IMAR Competition 2008 – Solutions

Problem 1. An array $n \times n$ is given, consisting of n^2 unit squares. A *pawn* is placed arbitrarily on a unit square. The pawn can move from a square of the k-th column to any square of the k-th row. Show that there exists a sequence of n^2 moves of the pawn so that all the unit squares of the array are visited once, the pawn returning to its original position.

D. Şerbănescu

Solution. Label the unit squares (i, j). A move will be denoted $(i, k) \to (k, t)$. We proceed by induction on n. As the base case is trivial, assume that a circuit exists for n and let (i, a_i) be the next square visited after (i, i) in this circuit. Now replace the move $(n, n) \to (n, a_n)$ with the moves

$$(n, n) \to (n, n+1) \to (n+1, n+1) \to (n+1, n) \to (n, a_n),$$

and replace $(i, i) \to (i, a_i), i = 1, 2, \ldots n-1$ by the sequence

$$(i, i) \to (i, n+1) \to (n+1, i) \to (i, a_i).$$

With the rest of the moves unaltered, notice that all the $3 + 2(n-1) = 2n+1 = (n+1)^2 - n^2$ new squares of the $n+1 \times n+1$ array are visited, so we are done.

P.S. The inductive step begins with the pawn placed on a square of the initial $n \times n$ array, and it is obvious that this statement can be achieved. ■

Problem 2. A point P of integer coordinates in the Cartesian plane is said *visible* if the segment OP does not contain any other points with integer coordinates (except its ends). Prove that for any $n \in \mathbb{N}^*$ there exists a visible point P_n, at distance larger than n from any other visible point.

D. Schwarz[1]

Solution. It is clear that $P(x, y)$, with $x, y \in \mathbb{N}^*$, is visible if and only if $\gcd(x, y) = 1$. The idea is to place $P_n(a, b)$ at the center of a $(2n+1) \times (2n+1)$ square having all lattice points in its interior or on its sides not visible.

[1] A strengthening of an 1977 AMM problem, also submitted in a 2002 Romanian IMO selection test.

To this purpose, consider a set of $(2n+1)^2-1$ distinct prime numbers p_{ij} with $-n \le i,j \le n$ (but i,j not simultaneously null), all larger than n. The systems of congruences $a \equiv -i \pmod{p_{ij}}$ and $b \equiv -j \pmod{p_{ij}}$, for all i,j, have positive integer solutions $a,b > n$, according to the Chinese Remainder Theorem.

Therefore all points $(a+i, b+j)$ (except maybe $P_n(a,b)$) are not visible, since $p_{ij} \mid \gcd(a+i, b+j)$. Hence the distance from $P_n(a,b)$ to any other visible point is larger than n.

The original part comes now. If $\gcd(a,b) = 1$, then P_n is visible, and we are done. If not, denote by $\pi = \prod p_{ij}$, $\delta =$ the product of all the primes dividing both a and b, and $\rho =$ the product of all the other primes that divide a. Then $\gcd(a,b,\pi) = 1$, since p_{ij} cannot divide both i and j, being larger in absolute value, and i,j not being simultaneously null. Take $b' = b+(\delta+1)\rho\pi$. Clearly we can substitute b' for b, since they differ by a multiple of π. Now, for a prime $d \mid \delta$, we have $d \mid a$ and $d \nmid b'$, since $d \mid b$ but $d \nmid (\delta+1)\rho\pi$. For a prime $r \mid \rho$, we have $r \mid a$ and $r \nmid b'$, since $r \mid (\delta+1)\rho\pi$ but $r \nmid b$.

Since any prime dividing a is necessarily of one of the two types above, it follows that $\gcd(a,b') = 1$, so this new point $P_n(a,b')$ is visible. ■

Problem 3. Two circles γ_1 and γ_2 meet at points X and Y. Consider the parallel through Y to the nearest common tangent of the two circles. This parallel meets again γ_1 and γ_2 at A, and B respectively. Let O be the center of the circle tangent to γ_1, γ_2 and the circle AXB, situated outside γ_1 and γ_2 and inside the circle AXB. Prove that XO is the bisector line of the angle $\angle AXB$.

R. Gologan

Solution. Perform an inversion of pole X and constant equal to the power of point X with respect to the circle of center O considered in the problem. It is easy to check, by considering regions determined by circles and lines involved, that the transformed configuration is the following: circle $\mathcal{C}_1(O_1)$ (image of line AB by inversion) is tangent and interior to circle $\mathcal{C}_2(O_2)$ (image of the common tangent to γ_1 and γ_2) at point X. In what follows, letters $A, B, Y, \ldots$ will denote the images of the corresponding points through the inversion. From Y, belonging to $\mathcal{C}_2$, draw the two tangents to $\mathcal{C}_1$ which meet the circle $\mathcal{C}_2$ at A and B respectively, the tangent points being T_1 and T_2 respectively. Let I be the incenter of triangle ABY. The conclusion of the problem translates to the fact that XI is the bisector line of angle $\angle AXB$.

For a proof of this folklore result, consider the following two classic lemmas.

Lemma 1. XT_1 and XT_2 are the bisector lines of angles $\angle AXY$ and $\angle BXY$ respectively.

Outline of the proof. It is an easy consequence of the fact that X, O_1, O_2 are collinear, $O_1T_1 \perp YA$ and the line bisector of segment YA passes through O_2 and meets the circle $\mathcal{C}_2(O_2)$ at the midpoint of the arc AY.

Lemma 2. T_1, T_2, I are collinear and I is the midpoint of T_1T_2. Moreover, AB_1 and BA_1, being bisector lines, meet at I.

Proof. Let A_1, B_1 be the midpoints of arcs AY and BY respectively. By the preceeding lemma, lines XA_1 and XB_1 meet AY and BY at T_1 and T_2 respectively.

The conclusion then follows by considering Pascal's theorem with the following pairing of diagonals (AY, XA_1), (BY, XB_1) and (BA_1, AB_1). So points T_1, T_2, I are collinear. The fact that I is midpoint follows easily from the fact that YT_1 and YT_2 are tangent lines.

$\blacksquare$

Problem 4. Show that for any function $f : (0, +\infty) \to (0, +\infty)$ there exist real numbers $x > 0$ and $y > 0$ such that

$$f(x + y) < yf(f(x)).$$

D. Schwarz[2]

Furthermore, if we now take $x \geq \frac{ab+2}{a-1} > b$, it results $f(x) = f(b + (x - b)) \geq (x - b)f(f(b)) \geq (x - b)a \geq x + 2$, hence $f(x) > x + 1$.

Solution. Assume $f(x + y) \geq yf(f(x))$ for all $x, y > 0$. Take $a > 1$ and $t = f(f(a)) > 0$. Then for $b \geq a(1 + \frac{1}{t} + \frac{1}{t^2}) > a$ we have

$$f(b) = f(a + (b - a)) \geq (b - a)f(f(a)) = (b - a)t \geq a\left(1 + \frac{1}{t}\right) > a.$$

Then we have the relation $f(f(b)) = f(a+(f(b)-a)) \geq (f(b)-a)t \geq a$. But now $f(f(x)) = f(x + (f(x) - x)) \geq (f(x) - x)f(f(x))$ implies $1 \geq f(x) - x$, i.e. $f(x) \leq x + 1$, contradiction. $\blacksquare$

Alternative Solution. Suppose, if possible, that $f(x + y) \geq yf(f(x))$, whatever $x > 0$ and $y > 0$. Fix a positive real number a, write $\alpha = f(f(a))$ to get

$$f(t) = f(a + (t - a)) \geq \alpha(t - a), \quad \text{for } t > a,$$

[2]Adapted after Italian Olympiad. The weaker relation $f(x+y) < f(x)+yf(f(x))$ was asked there, which, when negated, immediately implies that f is strictly increasing, thus facilitating the solution.

and deduce that $f(t) > a$, for all large enough t. For such t, write

$$f(f(t)) = f(a + (f(t) - a)) \geq \alpha(f(t) - a) \geq \alpha(\alpha(t - a) - a),$$

to infer that $\beta = f(f(b)) > 1$, for some large enough b. Finally, for $t > b + (b - 1)/(\beta - 1)$,

$$f(t) = f(b + (t - b)) \geq \beta(t - b) > t + 1, \qquad (*)$$

so

$$f(f(t)) = f(t + (f(t) - t)) \geq (f(t) - t)f(f(t));$$

that is, $f(t) \leq t + 1$, which contradicts $(*)$. ■

Remark. The idea of the solution comes from graphic considerations; the quantity $f(x + y)/y$ is the slope of a line passing through the point $X(x, 0)$, needing to be not less than the quantity $f(f(x))$, seen as the slope of a reference line. Our first goal was to find points X where this reference slope is larger than 1, in order to later reach a contradiction.

The problem could be extended by asking to satisfy the relation

$$f(x + y) < yf^n(x),$$

where f^n is the n^{th} iterate of f, for $n \geq 2$. This works because we can take $t = f^n(a)$ and $b > a(1 + \frac{1}{t} + \cdots + \frac{1}{t^n})$, and in a similar, step-by-step way, reach $f^n(b) \geq a$ and $f(x) > x + 1$. Now we obtain $f^2(x) > f(x) + 1 > x + 1$ (since $f(x) > x \geq \frac{ab+2}{a-1}$, so we can repeat the same procedure), etc., up to $f^{n-1}(x) > f^{n-2}(x) + 1 > \cdots > x + 1$, with analogous road to contradiction.

Notice however that the result is not true for $n < 2$. For $n = 0$ the relation is $f(x + y) < yx$, which is always false for $f(z) = z^2$, since we have $(x + y)^2 > yx$, or $x^2 + xy + y^2 > 0$, patently true. For $n = 1$ the relation is $f(x + y) < yf(x)$, which is always false for $f(z) = e^z$, since we have $e^{x+y} > ye^x$, or $e^y > y$, patently true.[3]

[3]This example also works for the relation of the original problem, which would now read $f(x + y) < f(x) + yf(x)$, since $e^y > 1 + y$ for all $y > 0$.