THE Ninth IMAR MATHEMATICAL COMPETITION

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Problem 1. Let $A_0A_1A_2$ be a triangle and let P be a point in the plane, not situated on the circle $A_0A_1A_2$. The line PA_k meets again the circle $A_0A_1A_2$ at point B_k, $k = 0, 1, 2$. A line ℓ through the point P meets the line $A_{k+1}A_{k+2}$ at point C_k, $k = 0, 1, 2$. Show that the lines B_kC_k, $k = 0, 1, 2$, are concurrent and determine the locus of their concurrency point as the line ℓ turns about the point P.

Solution. The lines B_kC_k and $B_{k+1}C_{k+1}$ meet projectively at a point Q_k. Read PC_kC_{k+1}, the line ℓ is the Pascal line of the hexagram

$$A_kB_kQ_kB_{k+1}A_{k+1}A_{k+2},$$

so the vertices of the latter lie on a conic. Notice that this conic and the circle $A_0A_1A_2$ share five points, to deduce that that the two coincide and thereby conclude that the three lines B_kC_k are concurrent at a point $Q = Q_k$ situated on the circle $A_0A_1A_2$.

Conversely, given a point Q on the circle $A_0A_1A_2$, the three hexagrams $A_kB_kQB_{k+1}A_{k+1}A_{k+2}$ share the same Pascal line $\ell = PC_kC_{k+1}$, so the required locus is the circle $A_0A_1A_2$.

Problem 2. The area of a convex polygon in the plane is equally shared by the four standard quadrants, and all non-zero lattice points lie outside the polygon. Show that the area of the polygon is less than 4.

Solution. Throughout the proof subscripts are reduced modulo 4. Let K be a convex polygon in the plane satisfying the conditions in the statement and let K_i be the intersection of K and the i-th standard quadrant. It is sufficient to show that one of the K_i has an area less than 1.

Since K is convex and the K_i have equal areas, the origin, $O = 0 \times 0$, is interior to K; it is a vertex of each K_i. Consider the lattice points $A_0 = 1 \times 0$, $A_1 = 0 \times 1$, $A_2 = -1 \times 0$ and $A_3 = 0 \times (-1)$. Since K is closed and convex and contains no non-zero lattice points, there exists a line ℓ_i through A_i such that K lies in one of the open half-planes determined by ℓ_i, a positive distance away from ℓ_i.

If one of the lines ℓ_i meets the closed segment $A_{i+1}A_{i+3}$, then the triangle determined by the lines ℓ_i, A_iA_{i+2}, $A_{i+1}A_{i+3}$ contains either K_i or K_{i+3}, its area does not exceed $1/2$, so either area $K_i < 1/2$ or area $K_{i+3} < 1/2$ (the inequality is strict, for K is a positive distance away from ℓ_i).

If no ℓ_i meets the corresponding closed segment $A_{i+1}A_{i+3}$, then the lines in each pair (ℓ_i, ℓ_{i+1}) meet at some point B_i situated in the i-th quadrant. Notice that K_i is contained in the convex quadrangle $OA_iB_iA_{i+1}$, a positive distance away from $\ell_i \cup \ell_{i+1}$, to write

$$\text{area}\, K_i < \text{area}\, OA_iB_iA_{i+1} = \text{area}\, OA_iA_{i+1} + \text{area}\, A_iB_iA_{i+1} = \frac{1}{2} + \text{area}\, A_iB_iA_{i+1}.$$

It is therefore sufficient to show that the area of one of the four triangles $A_iB_iA_{i+1}$ does not exceed $1/2$. To this end, notice that the convex quadrangle $B_0B_1B_2B_3$ has at least one non-acute internal angle, say at B_i, to deduce that B_i does not lie outside the closed disc of diametre A_iA_{i+1}, so the area of the triangle $A_iB_iA_{i+1}$ is indeed at most $1/2$. This completes the proof.

Remark. It should now be clear that the conclusion holds for any closed convex planar set satisfying the conditions in the statement.

Problem 3. Given an integer number $n \geq 2$, show that there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) + f(2x) + \cdots + f(nx) = 0$, for all $x \in \mathbb{R}$, and $f(x) = 0$ if and only if $x = 0$.

Solution. If $1 \leq x < n$, set $f(x) = 1$. Let $a = n/(n-1)$. If $na^k \leq x < na^{k+1}$, $k = 0, 1, 2, \cdots$, define $f(x)$ by

$$f(x) = -\sum_{r=1}^{n-1} f(rx/n). \tag{1}$$

If $2^{-k-1} \leq x < 2^{-k}$, $k = 0, 1, 2, \cdots$, define $f(x)$ by

$$f(x) = -\sum_{r=2}^{n} f(rx). \tag{2}$$

Finally, set $f(0) = 0$ and $f(x) = -f(-x)$, if $x < 0$. Then f satisfies the required relation and it is clear from (1) and (2) that $f(x) \equiv 1 \pmod{n}$ if $x > 0$. Thus $f(x) = 0$ if only if $x = 0$.

Problem 4. Given an integer number $n \geq 3$, show that the number of lists of jointly coprime positive integer numbers that sum to n is divisible by 3. (For instance, if $n = 4$, there are six such lists: $(3, 1)$, $(1, 3)$, $(2, 1, 1)$, $(1, 2, 1)$, $(1, 1, 2)$ and $(1, 1, 1, 1)$.)

Solution. Given an integer number $n \geq 1$, a list of k positive integer numbers that sum to n will be called a *decomposition* of n into k parts. Let $f(n)$ be the number of decompositions of n into jointly coprime parts, with the convention that the 1-element list (n) is a decomposition into 'jointly coprime parts' if and only if $n = 1$.

The number of unrestricted decompositions of n is 2^{n-1}, for in a string of n dots, we may insert dividing bars at any subset of the $n-1$ places between two 1's. Consider the decompositions into parts with a common divisor, grouped by the greatest common divisor d, to get

$$\sum_{d|n} f(n/d) = 2^{n-1}.$$

Thus, $f(1) = f(2) = 1$ and $f(3) = 3$. Now proceed by induction on n. If $n > 3$, then $f(n) \equiv 2^{n-1} - f(1) - f(2)$ (mod 3) when n is even and $f(n) \equiv 2^{n-1} - f(1)$ (mod 3) when n is odd. Since $2^{n-1} \equiv 2$ (mod 3) when n is even and $2^{n-1} \equiv 1$ (mod 3) when n is odd, the conclusion follows.

Remark. By the Möbius inversion formula, $f(n) = \sum_{d|n} \mu(n/d)2^{d-1}$, where μ is the Möbius function. As a corollary, $f(n)$ is odd if and only if n is squarefree.