IMAR 2012 — Solutions

Problem 1. Let K be a convex planar set, symmetric about a point O, and let X, Y, Z be three points in K. Show that K contains the head of one of the vectors $\overrightarrow{OX} \pm \overrightarrow{OY}$, $\overrightarrow{OX} \pm \overrightarrow{OZ}$, $\overrightarrow{OY} \pm \overrightarrow{OZ}$.

Solution 1. If two of the vectors $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ are linearly dependent, the conclusion is clear. So suppose $\alpha \cdot \overrightarrow{OX} + \beta \cdot \overrightarrow{OY} + \gamma \cdot \overrightarrow{OZ} = \mathbf{0}$, where $\alpha\beta\gamma \neq 0$, and let $|\gamma| = \min(|\alpha|, |\beta|, |\gamma|)$ to write $\overrightarrow{OZ} = \alpha' \cdot \overrightarrow{OX} + \beta' \cdot \overrightarrow{OY}$, where $|\alpha'| \geq 1$ and $|\beta'| \geq 1$. Since K is symmetric about O, we may (and will) assume that $\alpha' \geq 1$ and $\beta' \geq 1$. Write

$$\overrightarrow{OX} + \overrightarrow{OY} = \frac{\beta' - 1}{\alpha' + \beta' - 1} \cdot \overrightarrow{OX} + \frac{\alpha' - 1}{\alpha' + \beta' - 1} \cdot \overrightarrow{OY} + \frac{1}{\alpha' + \beta' - 1} \cdot \overrightarrow{OZ}\,,$$

to deduce that the head of the vector $\overrightarrow{OX} + \overrightarrow{OY}$ lies in the triangle XYZ which is the convex hull of the points X, Y, Z. Since K is convex, the conclusion follows.

Solution 2. This is merely a translation of the previous algebraic solution into geometric language. Discard again the trivial cases and consider a generic configuration. The proof is based on the following two facts:

(1) Any one of the three given points and some pair of consecutive vertices of the parallelogram determined by the other two and their reflections in O, form a triangle Δ of which O is an interior point; and

(2) One of the vertices of Δ and the reflections in O of the other two form a triangle Δ' which contains the head of one of the six vectors in the statement or its reflection in O.

Since K is convex and symmetric about O, the conclusion follows.

We now proceed to prove **(1)** and **(2)**. Let X', Y', Z' be the reflections in O of X, Y, Z, respectively.

To prove **(1)**, consider one of the three given points, say Y. The line XOX' separates Y from one of the points Z and Z', say Z', so the point O is interior to the convex hull of X, X', Y, Z'. Similarly, the line ZOZ' separates Y from one of the points X and X', say X, so the point O is interior to the convex hull of X, Y, Z, Z'. Consequently, O is interior to the intersection of the two convex hulls which is the triangle $\Delta = XYZ'$. This establishes **(1)**.

To prove **(2)**, amongst the three triangles formed by O and two vertices of Δ, consider one of minimal area, say OYZ' in our case. Further, let the parallel through X to the line ZOZ' (respectively, YOY') meet the line YOY' (respectively, ZOZ') at U (respectively, V). Since O is interior to Δ, each vertex of Δ is interior to the angle formed by the rays from O through the reflections in O of the other two vertices. In particular, X is interior to the angle $Y'OZ$, so U lies on the ray OY', and V lies on the ray OZ. We now proceed to show that the minimality condition on areas forces Y' to lie on the segment OU, and Z on the segment OV. Since $OZ = OZ'$ and

$$\operatorname{area} OXZ' \geq \operatorname{area} OYZ' = \operatorname{area} OY'Z,$$

it follows that X is farther than Y' from the line ZOZ', so Y' lies on the segment OU. Similarly, Z lies on the segment OV, so the head W of the vector $\overrightarrow{OY'} + \overrightarrow{OZ} = -\overrightarrow{OY} + \overrightarrow{OZ}$ lies in the parallelogram $OUXV$. Finally, it is easily seen that W actually lies in the triangle $\Delta' = XY'Z$. This establishes **(2)** and completes the proof.

Problem 2. Given an integer $n \geq 2$, evaluate

$$\sum \frac{1}{pq},$$

where the summation is over all coprime integers p and q such that $1 \leq p < q \leq n$ and $p + q > n$.

Solution 1. The required sum is $1/2$, for it is the sum of the distances between successive terms in the first half of the Farey series of order n — for instance, see G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford at the Clarendon Press, 1956, Chap. III.

Solution 2. Let S_n denote the required sum and compare S_n and S_{n-1}, $n \geq 3$. The summands that occur in S_n but not in S_{n-1} are precisely those of the form $1/(pn)$, where $1 \leq p < n$ and p and n are coprime. On the other hand, the summands that occur in S_{n-1} but not in S_n are precisely those of the form $1/(p(n-p))$, where $1 \leq p < n - p$ and p and $n - p$ are coprime; that is, $1 \leq p < n/2$ and p and n are coprime. Consequently,

$$S_n - S_{n-1} = \sum_{\substack{1 \leq p < n \\ (p,n)=1}} \frac{1}{pn} - \sum_{\substack{1 \leq p < n/2 \\ (p,n)=1}} \frac{1}{p(n-p)} = \sum_{\substack{1 \leq p < n/2 \\ (p,n)=1}} \left(\frac{1}{pn} + \frac{1}{(n-p)n} - \frac{1}{p(n-p)} \right) = 0.$$

Since $S_2 = 1/2$, the conclusion follows.

Problem 3. Given a triangle ABC, let D be a point different from A on the external bisectrix ℓ of the angle BAC, and let E be an interior point of the segment AD. Reflect ℓ in the internal bisectrices of the angles BDC and BEC to obtain two lines that meet at some point F. Show that the angles ABD and EBF are congruent.

Solution. Reflect B in the lines DF and EF to obtain the points B' and B'', respectively. The lines $B'C$ and DF meet at M, and the lines $B''C$ and EF meet at N. We shall prove that the lines BD, BE and BF bisect the angles ABM, ABN and MBN, respectively. The conclusion then follows at once:

$$\angle ABD = \frac{1}{2} \angle ABM = \frac{1}{2} (\angle ABN + \angle MBN) = \angle EBN + \angle FBN = \angle EBF.$$

To show that the line BD bisects the angle ABM, let C' be the reflection of C in the line ℓ and notice that the triangles DBC' and $DB'C$ are congruent, for $DB = DB'$ and $DC' = DC$, both by symmetry, and $\angle BDC' = \angle B'DC$, by symmetry and isogonality. Consequently, $\angle DBC' = \angle DB'C$ and $BC' = B'C$. Since ℓ is the external bisectrix of the angle BAC, the points A, B and C' are collinear, so $\angle DBA = \angle DBC'$. On the other hand, $\angle DB'C = \angle DB'M = \angle DBM$ (the latter equality holds by symmetry), so BD is indeed the internal bisectrix of the angle ABM.

A similar argument shows that BE is the internal bisectrix of the angle ABN and $BC' = B''C$.

Finally, to show that the line BF bisects the angle MBN, let C'' be the reflection of C in the line EF and notice that the triangles FBC'' and $FB'C$ are congruent, for $FB = FB'$ and $FC'' = FC$, both by symmetry, and $BC'' = B''C = BC' = B'C$, where the first equality holds by symmetry, and the last two by the preceding paragraphs. Consequently, $\angle FBC'' = \angle FB'C$, so $\angle FBN = \angle FBC'' = \angle FB'C = \angle FB'M = \angle FBM$ (the latter eqality holds by symmetry); that is, BF is the internal bisectrix of the angle MBN.

Remarks. If D lies between A and E, the angles ABD and EBF are still congruent. But if D and E lie on opposite sides of A, the two angles turn out to be supplementary to one another. In all cases, the argument is essentially the same. Consequently, up to orientation, $\angle EBF \equiv \angle ABD$ modulo π, as E traces ℓ. Mutatis mutandis, the same holds if B is replaced by C throughout or if ℓ is the internal bisectrix of the angle BAC.

The problem reveals a remarkable configuration of pairs of isogonal lines. Clearly, the lines DF and EF are the isogonals of the bisectrix ℓ relative to the pairs of lines (DB, DC) and (EB, EC), respectively. The conclusion states that (BA, BD) and (BE, BF) are pairs of isogonal lines; and so are the pairs (CA, CD) and (CE, CF).

Notice further that ℓ may be looked upon as self-isogonal relative to the pair of lines (AB, AC). A similar configuration may be obtained by considering D and E each on a line in a pair of isogonals relative to (AB, AC).

Those acquainted with conics may have noticed here the following remarkable focal property which goes back to Poncelet and Steiner: The focal angular span of the segment intercepted by two fixed tangents to a conic on a variable tangent to that conic is constant modulo π and orientation.

Other remarkable properties involved here are:

(1) A tangent to a conic bisects internally or externally the angle formed by the focal rays at the point of contact.

(2) Two tangents to a conic are isogonal relative to the focal rays of the point where they meet (Poncelet); moreover, each focal ray bisects the corresponding focal angle determined by the points of contact of the conic and the two tangents.

In our case, the points A, M and N lie on an ellipse of foci B and C, externally tritangent (escribed) to the triangle DEF: the lines DE, DF and EF are tangent to the ellipse at A, M and N, respectively. If D and E lie on opposite sides of A, then the ellipse is internally tritangent (inscribed) to the triangle DEF at A, M and N. Finally, had we considered the case where ℓ is the internal bisectrix of the angle BAC, the conic in question would have been a hyperbola of foci B and C, tritangent to the triangle DEF at A, M and N.

Problem 4. Design a planar finite non-empty set S satisfying the following two conditions:

(a) every line meets S in at most four points; and

(b) every 2-colouring of S — that is, each point of S is coloured one of two colours — yields (at least) three monochromatic collinear points.

Solution. Call a planar set a *partial n-point set* if it meets every line in at most n points. We are required to design a finite partial 4-point set that cannot be split into two partial 2-sets.

We shall prove that the set

$$S = \{\pm 1, \pm 3\} \times \{\pm 1, \pm 3\} \cup \{\pm 2, \pm 4\} \times \{0\} \cup \{0\} \times \{\pm 4\}$$

is one such.

Inspection of S shows that it is indeed a partial 4-point set. Suppose, if possible, that S can be partitioned into two partial 2-sets: one red, and the other blue.

There are exactly three ways to split the row $\{\pm 1, \pm 3\} \times \{-1\}$ after accounting for symmetries:

(1) $(\pm 3, -1)$ are red, and $(\pm 1, -1)$ are blue;

(2) $(-3, -1)$ and $(1, -1)$ are red, and $(-1, -1)$ and $(3, -1)$ are blue; and

(3) $(-3, -1)$ and $(-1, -1)$ are red, and $(1, -1)$ and $(3, -1)$ are blue.

In cases **(2)** and **(3)** we may (and will) assume that the point $(0,-4)$ is coloured red.

Notice that if two points in S share one colour, then the other points of S on that line must share the other.

In case **(1)**, consider the two possibilities for the point $(-3,-3)$.

If $(-3,-3)$ is red, then $(-3,1)$ and $(-3,3)$ must be blue, for they both lie on the vertical through the reds $(-3,-3)$ and $(-3,-1)$. Therefore, $(1,3)$ and $(3,-3)$ must be red, for the former is on the line through the blues $(-3,1)$ and $(-1,-1)$, and the latter is on the line through the blues $(-3,3)$ and $(1,-1)$. Consequently, $\{-3,1,3\} \times \{-3\}$ is red — a contradiction.

If $(-3,-3)$ is blue, then $(1,1)$ and $(3,3)$ must be red, for they both lie on the line through the blues $(-3,3)$ and $(-1,-1)$. Therefore, $(2,0)$ and $(3,1)$ must be blue, for the former is on the line through the reds $(1,1)$ and $(3,-1)$, and the latter is on the vertical through the reds $(3,-1)$ and $(3,3)$. Consequently, $(1,-1)$, $(2,0)$ and $(3,1)$ are three collinear blues — a contradiction. This establishes case **(1)**.

In case **(2)**, recall that $(0,-4)$ is red. The points $(-4,0)$ and $(-1,-3)$ lie on the line through the reds $(-3,-1)$ and $(0,-4)$, so they must be blue. The point $(-1,1)$ lies on the vertical through the blues $(-1,-3)$ and $(-1,-1)$, so it must be red. The line through the reds $(-3,-1)$ and $(-1,1)$ contains $(-2,0)$, so the latter must be blue. The point $(1,-3)$ lies on the line through the blues $(-2,0)$ and $(-1,1)$, so it must be red. Finally, the line through the reds $(0,-4)$ and $(1,-3)$ contains $(4,0)$ which must therefore be blue. Consequently, $\{-4,-2,4\} \times \{0\}$ is blue — a contradiction which establishes case **(2)**.

In case **(3)**, recall that $(0,-4)$ is red and consider the two possible choices for the point $(1,-3)$.

If $(1,-3)$ is red, then $(-2,0)$ must be blue, for it is on the line through the reds $(1,-3)$ and $(-1,-1)$. The points $(\pm 4,0)$ both lie on lines through pairs of reds, so they must be blue: $(-4,0)$ is on the line through the reds $(0,-4)$ and $(-3,-1)$, and $(4,0)$ is on the line through the reds $(0,-4)$ and $(1,-3)$. Consequently, $\{-4,-2,4\} \times \{0\}$ is blue — a contradiction.

If $(1,-3)$ is blue, then $(1,1)$ must be red, for it lies on the vertical through the blues $(1,-3)$ and $(1,-1)$. The line through the reds $(0,-4)$ and $(-3,-1)$ contains $(-1,-3)$ which must therefore be blue. Finally, $(-3,-3)$ lies on the line through the blues $(1,-3)$ and $(-1,-3)$, so it must be red. Consequently, $(-3,-3)$, $(-1,-1)$ and $(1,1)$ are three collinear reds — a contradiction. This establishes case **(3)** and concludes the proof.