Explorations of the Common Core in Algebra: A Training Ground for Gifted Students

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We see these challenges if we study the empirical data published by MAA from the AMC 8 or AMC 10. How can we, as math educators, address these challenges?

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Why did so many students omit this problem?

What can we do to target this issue?

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- Mathematical Circles are a form of outreach that bring mathematicians together with K-12 students.
- These students, and sometimes their teachers or parents, meet together with a mathematician, undergraduate, or graduate student in an informal setting to work on interesting topics or problems in mathematics.
- These informal meetings are held outside of school hours, either at night or on the weekends.

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Additionally, we cover problems from:

- Abacus International Challenge (students in grades 2-4 and students in grades 5-6)
- American Mathematical Competition (AMC 8, AMC 10, AMC 12, AIME, USAMO)
- Monographs published in the MSRI's Math Circle series
- We host Math Kangaroo every March

At the Fullerton Math Circle, we are mostly interested in student's enrichment.

There is a continuous feedback process and we listen to students' suggestions.

We have organized so far four sessions when the students were the speakers.

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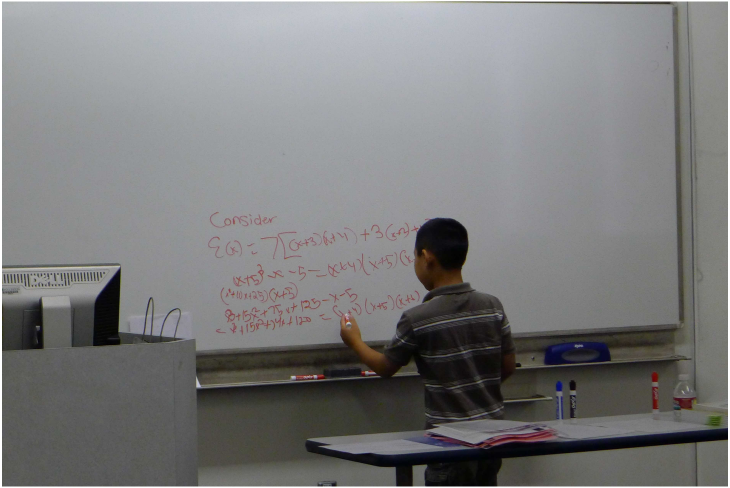
How can our students practice such manipulations on non-standard algebra problems?

This is when we turn to our literature for other examples.

Where can we find approachable training problems?

Can our Math Circle students solve them?



Fullerton Math Circle, August 25, 2012.

Problem S:E12.343, January 2012.

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I remarked that the difference between the numbers 3, 9, 15 is 6.

Although the problem was done, I continued the exploration and I have noticed that all the numbers at a distance of 6 units on the real axis satisfy this properly, namely:

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and so on, the pattern is satisfied.

Problem S:E12.356, January 2012.

The arithmetic mean of the numbers a and b is 4.5; the arithmetic mean of the numbers b and c is 12.5 and the arithmetic mean of the numbers c and a is 16.5. Find the arithmetic mean of the numbers a, b and c.

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Thus $a + b + c = 33.5$ and their arithmetic mean is $\frac{33.5}{3}$.

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Let n be a natural number. Is it possible that the fraction $\frac{2n+3}{3n+2}$ represents a natural number?

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Solution: Remark that for $n = 1$ the value of the fraction is 1.

For any other value of n we notice that the fraction is less than 1, thus it can not be a natural number.

Problem S:E12.480, April 2012.

Let a, b, c be three nonzero integers. Prove that the numbers $a + 2b + 3c$, $2a + 3b + c$, $3a + b + 2c$ can not be all three odd.

Problem S:E12.480, April 2012.

Let a, b, c be three nonzero integers. Prove that the numbers $a + 2b + 3c$, $2a + 3b + c$, $3a + b + 2c$ can not be all three odd.

Solution: This requires a discussion by cases.

Kyle Kishimoto
Fairmont Private School
Fullerton, CA

Problem S: E12.480

Let a, b, c be 3 nonzero integers. Prove the numbers a+2b+3c, 2a+3b+c, and 3a+b+2c cannot be all three odd.

Possible cases

a	b	c	a+2b+3c	2a+3b+c	3a+b+2c
even	even	even	even	even	even
even	even	odd	odd	odd	even
odd	even	even	odd	even	odd
even	odd	even	even	odd	odd
odd	odd	even	odd	odd	even
even	odd	odd	odd	even	odd
odd	even	odd	even	odd	odd
odd	odd	odd	even	even	even

These are all the combinations of numbers

None of the combinations result in all odd. ∎

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This is the circle centered at $(-1, 2)$ of radius 5.

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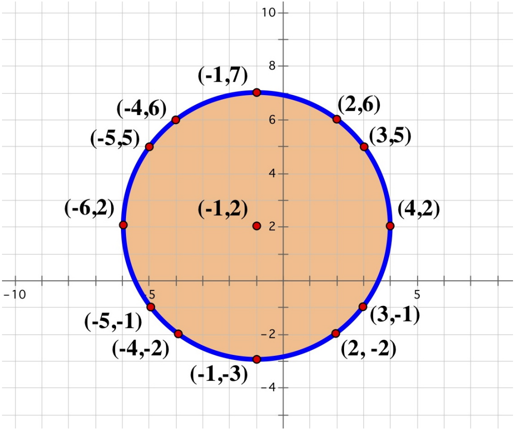
We draw the figure.

Represent the circle:



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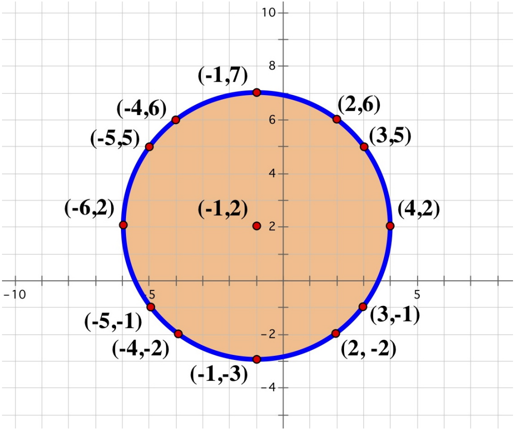

I found twelve pairs of integers satisfying the given equation: $(-1,7)$, $(2,6)$, $(3,5)$, $(4,2)$, $(3,-1)$.$(2,-2)$,$(-1,-3)$, $(-4,-2)$, $(-5,-1)$, $(-6,2)$, $(-5,5)$, $(-4,6)$. One can see on the graph that these are all the possible solutions.

Problem S:E12.365, January 2012

Prove that the triangle ABC having the lengths of sides equal to
$\sqrt{45}+\sqrt{243}+\sqrt{450}$;
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By the converse of Pythagorean Theorem, since
$(3M)^2 + (4M)^2 = (5M)^2$, we conclude that the triangle is right.

What do the undergraduate facilitators learn from these problems?

- For some of us as undergraduate students, Math Circle sessions are among our first experiences as teachers
- We are seeing at work extremely gifted students
- We develop our teaching skills and learn how to present to an audience that looks forward to be challenged by interesting problems
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- We understand that education of gifted students is a very interesting topic in mathematical education
- We understand the importance of educators' and mathematicians' reflection on how they could best serve the development of gifted K-12 students

We thank for support to:

- MAA supporting Fullerton Mathematical Circle through a MAA Dolciani Enrichment Grant

- College of Natural Sciences and Mathematics at Cal State Fullerton

- Department of Mathematics at Cal State Fullerton

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