then $kp - n$ is improper.

The first claim follows from the fact that the product of two proper integers is again proper. For the second, notice that p does not divide kn^{p-2} to deduce that

$$(kp-n)^{p-1} \equiv n^{p-1} - (p-1)kpn^{p-2} \equiv 1 + kpn^{p-2} \not\equiv 1,$$

so $kp - n$ is indeed improper (since n is coprime to p, so is $kp - n$).

Since ± 1 are both proper, letting $k \in \{1, 2\}$ and $n = \pm 1$ in **(2)** shows that $p \pm 1$ and $2p \pm 1$ are all improper, so each has an improper prime divisor by **(1)**.

Since $p \geq 5$, the prime factors of $p \pm 1$ are all less than $p - 1$; and since 2 is the highest common factor of $p - 1$ and $p + 1$, the conclusion follows, provided that 2 is proper.

Otherwise, look for an improper odd prime in the required range. To this end, notice that one of the numbers $2p \pm 1$ is divisible by 3, so its prime factors are all less than $p - 1$, for $p \geq 5$; clearly, they are all odd, and the conclusion follows.

Solution 2. If $p = 5$, the primes $q = 2$ and $r = 3$ satisfy the required conditions, so let $p \geq 7$. In the setting of the previous solution, distinguish the following two cases:

If $p-2$ is improper, it has an improper prime divisor q by **(1)**. On the other hand, since 1 is proper, setting $k = n = 1$ in **(2)** shows that $p - 1$ is improper, so it has an improper prime divisor r by **(1)**. Clearly, q and r both lie in the required range, and they are distinct since $p - 2$ and $p - 1$ are coprime.

If $p - 2$ is proper, set $k = 1$ and $n = p - 2$ in **(2)** to deduce that 2 is improper. On the other hand, since $(p-2)^2$ is proper, so is $-4p + 4$. Setting $k = -3$ and $n = -4p + 4$ in **(2)** shows $p-4$ improper, so it has an improper prime divisor s by **(1)**. Finally, since $p-4$ is odd, so is s, and it should now be clear that the primes 2 and s satisfy the required conditions.

Problem 2. For every non-negative integer n, let s_n be the sum of the digits in the decimal expansion of 2^n. Is the sequence $(s_n)_{n\in\mathbb{N}}$ eventually increasing?

Solution. The answer is in the negative. To prove this, begin by noticing that the sequence is periodic modulo 9, of period 6, the first block of values it takes on being 1, 2, 4, 8, 7, 5.

Suppose, if possible, that the sequence is eventually increasing, say from some rank n_0 on. Fix a non-negative integer m such that $6m \geq n_0$ to write

$$\begin{aligned} &s_{6m+1} \geq s_{6m} + 1, &&s_{6m+2} \geq s_{6m+1} + 2, &&s_{6m+3} \geq s_{6m+2} + 4, \\ &s_{6m+4} \geq s_{6m+3} + 8, &&s_{6m+5} \geq s_{6m+4} + 7, &&s_{6m+6} \geq s_{6m+5} + 5, \end{aligned}$$

and deduce thereby that $s_{6m+6} \geq s_{6m} + 27$, so

$$s_{6m+6n} \geq s_{6m} + 27n, \quad n \in \mathbb{N}. \qquad (*)$$

On the other hand, the number of non-vanishing digits in the decimal expansion of 2^{6m+6n} does not exceed $\lceil (6m+6n)\log_{10} 2 \rceil < 2m+2n$, so $s_{6m+6n} \le 18m+18n$, contradicting $(*)$ for n large enough. The conclusion follows.

Problem 3. The closure (interior and boundary) of a convex quadrangle is covered by four closed discs centred at each vertex of the quadrangle each. Show that three of these discs cover the closure of the triangle determined by their centres.

Solution. Suppose, if possible, that the conclusion does not hold. Then no three discs meet, and each disc contains points of the closure of the triangle determined by the centres of the other three discs, not covered by the latter.

Amongst the four discs, choose one, say Δ_0, containing the point O where the diagonals of the quadrangle cross one another. Let A_0 be the centre of Δ_0, label the other three centres in circular order, A_1, A_2, A_3, so that the opposite angles A_0OA_1 and A_2OA_3 be not obtuse, and let Δ_i denote the disc centred at A_i.

Before proceeding, we take time out to state a simple, but quite useful lemma whose proof is postponed for the sake of clarity.

Lemma. *Let $ABCD$ be a convex quadrangle, let Δ be a disc centred at A, and let E be the point where the ray AC emanating from A crosses the boundary of Δ. If the orthogonal projection of B on the line AC falls on the closed ray EA emanating from E, then* $\operatorname{dist}(B, [ACD] \setminus \Delta) \ge BE$, *where $[ACD]$ is the closure of the triangle ACD.*

We now apply the lemma to show that O is also covered by Δ_1. To this end, let B_0 be the point where the ray A_0O emanating from A_0 crosses the the boundary of Δ_0. Since Δ_0 contains O, the latter lies on the closed segment A_0B_0, and since the angle A_0OA_1 is not obtuse, it follows that $A_1O \le A_1B_0$. On the other hand, Δ_1 contains points of the closure $[A_0A_2A_3]$ of the triangle $A_0A_2A_3$ not covered by Δ_0, so the radius of Δ_1 is greater than or equal to $\operatorname{dist}(A_1, [A_0A_2A_3] \setminus \Delta_0)$, which in turn is greater than or equal to A_1B_0 by the lemma. Consequently, O is indeed covered by Δ_1.

Recall that no three discs meet to deduce that neither Δ_2, nor Δ_3 contains O. It follows, for $i = 2, 3$, that the open segment A_iO crosses the boundary of Δ_i at some point B_i. The open segments A_2B_3 and A_3B_2 cross each other, so $r_2 + r_3 = A_2B_2 + A_3B_3 < A_2B_3 + A_3B_2$, where r_i is the radius of the disc Δ_i, $i = 2, 3$.

We are presently going to show that $r_2 \ge A_2B_3$ and $r_3 \ge A_3B_2$ and reach thereby the contradiction we were heading for. Only the first inequality will be dealt with; the argument applies mutatis mutandis to the other. Since the angle A_2OA_3 is not obtuse, the orthogonal projection A_2' of A_2 on the line A_1A_2 falls on the closed ray OA_3 emanating from O. If A_2' fell on the closed segment B_3O, then the image of the line A_2A_2' under a slight rotation about the midpoint of the segment A_2A_2' would separate the disc Δ_3 and the closure $[A_0A_1A_2]$ of the triangle $A_0A_1A_2$, in contradiction with the second remark in the opening paragraph. Hence A_2' lies on the open ray B_3A_3 emanating from B_3, so $\operatorname{dist}(A_2, [A_0A_1A_3] \setminus \Delta_3) \ge A_2B_3$ by the lemma. Finally, recall that Δ_2 covers points in $[A_0A_1A_3] \setminus \Delta_3$, to conclude that $r_2 \ge A_2B_3$.

Proof of the lemma. Since the quadrangle $ABCD$ is convex, the whole configuration of points lies on one side of the line AB, say $\mathcal{H}$. Let F be the point where the ray AD emanating from A crosses the boundary of Δ, let α denote the arc EF of the boundary of Δ situated in $\mathcal{H}$, and let r be the ray emanating from E along the line AC, not containing A.

Notice that, if X is a point in $[ACD] \setminus \Delta$, then the closed segment BX meets either α or r (this fails to hold if the quadrangle $ABCD$ is not convex at D), so it is sufficient to consider only points X in $\alpha \cup r$.

Now, as a point X traces α from E to F, the length of the segment BX varies increasingly by the cosine law in the triangle ABX (this fails to hold if the quadrangle $ABCD$ is not convex at A or at B), so $BX \geq BE$.

Finally, since the orthogonal projection of B on the line AC is not interior to r, the length of the segment BX varies again increasingly, as X runs along r away from E, so $BX \geq BE$ again. This ends the proof of the lemma and completes the solution.

Remarks. Since the distance to the empty set may take on any value, the conclusion of the lemma still holds if Δ covers $[ACD]$.

Under the conditions in the lemma, it may very well happen that $\operatorname{dist}(B, [ACD]\backslash\Delta) > BE$, in which case C is certainly interior to Δ. Such configurations are easily produced.

Finally, it is not hard to see that the conclusion of the lemma may fail to hold if the quadrangle $ABCD$ is not convex at one of the vertices A, B, D or the projection of B on the line AC does not fall on the closed ray EA emanating from E.

Problem 4. Given a triangle ABC, a circle centred at some point O meets the segments BC, CA, AB in the pairs of points X and X', Y and Y', Z and Z', respectively, labelled in circular order: X, X', Y, Y', Z, Z'. Let M be the Miquel point of the triangle XYZ (i.e., the point of concurrence of the circles AYZ, BZX, CXY), and let M' be that of the triangle $X'Y'Z'$. Prove that the segments OM and OM' have equal lengths.

Solution. We begin by reviewing some basic facts on conics. For an ellipse Σ with centre N, foci M and M', semiaxes a and b, it is known that the orthogonal projections P and P' of M and M' on any line t tangent to Σ lie on the major auxiliary circle of Σ, so that $NP = a = NP'$. Application to triangle MNP (respectively, $M'NP'$) of a rotation θ (respectively, $-\theta$) about M (respectively, M') and a homothety of ratio $\sec\theta$ with centre M (respectively, M') yields triangle MOX (resprectively, $M'OX'$), where $MO = M'O$, $NO = \frac{1}{2} \cdot MM' \cdot \tan\theta$, $OX = a\sec\theta = OX'$, and X, X' both lie on t. If t varies and θ is constant, the locus of X and X' is then a circle Γ centred at O. By cartesian geometry it is readily checked that Γ and Σ are bitangent, and the line ℓ supporting their common chord is also the radical axis of the circles Γ and OMM', with this real geometrical significance even if the bitangency is not real. Since ℓ and the circle OMM' are mutually inverse in Γ, the inverse points of M and M' in Γ both lie on ℓ. Finally, the distance d between the parallel lines ℓ and MM' is given by $d \cdot MM' = 2b^2 \tan\theta$. Similar considerations hold for a hyperbola Σ.

Consider now an isopair M, M' (two isogonally conjugate in the triangle ABC, the foci of a conic Σ touching its sides), and take points X, Y, Z (respectively, X', Y', Z') on lines BC, CA, AB, respectively, so that the lines MX, MY, MZ (respectively, $M'X'$, $M'Y'$, $M'Z'$) make the same directed angle θ (respectively, $-\theta$) with the perpendiculars to BC, CA, AB, respectively; then the isopedal triangles XYZ, $X'Y'Z'$ of angles θ, $-\theta$ for the isopair M, M' have their Miquel points at M, M' and are inscribed in a common isopedal circle Γ bitangent to Σ, centred at a point O on the perpendicular bisector of the segment MM'.

Conversely, for any pair of triangles inscribed in a triangle ABC and in a circle Γ (as in the statement of the problem), the Miquel points M, M' are an isopair and Γ is an isopedal circle of M, M'. (If M, M' are the Brocard points, Σ is the Brocard ellipse and Γ is a Tucker circle.)