Bull. Math. Soc. Sc. Math. Roumanie
Tome 48(96) No. 2, 2005, 243–258

The one-dimensional obstacle problem as approximation of the three-dimensional Signorini problem

by
JUAN M. VIAÑO

To J. C. Paumier, In Memoriam

Abstract

In this work, we apply the asymptotic expansion method to the three-dimensional Signorini problem for an elastic beam with unilateral frictionless contact conditions in one part of its lateral boundary. This allows us to obtain a one-dimensional model which includes the classical flexion model of an elastic beam on a rigid foundation (the one-dimensional obstacle problem).

Key Words: Asymptotic methods, Signorini problem, elastic beam, rigid foundation, contact.
2000 Mathematics Subject Classification: Primary: 35J85 , Secondary: 74K10, 74M15.

1 Introduction

In this work, we apply the asymptotic expansion method to the three-dimensional elasticity problem with unilateral frictionless contact conditions in one part of the boundary (Signorini problem). This allows us to obtain a one-dimensional model which generalizes the classical flexion model of an elastic beam on a rigid foundation also called the one-dimensional obstacle problem (see [3],[5]). The methodology is that developed in [1], [7], [10] for beams and [9] for plates: variational formulation, change of variable to a reference domain and scaling of unknowns, asymptotic expansion with respect to the diameter of cross-section and identification of the first terms in such expansion. The major difference here resides in the unilateral contact conditions in the three-dimensional elasticity problem, which give rise to nonlinear problems and to variational inequalities, whose treatment is different from the one used so far.

2 The Signorini problem in linear elastic rods

Let ω be an open, bounded and connected set in $\mathbb{R}^2$ with area $A(\omega)$. With no loss of generality, it is usually supposed that $A(\omega) = 1$, but in order to preserve the physical meaning of the equations we maintain the notation $A(\omega)$. Let $\gamma = \partial\omega$ be its boundary. The coordinate system Ox_1x_2 will be assumed a principal system of inertia associated with the section ω, which means that

$$\int_\omega x_1 d\omega = \int_\omega x_2 d\omega = \int_\omega x_1 x_2 d\omega = 0. \tag{1}$$

Given $\varepsilon \in \mathbb{R}, 0 < \varepsilon \leq 1$, and $L > 0$, we define

$$\omega^\varepsilon = \varepsilon\omega, \quad \gamma^\varepsilon = \partial\omega^\varepsilon = \varepsilon\partial\omega,$$

and we note by $\Omega^\varepsilon = \omega^\varepsilon \times (0, L)$ the prismatic set that we will identify as the reference configuration of the actual rod.
We denote by $x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) = (\varepsilon x_1, \varepsilon x_2, x_3)$ an arbitrary point in Ω^ε and by $n^\varepsilon = (n_i^\varepsilon)$ the outer unit normal vector on $\partial\Omega^\varepsilon$. The parameter ε identifies the size of the diameter of the transversal section ω^ε, that has area $A(\omega^\varepsilon) = \varepsilon^2 A(\omega)$.
We denote the edges of Ω^ε by:

$$\Gamma_0^\varepsilon = \omega^\varepsilon \times \{0\}, \quad \Gamma_L^\varepsilon = \omega^\varepsilon \times \{L\}.$$

We also assume the boundary γ^ε is divided into two nonempty disjoint parts denoted by γ_C^ε and γ_N^ε. Consequently, we denote $\Gamma^\varepsilon = \Gamma_N^\varepsilon \cup \Gamma_C^\varepsilon$, with $\Gamma_N^\varepsilon = \gamma_N^\varepsilon \times (0, L)$ and $\Gamma_C^\varepsilon = \gamma_C^\varepsilon \times (0, L)$. The part Γ_C^ε of the lateral boundary can become in contact without friction with a rigid foundation. We denote by $s^\varepsilon(x^\varepsilon)$ the distance of the point $x^\varepsilon \in \Gamma_C^\varepsilon$ to the obstacle measured in the normal direction of vector n^ε. We assume that the function $s^\varepsilon : \Gamma_C^\varepsilon \to \mathbb{R}$ satisfies $s^\varepsilon \in L^\infty(\Gamma_C^\varepsilon)$. Obviously $s^\varepsilon \geq 0$. We remark that we drop the superindex ε when $\varepsilon = 1$, i.e.:

$$\Omega \equiv \Omega^1, \quad \Gamma_0 \equiv \Gamma_0^1, \ldots$$

The material which constitutes the rod is assumed to be homogeneous and isotropic with Young's modulus E and Poisson's ratio ν, both independent of ε. Also, Lamé's coefficients λ and μ will be employed, related with E and ν by the formulae

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \tag{2}$$

or their inverses

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}. \tag{3}$$

If it is not explicitly mentioned, in what follows we will use the summation convention on repeated indices (ε excluded); moreover, Latin indices take their values in the set $\{1, 2, 3\}$ and Greek indices (except ε) in $\{1, 2\}$.

We suppose that the rod is clamped in both ends Γ_0^ε and Γ_L^ε and submitted to the action of body forces of volume density $f^\varepsilon = (f_i^\varepsilon)$ and surface forces acting on Γ_N^ε of density $g^\varepsilon = (g_i^\varepsilon)$. We assume the following regularity for the forces:

$$f_i^\varepsilon \in L^2(\Omega^\varepsilon), \quad g_i^\varepsilon \in L^2(\Gamma_N^\varepsilon). \tag{4}$$

The classical model used in linear elasticity for this situation is know as the Signorini problem and it is written as (see [5]): Find $u^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ such that:

$$\begin{array}{rcll} -\partial_j \sigma_{ij}(u^\varepsilon) & = & f_i^\varepsilon, & \text{in } \Omega^\varepsilon, \\ \sigma_{ij}(u^\varepsilon) n_j^\varepsilon & = & g_i^\varepsilon, & \text{on } \Gamma_N^\varepsilon, \\ u_i^\varepsilon & = & 0, & \text{on } \Gamma_0^\varepsilon \cup \Gamma_L^\varepsilon, \\ u_n^\varepsilon \leq s^\varepsilon, \quad \sigma_n^\varepsilon \leq 0, \quad \sigma_{ti}^\varepsilon & = & 0, & \text{on } \Gamma_C^\varepsilon \\ \sigma_n^\varepsilon (u_n^\varepsilon - s^\varepsilon) & = & 0, & \text{on } \Gamma_C^\varepsilon, \end{array} \tag{5}$$

where

- $\sigma^\varepsilon = \sigma(u^\varepsilon) = (\sigma_{ij}(u^\varepsilon))$ is the stress tensor, related with the displacement field $u^\varepsilon = (u_i^\varepsilon)$ by the Hooke's generalized law

$$\sigma_{ij}(u^\varepsilon) = \frac{\nu E}{(1+\nu)(1-2\nu)} e_{pp}(u^\varepsilon)\delta_{ij} + \frac{E}{1+\nu} e_{ij}(u^\varepsilon),$$

- $e(u^\varepsilon) = (e_{ij}(u^\varepsilon))$ is the linearized strain tensor

$$e_{ij}(u^\varepsilon) = \frac{1}{2}(\partial_i u_j^\varepsilon + \partial_j u_i^\varepsilon),$$

- $u_n^\varepsilon = u_i^\varepsilon n_i^\varepsilon, \sigma_n^\varepsilon = \sigma_{ij}(u^\varepsilon) n_i^\varepsilon n_j^\varepsilon$ and $\sigma_{ti}^\varepsilon = \sigma_{ij}(u^\varepsilon) n_j^\varepsilon - \sigma_n^\varepsilon n_i^\varepsilon$.

The two last conditions in (5) describes the well-known unilateral contact without friction (Signorini's problem). By introducing the space of admissible displacements,

$$V(\Omega^\varepsilon) = \left\{ v^\varepsilon = (v_i^\varepsilon) \in [H^1(\Omega^\varepsilon)]^3 : v^\varepsilon = 0 \text{ in } \Gamma_0^\varepsilon \cup \Gamma_L^\varepsilon \right\},$$

the following variational formulation of problem (5) is obtained by a classical procedure:

$$\begin{array}{l} u^\varepsilon \in K(\Omega^\varepsilon) := \{ v^\varepsilon \in V(\Omega^\varepsilon) : v_n^\varepsilon \leq s^\varepsilon \text{ a.e. on } \Gamma_C^\varepsilon \}, \\ \displaystyle\int_{\Omega^\varepsilon} \sigma_{ij}(u^\varepsilon) e_{ij}(v^\varepsilon - u^\varepsilon) dx^\varepsilon \geq \int_{\Omega^\varepsilon} f_i^\varepsilon (v_i^\varepsilon - u_i^\varepsilon) dx^\varepsilon + \int_{\Gamma_N^\varepsilon} g_i^\varepsilon (v_i^\varepsilon - u_i^\varepsilon) da^\varepsilon, \\ \text{for all } v^\varepsilon \in K(\Omega^\varepsilon). \end{array} \tag{6}$$

The problem (6) is written as a classical variational inequality of the following form:

$$\left\{ \begin{array}{l} u^\varepsilon \in K(\Omega^\varepsilon), \\ a_\varepsilon(u^\varepsilon, v^\varepsilon - u^\varepsilon) \geq l_\varepsilon(v^\varepsilon - u^\varepsilon), \text{for all } v^\varepsilon \in K(\Omega^\varepsilon), \end{array} \right. \tag{7}$$

where, for all $w^\varepsilon, v^\varepsilon \in [H^1(\Omega^\varepsilon)]^3$ we note:

$$a_\varepsilon(w^\varepsilon, v^\varepsilon) = \int_{\Omega^\varepsilon} \sigma_{ij}(w^\varepsilon) e_{ij}(v^\varepsilon) dx^\varepsilon, \quad l_\varepsilon(v^\varepsilon) = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon dx^\varepsilon + \int_{\Gamma_N^\varepsilon} g_i^\varepsilon v_i^\varepsilon da^\varepsilon.$$

Now, because of the continuity of the linear form l_ε and the continuity and $V(\Omega^\varepsilon)$–ellipticity of the bilinear form a_ε (due to the Korn's inequality), the problem (7) has a unique solution for each ε (see [2], [6]).
The geometric characteristic of the rod is that the area of its cross section, $\varepsilon^2 A(\omega)$, is very small compared with its length because usually we have $\varepsilon << 1$. As a consequence, the three-dimensional model is ill-conditioned and difficult to solve by the usual finite element methods. Hence, we take advantage of the geometrical property of the rod to approximate $(u^\varepsilon, \sigma^\varepsilon)$, when ε is small, using the asymptotic techniques for problems depending on a small parameter, essentially as in [1] and [10]. In sections 4 and 5, the distinct stages of this process are discussed.
An important task of this work is to compare the limit model obtained via the asymptotic method with the well-known classical one-dimensional models for flexion of elastic beams on a rigid foundation (one-dimensional obstacle problem). In the next section we recall this model in a particular case (see [3]).

3 The classical flexion model of elastic beams on a rigid foundation

In practise, the most known one-dimensional model for flexion of elastic beams on a foundation corresponds to assume that each point $(0, 0, x_3^\varepsilon)$ of the central line of the beam is situated initially to a distance $\hat{s}^\varepsilon(x_3^\varepsilon)$ of the obstacle, measured in the direction Ox_α^ε and the total loading applied in this direction at the same point is $F_\alpha^\varepsilon(x_3^\varepsilon)$. Then the model is written as follows (no sum on α) (see [3]):

$$\begin{array}{l} EI_\alpha^\varepsilon(\xi_\alpha^\varepsilon)^{(4)} = F_\alpha^\varepsilon + q_\alpha^\varepsilon, \text{ in } (0, L) \\ \xi_\alpha^\varepsilon(0) = \xi_\alpha^\varepsilon(L) = 0, \\ (\xi_\alpha^\varepsilon)'(0) = (\xi_\alpha^\varepsilon)'(L) = 0, \\ \xi_\alpha^\varepsilon \geq \hat{s}^\varepsilon, \; q_\alpha^\varepsilon \geq 0, \; (\xi_\alpha^\varepsilon - \hat{s}^\varepsilon) q_\alpha^\varepsilon = 0, \text{ in } (0, L), \end{array} \tag{8}$$

where we use the notation $\chi', \chi'', \ldots, \chi^{(n)}$ for derivatives with respect to variable $x_3 = x_3^\varepsilon$ of a function χ only depending on variable $x_3 = x_3^\varepsilon \in (0, L)$ and

- ξ_α^ε is the flexion of the central line on the direction Ox_α^ε,
- q_α^ε is the (unknown) reaction of the foundation on the direction Ox_α^ε,
- I_α^ε is the inertia moment of ω^ε with respect to the axis Ox_α^ε defined by

$$I_\alpha^\varepsilon = \int_{\omega^\varepsilon} (x_\alpha^\varepsilon)^2 d\omega^\varepsilon. \tag{9}$$

The last line in (8) traduces the non penetration condition and that the reaction is strictly positive only when the contact is produced.
The variational formulation of problem (8) is also well-known (no sum on α):

$$\left\{\begin{array}{l} \xi_\alpha^\varepsilon \in U_\alpha^\varepsilon := \{\chi_\alpha^\varepsilon \in H_0^2(0,L) : \chi_\alpha^\varepsilon \geq \hat{s}^\varepsilon \text{ a.e. in } (0,L)\}, \\ EI_\alpha^\varepsilon \int_0^L (\xi_\alpha^\varepsilon)''(\chi_\alpha^\varepsilon - \xi_\alpha^\varepsilon)'' dx_3^\varepsilon \geq \int_0^L F_\alpha^\varepsilon(\chi_\alpha^\varepsilon - \xi_\alpha) dx_3, \text{ for all } \chi_\alpha^\varepsilon \in U_\alpha^\varepsilon. \end{array}\right. \tag{10}$$

¿From classical theory of elliptic variational inequalities (see [2], [6]), the existence and uniqueness of solution of the problem (10) is obtained.

4 The asymptotic method

Following [1] and [10], we introduce the change of variable

$$\Pi^\varepsilon : \Omega \longrightarrow \Omega^\varepsilon, \quad (x_1, x_2, x_3) \to x^\varepsilon = (\varepsilon x_1, \varepsilon x_2, x_3). \tag{11}$$

Also, we scale the unknown and the test displacements:

$$u_\alpha(\varepsilon)(x) = \varepsilon u_\alpha^\varepsilon(x^\varepsilon), \qquad u_3(\varepsilon)(x) = u_3^\varepsilon(x^\varepsilon), \tag{12}$$

$$v_\alpha(\varepsilon)(x) = \varepsilon v_\alpha^\varepsilon(x^\varepsilon), \qquad v_3(\varepsilon)(x) = v_3^\varepsilon(x^\varepsilon), \text{ for all } v^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}^3. \tag{13}$$

We suppose the following order of magnitude for the forces and distance s^ε which warrant the hypothetical limit displacements have the most general form (see (40)):

$$\begin{array}{l} f_\alpha^\varepsilon(x^\varepsilon) = \varepsilon f_\alpha(x), \quad f_3^\varepsilon(x^\varepsilon) = f_3(x), \\ g_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 g_\alpha(x), \quad g_3^\varepsilon(x^\varepsilon) = \varepsilon g_3(x), \\ s^\varepsilon = \varepsilon^{-1} s(x), \end{array} \tag{14}$$

where the functions

$$f_i \in L^2(\Omega),\ g_i \in L^2(\Gamma_N),\ s \in L^\infty(\Gamma_C) \tag{15}$$

are independent of the parameter ε.

Remark We note that if the components of applied forces have another orders of magnitude with respect to ε the linearity of equations allows us to decompose the problem as a sum of problems having this property.
So, an elementary calculation based on the change of variable for integration gives the following result.

Theorem 4.1 *The scaled displacement $u(\varepsilon)$ obtained by means the transformation (12) of the solution u^ε of problem (7) is the unique solution of the following variational problem in Ω:*

$$\left\{\begin{array}{l} u(\varepsilon) \in K(\Omega) = \{v \in V(\Omega) : v_n \leq s \textit{ a.e. on } \Gamma_C\}, \\ c_0(u(\varepsilon), v - u(\varepsilon)) + \varepsilon^2 c_2(u(\varepsilon), v - u(\varepsilon)) + \varepsilon^4 c_4(u(\varepsilon), v - u(\varepsilon)) \\ \geq \varepsilon^4 \left[\int_\Omega f_i(v_i - u_i(\varepsilon)) dx + \int_{\Gamma_N} g_i(v_i - u_i(\varepsilon)) da\right], \\ \textit{for all } v \in K(\Omega), \end{array}\right. \tag{16}$$

where for all $w, v \in V(\Omega)$ the bilinear forms c_0, c_2 and c_4 are defined by

$$\begin{aligned} c_0(w,v) &= \int_\Omega [\lambda e_{\alpha\alpha}(w)e_{\beta\beta}(v) + 2\mu e_{\alpha\beta}(w)e_{\alpha\beta}(v)]dx, \\ c_2(w,v) &= \int_\Omega [\lambda e_{\alpha\alpha}(w)e_{33}(v) + 4\mu e_{3\alpha}(w)e_{3\alpha}(v) + \lambda e_{33}(w)e_{\alpha\alpha}(v)]dx, \\ c_4(w,v) &= \int_\Omega (\lambda + 2\mu)e_{33}(u)e_{33}(v)dx. \end{aligned}$$

Now, the polynomial expression in powers of ε^2 in (16) lead us in a natural way to use asymptotical techniques (Lions [8]) to approach $u(\varepsilon)$, when ε is small, by means of an expansion of the form

$$u(\varepsilon) = u^{(0)} + \varepsilon^2 u^{(2)} + \varepsilon^4 u^{(4)} + \dots \tag{17}$$

where

$$u^{(2p)} \in V(\Omega),\ u^{(0)} \in K(\Omega),\ u_n^{(2p)} \le 0,\ p = 1, 2, \dots \tag{18}$$

Remark Let us mention that the natural parameter in (16) is $\delta = \varepsilon^2$, and this motivates the even expansion (17). Moreover, it is easy to prove (see [10]) that if we introduce odd powers in (17), then the coefficients $u^{(2p+1)}$ vanish.

In a general way, given a symmetric bilinear form $c : V(\Omega) \times V(\Omega) \to \mathbb{R}$, for all $v \in V(\Omega)$ we obtain the following decomposition according to the expansion (17)-(18):

$$\begin{aligned} c(u(\varepsilon), v - u(\varepsilon)) = c(u^{(0)}, v - u^{(0)}) + \varepsilon^2 c(u^{(2)}, v - 2u^{(0)}) + \\ +\varepsilon^4 [c(u^{(4)}, v - 2u^{(0)}) - c(u^{(2)}, u^{(2)})] + O(\varepsilon^6). \end{aligned} \tag{19}$$

Substituting the expansion (17) into (16) and taking into account (19) successively for the bilinear forms c_0, c_2, c_4, we obtain the following inequality:

$$\begin{aligned} &c_0(u^{(0)}, v - u^{(0)}) + \varepsilon^2 c_0(u^{(2)}, v - 2u^{(0)}) + \varepsilon^4 [c_0(u^{(4)}, v - 2u^{(0)}) - c_0(u^{(2)}, u^{(2)})] \\ &+\varepsilon^2 c_2(u^{(0)}, v - u^{(0)}) + \varepsilon^4 c_2(u^{(2)}, v - 2u^{(0)}) + \varepsilon^4 c_4(u^{(0)}, v - u^{(0)}) \\ &\ge \varepsilon^4 \int_\Omega f_i(v_i - u_i^0)dx + \varepsilon^4 \int_{\Gamma_N} g_i(v_i - u_i^0)da + O(\varepsilon^6),\ \text{for all } v \in K(\Omega). \end{aligned} \tag{20}$$

We will see that the previous inequality (20) determines in a unique way $u^{(0)} \in K(\Omega)$ and gives the form (non unique) of $u^{(2)} \in [H^1(\Omega)]^3$ satisfying $u_n^{(2)} \le 0$ but not in $V(\Omega)$ except for some particular cases. This fact causes a boundary layer phenomenon similar to those seen for the elasticity problem with no contact condition in [10], [4] and [7], among others.

5 The first order terms in the asymptotic expansion

We introduce some constants and functions which only depend on the geometry of the transversal section ω^ε (see [10]). For simplicity in the notations we assume $\varepsilon = 1$.

- Second moments of area of ω: $I_\alpha = \displaystyle\int_\omega x_\alpha^2 d\omega$.
- Functions $\Phi_{\alpha\beta}$ and δ_α :

$$\begin{aligned} &\Phi_{11}(x_1,x_2) = \tfrac{1}{2}(x_1^2 - x_2^2) = -\Phi_{22}(x_1,x_2), \\ &\Phi_{12}(x_1,x_2) = \Phi_{21}(x_1,x_2) = x_1x_2, \\ &\delta_1(x_1,x_2) = x_2, \quad \delta_2(x_1,x_2) = -x_1. \end{aligned} \tag{21}$$

- Warping function w is the unique solution of the following problem:

$$\begin{aligned} &w \in H^1(\omega), \quad \int_\omega w d\omega = 0, \\ &\int_\omega \partial_\alpha w \partial_\alpha \varphi d\omega = \int_\omega (x_2\partial_1\varphi - x_1\partial_2\varphi)d\omega, \text{ for all } \varphi \in H^1(\omega). \end{aligned} \tag{22}$$

- Timoshenko's functions η_β and $\theta_\beta$are the unique solution of the following problems, respectively:

$$\begin{aligned} &\eta_\beta \in H^1(\omega), \quad \int_\omega \eta_\beta d\omega = 0, \\ &\int_\omega \partial_\alpha \eta_\beta \partial_\alpha \varphi d\omega = -2\int_\omega x_\beta \varphi d\omega, \text{ for all } \varphi \in H^1(\omega). \end{aligned} \tag{23}$$

$$\begin{aligned} &\theta_\beta \in H^1(\omega), \quad \int_\omega \theta_\beta d\omega = 0, \\ &\int_\omega (\partial_\alpha \theta_\beta + \Phi_{\alpha\beta})\partial_\alpha \varphi d\omega = 0, \text{ for all } \varphi \in H^1(\omega). \end{aligned} \tag{24}$$

Furthermore, we use in this section the spaces

$$V_1(\Omega) = \{v \in V(\Omega) : e_{\alpha\beta}(v) = 0\}, \tag{25}$$

$$V_2(\Omega) = V_{BN}(\Omega) = \{v \in V(\Omega) : e_{\alpha\beta}(v) = e_{3\alpha}(v) = 0\}. \tag{26}$$

The elements of $V_{BN}(\Omega)$ are called the Bernoulli-Navier displacements. We have the following equivalent definitions for for $V_1(\Omega)$ and $V_2BN(\Omega)$.

Lemma 5.1 (Trabucho–Viaño [10]). *The following characterization for the spaces $V_1(\Omega)$ and $V_2(\Omega)$ hold:*

$$\begin{aligned} &V_1(\Omega) = \big\{v \in [H^1(\Omega)]^3 : \\ &v_\alpha(x_1,x_2,x_3) = \chi_\alpha(x_3) + \delta_\alpha(x_1,x_2)\chi_3(x_3),\ \chi_3, \chi_\alpha \in H_0^1(0,L)\big\}, \end{aligned} \tag{27}$$

$$\begin{aligned} &V_2(\Omega) = V_{BN}(\Omega) = \big\{v \in [H^1(\Omega)]^3 : v_\alpha(x_1,x_2,x_3) = \chi_\alpha(x_3), \\ &v_3(x_1,x_2,x_3) = \chi_3(x_3) - x_\alpha\chi_\alpha'(x_3),\ \chi_3 \in H_0^1(0,L), \chi_\alpha \in H_0^2(0,L)\big\}. \end{aligned} \tag{28}$$

We also use the following sets of functions with separated variables.

$$\begin{aligned} W_T(\Omega) = \{v = (v_1, v_2, 0) \in [H^1(\Omega))]^3 : \\ v_\alpha(x_1, x_2, x_3) = \varphi_\alpha(x_1, x_2)\chi(x_3),\ \varphi_\alpha \in H^1(\omega),\ \chi \in H_0^1(0, L)\} \\ \equiv \{(v_1, v_2) \in [H^1(\Omega)]^2 : \\ v_\alpha(x_1, x_2, x_3) = \varphi_\alpha(x_1, x_2)\chi(x_3),\ \varphi_\alpha \in H^1(\omega),\ \chi \in H_0^1(0, L)\} \end{aligned} \tag{29}$$

$$\begin{aligned} W_L(\Omega) = \{v = (0, 0, v_3) \in [H^1(\Omega))]^3 : \\ v_3(x_1, x_2, x_3) = \varphi(x_1, x_2)\chi(x_3),\ \varphi \in H^1(\omega),\ \chi \in H_0^1(0, L)\} \\ \equiv \{v_3 \in H^1(\Omega) : \\ v_3(x_1, x_2, x_3) = \varphi(x_1, x_2)\chi(x_3),\ \varphi \in H^1(\omega),\ \chi \in H_0^1(0, L)\}. \end{aligned} \tag{30}$$

Having in mind that in $\Gamma = \Gamma_C \cup \Gamma_N$ the outward unit normal vector is of the form $(n_1, n_2, 0)$, we have

$$K(\Omega) = K_2(\Omega) \times W_1(\Omega), \tag{31}$$

where

$$K_2(\Omega) = \{(v_\beta) \in [H^1(\Omega)]^2 : v_\beta = 0 \text{ on } \Gamma_0 \cup \Gamma_L,\ v_\beta n_\beta \le s \text{ a. e. on } \Gamma_C\}, \tag{32}$$

$$W_1(\Omega) = \{v_3 \in H^1(\Omega) : v_3 = 0 \text{ on } \Gamma_0 \cup \Gamma_L\}. \tag{33}$$

Now, we try to characterize the solutions of problem (20). We follow the process developed by Trabucho&Viaño in [10] for the elasticity problem with no contact conditions. Firstly, the following transversal forces F_i^ε and moments M_i^ε, and the function $w^{\varepsilon(0)}$ are introduced (for simplicity in the notations, $\varepsilon = 1$ is assumed):

$$F_i = \int_\omega f_i d\omega + \int_{\gamma_N} g_i d\gamma, \quad M_\alpha = \int_\omega x_\alpha f_3 d\omega + \int_{\gamma_N} x_\alpha g_3 d\gamma, \tag{34}$$

$$\left\{ \begin{aligned} & w^{(0)} \in L^2(0, L; H^1(\omega)) \text{ and a.e. in } (0, L), \quad \int_\omega w^{(0)} d\omega = 0, \\ & \int_\omega \partial_\alpha w^{(0)} \partial_\alpha \varphi d\omega = \int_\omega f_3 \varphi d\omega + \int_{\gamma_N} g_3 \varphi d\gamma - \frac{1}{A(\omega)} F_3 \int_\omega \varphi d\omega, \\ & \text{for all } \varphi \in H^1(\omega). \end{aligned} \right. \tag{35}$$

The following result is a first step to determine the solutions of problem(20). We need some additional regularity for the forces.

Theorem 5.1 *Let us suppose the forces satisfy the basic condition (15) and also*

$$f_3 \in H^1(0, L; L^2(\omega)),\ \ g_3 \in H^1(0, L; L^2(\gamma_N)). \tag{36}$$

Then, the variational inequality (20) yields the following inequalities involving $u^{(0)}$*:*

$$c_0(u^{(0)}, v - u^{(0)}) \ge 0, \text{ for all } v \in K(\Omega), \tag{37}$$

$$c_0(u^{(2)}, v - 2u^{(0)} + c_2(u^{(0)}, v - u^{(0)}) \ge 0, \text{ for all } v \in K(\Omega), \tag{38}$$

$$\begin{aligned} & c_0(u^{(4)}, v - 2u^{(0)}) - c_0(u^{(2)}, u^{(2)}) + c_2(u^{(2)}, v - 2u^{(0)}) + c_4(u^{(0)}, v - u^{(0)}) \\ & \ge \int_\Omega f_i(v_i - u_i^{(0)}) dx + \int_{\Gamma_N} g_i(v_i - u_i^{(0)}) da, \text{ for all } v \in K(\Omega). \end{aligned} \tag{39}$$

The displacement $u^{(0)} \in K(\Omega)$ *is uniquely determined and the displacement* $u^{(2)} \in [H^1(\Omega)]^3, u_n^{(2)} \leq 0$ *is characterized in a non-unique way from equations (37)–(39). These characterizations are obtained as follows:*

(i) The first term $u^{(0)}$ *belongs to the space* $V_{BN}(\Omega)$ *and it has the following form*

$$u_\alpha^{(0)}(x_1, x_2, x_3) = \xi_\alpha(x_3), \quad u_3^{(0)}(x_1, x_2, x_3) = \xi_3(x_3) - x_\alpha \xi'_\alpha(x_3), \tag{40}$$

where the flexions (ξ_α) *are the only solution of the following coupled elliptic variational inequality:*

$$\left\{ \begin{array}{l} (\xi_\alpha) \in [H_0^2(0,L)]^2 \cap [H^4(0,L)]^2 \cap K_2(\Omega), \\ EI_\alpha \displaystyle\int_0^L \xi''_\alpha(\chi_\alpha - \xi_\alpha)'' dx_3 \geq \int_0^L F_\alpha(\chi_\alpha - \xi_\alpha) dx_3 \\ - \displaystyle\int_0^L M_\alpha(\chi_\alpha - \xi_\alpha)' dx_3, \ \textit{for all } (\chi_\alpha) \in [H_0^2(0,L)]^2 \cap K_2(\Omega), \end{array} \right. \tag{41}$$

and the stretching ξ_3 *is the only solution of the following variational problem:*

$$\left\{ \begin{array}{l} \xi_3 \in H_0^1(0,L) \cap H^2(0,L), \\ EA(\omega) \displaystyle\int_0^L \xi'_3 \chi' dx_3 = \int_0^L F_3 \chi dx_3, \ \textit{for all } \chi \in H_0^1(0,L). \end{array} \right. \tag{42}$$

(ii) The term $u^{(2)} \in [H^1(\Omega)]^3$ *is characterized in the following way:*

$$u_\alpha^{(2)}(x_1, x_2, x_3) = z_\alpha(x_3) + U_\alpha^{(2)}(x_1, x_2, x_3), \tag{43}$$

$$u_3^{(2)}(x_1, x_2, x_3) = z_3(x_3) - x_\alpha z'_\alpha(x_3) + U_3^{(2)}(x_1, x_2, x_3), \tag{44}$$

where $U^{(2)} = (U_i^{(2)})$ *has the following form*

$$U_\alpha^{(2)}(x_1, x_2, x_3) = \delta_\alpha r(x_3) - \nu[x_\alpha \xi'_3(x_3) - \Phi_{\alpha\beta} \xi''_\beta(x_3)], \tag{45}$$

$$\begin{aligned} U_3^{(2)}(x_1, x_2, x_3) &= -wr'(x_3) + \nu \left\{ \frac{1}{2}(x_1^2 + x_2^2) - \frac{1}{2A(\omega)}(I_1 + I_2) \right\} \xi''_3(x_3) \\ &+ [(1+\nu)\eta_\alpha + \nu\theta_\alpha] \xi'''_\alpha(x_3) + \frac{2(1+\nu)}{E} w^{(0)}, \end{aligned} \tag{46}$$

with $z_\alpha \in H^2(0,L)$, $r \in H^1(0,L)$ *and* $z_3 \in H_0^1(0,L)$.

Proof. We will present the proof in several steps, following the scheme developed in [7].

Step 1. Passing to the limit as ε tends to zero in inequality (20), we obtain (37), that is, for all $v \in K(\Omega)$:

$$\int_\Omega [\lambda e_{\alpha\alpha}(u^{(0)}) e_{\beta\beta}(v - u^{(0)}) + 2\mu e_{\alpha\beta}(u^{(0)}) e_{\alpha\beta}(v - u^{(0)})] dx \geq 0. \tag{47}$$

Taking succesivelly $v = 2u^{(0)}$ and $v = 0 \in K(\Omega)$ in (47) we have:

$$\int_\Omega [\lambda e_{\alpha\alpha}(u^{(0)})e_{\beta\beta}(u^{(0)}) + 2\mu e_{\alpha\beta}(u^{(0)})e_{\alpha\beta}(u^{(0)})]dx = 0,$$

and, consequently, $u^{(0)} \in V_1(\Omega) \cap K(\Omega)$:

$$e_{\alpha\beta}(u^{(0)}) = 0, \quad u_n^{(0)} \leq s \text{ on } \Gamma_C. \tag{48}$$

Condition (48) restricts the form of the transversal components of $u^{(0)}$ to the following one (see lemma 5.1):

$$u_\alpha^{(0)}(x_1, x_2, x_3) = \xi_\alpha(x_3) + \delta_\alpha(x_1, x_2)\xi(x_3), \quad \xi_\alpha, \xi \in H_0^1(0, L). \tag{49}$$

Hence, inequality (37) is equivalent to the equation

$$c_0(u^{(0)}, v - u^{(0)}) = 0, \text{ for all } v \in K(\Omega). \tag{50}$$

Step 2. Taking the limit as $\varepsilon \to 0$ on the combination of inequalities $\frac{1}{\varepsilon^2}[(20) - (50)]$ we obtain the limit inequality (38). Taking into account the conditions (48), the inequality (38) is written as:

$$\begin{aligned}&\int_\Omega [\lambda e_{\alpha\alpha}(u^{(2)})e_{\beta\beta}(v) + 2\mu e_{\alpha\beta}(u^{(2)})e_{\alpha\beta}(v)]dx \\ &+ \int_\Omega [\lambda e_{33}(u^{(0)})e_{\alpha\alpha}(v) + 4\mu e_{3\alpha}(u^{(0)})e_{3\alpha}(v - u^{(0)})dx \geq 0, \\ &\text{for all } v \in K(\Omega).\end{aligned} \tag{51}$$

Equation (51) evaluated successively in $v = 2u^{(0)}$ and $v = 0$ produces:

$$\int_\Omega 4\mu e_{3\alpha}(u^{(0)})e_{3\alpha}(u^{(0)})dx = 0,$$

which yields

$$e_{3\alpha}(u^{(0)}) = 0. \tag{52}$$

Properties (48) and (52) mean the term $u^{(0)}$ belongs to the space $V_2(\Omega) = V_{BN}(\Omega)$ of Bernoulli–Navier displacements and also to $K(\Omega)$. Specifically,

$$\begin{aligned}&u^{(0)} \in V_{BN}(\Omega) \cap K(\Omega), \\ &u_\alpha^{(0)}(x) = \xi_\alpha(x_3), \quad \xi_\alpha \in H_0^2(0, L), \\ &u_3^{(0)}(x) = \xi_3(x_3) - x_\alpha \xi_\alpha'(x_3), \quad \xi_3 \in H_0^1(0, L), \\ &\xi_\alpha(x_3)n_\alpha(x_1, x_2) \leq s(x_1, x_2, x_3) \text{ a.e. on } \Gamma_C.\end{aligned} \tag{53}$$

Then, by the corresponding identifications, we have

$$(\xi_\alpha) \in [H_0^2(0, L)]^2 \cap K_2(\Omega). \tag{54}$$

Now, by substituting (52) in inequality (51), we deduce, for all $v \in K(\Omega)$:

$$\int_\Omega [\lambda e_{\alpha\alpha}(u^{(2)})e_{\beta\beta}(v) + 2\mu e_{\alpha\beta}(u^{(2)})e_{\alpha\beta}(v)]dx + \int_\Omega \lambda e_{33}(u^{(0)})e_{\alpha\alpha}(v)dx \geq 0. \quad (55)$$

Hence, taking test functions $v \in W_T(\Omega) \cap K(\Omega)$ in (55) we have:

$$\begin{aligned}&\int_0^L \left\{ \int_\omega [\lambda e_{\alpha\alpha}(u^{(2)})e_{\beta\beta}(\varphi) + 2\mu e_{\alpha\beta}(u^{(2)})e_{\alpha\beta}(\varphi)]\, d\omega \right.\\ &\left. + \int_\omega \lambda e_{33}(u^{(0)})e_{\beta\beta}(\varphi)d\omega \right\} \chi\, dx_3 \geq 0, \text{ for all } \varphi = (\varphi_\alpha) \in [H^1(\omega)]^2,\\ &\text{and } \chi \in H_0^1(0,L) \text{ s.t. } \varphi_\alpha n_\alpha \leq s \text{ a.e. on } \Gamma_C,\ \chi \geq 0 \text{ a.e. in } (0,L).\end{aligned}$$

We conclude that the following equation holds a.e. in $(0,L)$:

$$\begin{aligned}&\int_\omega [\lambda e_{\alpha\alpha}(u^{(2)})e_{\beta\beta}(\varphi) + 2\mu e_{\alpha\beta}(u^{(2)})e_{\alpha\beta}(\varphi) + \lambda e_{33}(u^{(0)})e_{\alpha\alpha}(\varphi)]d\omega, \geq 0,\\ &\text{for all } \varphi = (\varphi_\alpha) \in [H^1(\omega)]^2 \text{ s.t. } \varphi_\alpha n_\alpha \leq s \text{ a.e. on } \Gamma_C.\end{aligned} \quad (56)$$

Taking $\varphi_\alpha \in \mathcal{D}(\omega)$ in (56) we conclude the following equality:

$$\lambda e_{\rho\rho}(u^{(2)})\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(u^{(2)}) = -\lambda e_{33}(u^{(0)})\delta_{\alpha\beta}, \text{ a.e. in } \Omega = \omega \times (0,L). \quad (57)$$

Since $e_{33}(u^{(0)}) = \xi_3' - x_\alpha \xi_\alpha''$ (see (53)), from (57) the expressions (43) and (45) of $u_\alpha^{(2)}$ are deduced (see [10],Th. 4.5). We note that conditions $z_\alpha, r \in H^1(0,L)$ are necessary but not sufficient in order to have $u_\alpha^{(2)} \in W_1(\Omega)$. Then $u_\alpha^{(2)} \in H^1(\Omega)$ but, in general, $u_\alpha^{(2)} \notin W_1(\Omega)$ and $(u_\alpha^{(2)}) \notin K_2(\Omega)$.
From (56, we see that $u^{(2)}$ is a solution of the following equation, for all $v \in K(\Omega)$:

$$\int_\Omega [\lambda e_{\alpha\alpha}(u^{(2)})e_{\beta\beta}(v) + 2\mu e_{\alpha\beta}(u^{(2)})e_{\alpha\beta}(v)]dx + \int_\Omega \lambda e_{33}(u^{(0)})e_{\alpha\alpha}(v)dx = 0, \quad (58)$$

and, finally, from (52) and (58) we deduce that

$$c_0(u^{(2)}, v - 2u^{(0)}) + c_2(u^{(0)}, v - u^{(0)}) = 0, \text{ for all } v \in K(\Omega). \quad (59)$$

Step 3. Passing to the limit as $\varepsilon \to 0$ in the combination of inequalities $\frac{1}{\varepsilon^4}[(20) - (50) - \varepsilon^2(59)]$ it is deduced the following limit inequality (see (39)):

$$\begin{aligned}&c_0(u^{(4)}, v - 2u^{(0)}) - c_0(u^{(2)}, u^{(2)}) + c_2(u^{(2)}, v - 2u^{(0)}) + c_4(u^{(0)}, v - u^{(0)})\\ &\geq \int_\Omega f_i(v_i - u_i^{(0)})dx + \int_{\Gamma_N} g_i(v_i - u_i^{(0)})da, \text{ for all } v \in K(\Omega).\end{aligned} \quad (60)$$

So, taking into account the properties deduced in the previous steps, mainly (57), the equation (60) is written as:

$$\begin{aligned}
&\int_\Omega [\lambda e_{\alpha\alpha}(u^{(4)})e_{\beta\beta}(v) + 2\mu e_{\alpha\beta}(u^{(4)})e_{\alpha\beta}(v)]dx \\
&+ \int_\Omega [\lambda e_{\alpha\alpha}(u^{(2)})e_{33}(v-u^{(0)}) + \lambda e_{33}(u^{(2)})e_{\alpha\alpha}(v)]dx \\
&+ \int_\Omega 4\mu e_{3\alpha}(u^{(2)})e_{3\alpha}(v)dx + \int_\Omega (\lambda+2\mu)e_{33}(u^{(0)})e_{33}(v-u^{(0)}) \\
&\geq \int_\Omega f_i(v_i - u_i^{(0)})dx + \int_{\Gamma_N} g_i(v_i - u_i^{(0)})da, \text{ for all } v \in K(\Omega).
\end{aligned} \tag{61}$$

Evaluating (61) in $v \in V_{BN}(\Omega) \cap K(\Omega)$ one has:

$$\begin{aligned}
&\int_\Omega \lambda e_{\alpha\alpha}(u^{(2)})e_{33}(v-u^{(0)})dx + \int_\Omega (\lambda+2\mu)e_{33}(u^{(0)})e_{33}(v-u^{(0)}) \\
&\geq \int_\Omega f_i(v_i - u_i^{(0)})dx + \int_{\Gamma_N} g_i(v_i - u_i^{(0)})da, \text{ for all } v \in V_{BN}(\Omega) \cap K(\Omega).
\end{aligned} \tag{62}$$

We notice that for any $v \in V_{BN}(\Omega) \cap K(\Omega)$ we have

$$\begin{aligned}
&v_\alpha(x_1,x_2,x_3) = \chi_\alpha(x_3),\ (\chi_\alpha) \in [H^2(0,L)]^2 \cap K_2(\Omega), \\
&v_3(x_1,x_2,x_3) = \chi_3(x_3) - x_\alpha \chi'_\alpha(x_3),\ \chi_3 \in H_0^1(0,L).
\end{aligned} \tag{63}$$

Now, as a consequence of (40) and (43), we have

$$\lambda e_{\alpha\alpha}(u^{(2)}) + (\lambda+2\mu)e_{33}(u^{(0)}) = E[\xi'_3 - x_\alpha \xi''_\alpha]. \tag{64}$$

Then, by substituting (64) into (62), the problems (41) and (42) are derived. Existence, unicity and regularity of solution of problem (41) are exhibited in Brezis–Stampacchia[2] (see also, [5],[6]).

Step 4. We restrict now (61) to $v \in K(\Omega) \cap V_1(\Omega)$, that is, $e_{\alpha\beta}(v) = 0$. We have:

$$\begin{aligned}
&\int_\Omega [\lambda e_{\alpha\alpha}(u^{(2)})e_{33}(v-u^{(0)}) + 4\mu e_{3\alpha}(u^{(2)})e_{3\alpha}(v-u^{(0)})]dx \\
&+ \int_\Omega (\lambda+2\mu)e_{33}(u^{(0)})e_{33}(v-u^{(0)}) \\
&\geq \int_\Omega f_i(v_i - u_i^{(0)})dx + \int_{\Gamma_N} g_i(v_i - u_i^{(0)})da, \\
&\text{for all } v \in K(\Omega) \text{ s.t. } e_{\alpha\beta}(v) = 0.
\end{aligned} \tag{65}$$

By taking in (65) respectively $v = (0,0,u_3^0 + v_3)$ and $v = (0,0,u_3^0 - v_3)$, $v_3 \in W_1(\Omega)$, one has

$$\begin{aligned}
&\int_\Omega \mu \partial_\alpha u_3^{(2)} \partial_\alpha v_3 dx = -\int_\Omega \lambda e_{\alpha\alpha}(u^{(2)})\partial_3 v_3 dx - \int_\Omega \mu \partial_3 u_\alpha^{(2)} \partial_\alpha v_3 dx \\
&- \int_\Omega (\lambda+2\mu)\partial_3 u_3^{(0)} \partial_3 v_3 dx + \int_\Omega f_3 v_3 dx + \int_{\Gamma_N} g_3 v_3 da, \\
&\text{for all } v_3 \in W_1(\Omega).
\end{aligned} \tag{66}$$

Evaluating (66) in $v \in W_L(\Omega)$ it results:

$$\begin{aligned}
&\int_0^L \left[\int_\omega \mu \partial_\alpha u_3^{(2)} \partial_\alpha \varphi d\omega\right] \chi dx_3 = -\int_0^L \left[\int_\omega \lambda e_{\alpha\alpha}(u^{(2)}) \varphi d\omega\right] \chi' dx_3 \\
&-\int_0^L \left[\int_\omega \mu \partial_3 u_\alpha^{(2)} \partial_\alpha \varphi d\omega\right] \chi dx_3 - \int_0^L \left[\int_\omega (\lambda + 2\mu) \partial_3 u_3^{(0)} \varphi d\omega\right] \chi' dx_3 \\
&+\int_0^L \left[\int_\omega f_3 \varphi d\omega\right] \chi dx_3 + \int_0^L \left[\int_{\gamma_N} g_3 \varphi d\gamma\right] \chi dx_3, \\
&\text{for all } \varphi \in H^1(\omega), \text{ for all } \chi \in H_0^1(0,L).
\end{aligned} \tag{67}$$

Using now equalities (64), (43) and (45), the following equation in the sense of distributions in $(0, L)$ is derived:

$$\begin{aligned}
&\int_\omega \partial_\alpha u_3^{(2)} \partial_\alpha \varphi d\omega = \frac{E}{\mu}\left[\xi_3'' \int_\omega \varphi d\omega - \xi_\alpha''' \int_\omega x_\alpha \varphi d\omega\right] - z_\alpha' \int_\omega \partial_\alpha \varphi d\omega \\
&-r' \int_\omega \delta_\alpha \partial_\alpha \varphi d\omega + \nu \xi_3'' \int_\omega x_\alpha \partial_\alpha \varphi d\omega - \nu \xi_\beta''' \int_\omega \Phi_{\alpha\beta} \partial_\alpha \varphi d\omega \\
&+\frac{1}{\mu} \int_\omega f_3 \varphi d\omega + \frac{1}{\mu} \int_{\gamma_N} g_3 \varphi d\gamma, \text{ for all } \varphi \in H^1(\omega).
\end{aligned} \tag{68}$$

For each $x_3 \in (0, L)$ the problem (68) is a Laplacian problem in ω with Neumann conditions in all boundary γ. The compatibility condition for φ such that $\partial_\alpha \varphi = 0$ is verified because of the equation (42) for the traction. Then, there exists a (non-unique) solution of (68) and it has the form given by expressions (44) and (46) (see Trabucho–Viaño[10], Sect. 8).

Remark Of course, the asymptotic expansion proposed in (17) is formal and its validation must be provided by a convergence result $u(\varepsilon) \to u^{(0)}$ in some functional space, which is under study by the author. Actually, we give another justification of the method: the displacement field $u^{0\varepsilon}$, obtained by de-scaling and undoing the change of variable, solves a one-dimensional inequality that generalizes the classical one-dimensional obstacle problem used to model the flexion of an elastic beam on a rigid foundation (see section 3).

6 The limit model to the current beam

Having in mind that $u^{(0)}$ is a first order approximation of $u(\varepsilon)$ in Ω (see (17)), we propose a first order approximation, $u^{0\varepsilon}$, of u^ε in Ω^ε, obtained by undoing the change of variable (11) and the scalings (12)-(13)and (14):

$$u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^{-1} u_\alpha(\varepsilon)(x) \sim \varepsilon^{-1} u_\alpha^{(0)}(x) =: u_\alpha^{0\varepsilon}(x^\varepsilon), \tag{69}$$

$$u_3^\varepsilon(x^\varepsilon) = u_3(\varepsilon)(x) \sim u_3^{(0)}(x) =: u_3^{0\varepsilon}(x^\varepsilon). \tag{70}$$

From (40) we immediately deduce that $u_\alpha^{0\varepsilon}$ and $u_3^{0\varepsilon}$ are of the following form:

$$u_\alpha^{0\varepsilon}(x^\varepsilon) = \varepsilon^{-1} u_\alpha^{(0)}(x) = \varepsilon^{-1} \xi_\alpha(x_3) =: \xi_\alpha^\varepsilon(x_3), \tag{71}$$

$$u_3^{0\varepsilon}(x^\varepsilon) = u_3^{(0)}(x) = \xi_3(x_3) - x_\alpha \xi_\alpha'(x_3) = \xi_3^\varepsilon(x_3) - x_\alpha^\varepsilon (\xi_\alpha^\varepsilon)'(x_3), \tag{72}$$

where we put $\xi_3^\varepsilon = \xi_3$.
Using now problems (41) and (42) we obtain a complete characterization of the first order displacements $u^{0\varepsilon}$ by means a well-posed "one-dimensional" model.

Theorem 6.1 *The first order displacements field $u^{0\varepsilon}$ defined by (71)-(72) is a Bernoulli-Navier displacement, i.e.:*

$$u_\alpha^{0\varepsilon}(x^\varepsilon) = \xi_\alpha^\varepsilon(x_3), \quad \xi_\alpha^\varepsilon \in H_0^2(0,L), \tag{73}$$
$$u_3^{0\varepsilon}(x^\varepsilon) = \xi_3^\varepsilon(x_3) - x_\alpha^\varepsilon(\xi_\alpha^\varepsilon)'(x_3), \quad \xi_3^\varepsilon \in H_0^1(0,L), \tag{74}$$

where
(i) The flexions $(\xi_1^\varepsilon, \xi_2^\varepsilon)$ are the only solution of the following coupled variational inequality:

$$\left\{ \begin{array}{l} (\xi_\alpha^\varepsilon) \in K^\varepsilon(0,L), \\ EI_\alpha^\varepsilon \displaystyle\int_0^L (\xi_\alpha^\varepsilon)''(\chi_\alpha^\varepsilon - \xi_\alpha^\varepsilon)'' dx_3 \geq \int_0^L F_\alpha^\varepsilon(\chi_\alpha^\varepsilon - \xi_\alpha^\varepsilon) dx_3 \\ -\displaystyle\int_0^L M_\alpha^\varepsilon(\chi_\alpha^\varepsilon - \xi_\alpha^\varepsilon)' dx_3, \ \textit{for all } (\chi_\alpha^\varepsilon) \in K^\varepsilon(0,L), \end{array} \right. \tag{75}$$

where

$$K^\varepsilon(0,L) := \left\{ (\chi_\alpha^\varepsilon) \in [H_0^2(0,L)]^2 : \chi_\alpha^\varepsilon n_\alpha^\varepsilon \leq s^\varepsilon \ \textit{a.e. on } \Gamma_C^\varepsilon = \gamma_C^\varepsilon \times (0,L) \right\}. \tag{76}$$

(ii) The stretching ξ_3^ε is the only solution of the following problem:

$$\left\{ \begin{array}{l} \xi_3^\varepsilon \in H_0^1(0,L) \cap H^2(0,L), \\ EA(\omega^\varepsilon) \displaystyle\int_0^L (\xi_3^\varepsilon)'(\chi^\varepsilon)' dx_3 = \int_0^L F_3^\varepsilon \chi^\varepsilon dx_3, \ \textit{for all } \chi^\varepsilon \in H_0^1(0,L). \end{array} \right. \tag{77}$$

Proof It is a direct consequence of equations (41)-(42) and definitions (71)-(72) and (34).

7 Conclusions

Equation(77) is the classical model for the stretching of a clamped beam without any restrictions due to the contact. The problem (75) is the most important one in this analysis. It represents a general bending model for a rod which may become in contact with a rigid foundation. It is a new model coupling both transverse bending displacements. Although it is governed by two one-dimensional equations, the convex set $K^\varepsilon(0,L)$ of the admissible displacements still retains some three-dimensional information. We remark that we can explicit the definition of $K^\varepsilon(0,L)$ as it follows:

$$\begin{array}{c} K^\varepsilon(0,L) = \left\{ (\chi_\alpha^\varepsilon) \in [H_0^2(0,L)]^2 : \chi_\alpha^\varepsilon(x_3) n_\alpha^\varepsilon(x_1^\varepsilon, x_2^\varepsilon) \leq s^\varepsilon(x_1^\varepsilon, x_2^\varepsilon, x_3), \right. \\ \left. \text{for all } x_3 \in (0,L) \text{ and a.e. } (x_1^\varepsilon, x_2^\varepsilon) \in \gamma_C^\varepsilon. \right\} \end{array} \tag{78}$$

Most of the well-known classical models, as that one we have seen in section 3, may be obtained as particular cases of (75)-(76). We show only just the example as the candidate contact surface Γ_C^ε is *plane and normal to one of the inertia axes of the beam* (Ox_1, to fix the ideas). Consequently, the outward unit normal vector to Γ_C^ε is constant and it has one of the form $(+1, 0, 0)$ or $(-1, 0, 0)$. Let us assume $n = (-1, 0, 0)$.
From (78) one deduces that the convex set $K^\varepsilon(0, L)$ for this case is:

$$K^\varepsilon(0,L) = U_1^\varepsilon(0,L) \times H_0^2(0,L), \tag{79}$$

where

$$\begin{aligned} U_1^\varepsilon(0,L) = \{\varphi^\varepsilon \in H_0^2(0,L) : \varphi^\varepsilon(x_3) \geq s^\varepsilon(x_1^\varepsilon, x_2^\varepsilon, x_3), \\ \text{for all } x_3 \in (0,L) \text{ and a.e. } (x_1^\varepsilon, x_2^\varepsilon) \in \gamma_C^\varepsilon\} \end{aligned} \tag{80}$$

We assume that the beam and the obstacle are regular enough in such a way the following function $\hat{s}^\varepsilon : [0, L] \to \mathbb{R}$, is well defined and $\hat{s}^\varepsilon \in L^\infty(0, L)$:

$$\hat{s}^\varepsilon(x_3) = \inf_{(x_1^\varepsilon, x_2^\varepsilon) \in \gamma_C^\varepsilon} s^\varepsilon(x_1^\varepsilon, x_2^\varepsilon, x_3), \quad x_3 \in (0, L).$$

Then, we have an equivalent definition of $U^\varepsilon(0, L)$ (compare with (10)):

$$U_1^\varepsilon(0,L) = \left\{\varphi^\varepsilon \in H_0^2(0,L) : \varphi^\varepsilon \geq \hat{s}^\varepsilon \text{ a.e in } (0,L)\right\}. \tag{81}$$

Setting in (75) successively $(\chi_1^\varepsilon, \chi_2^\varepsilon) = (\chi_1^\varepsilon, \xi_2^\varepsilon)$ and $(\chi_1^\varepsilon, \chi_2^\varepsilon) = (\xi_1^\varepsilon, \chi_2^\varepsilon)$, with $\chi_1^\varepsilon \in U^\varepsilon(0, L)$ and $\chi_2^\varepsilon \in H_0^2(0, L)$, we prove that, in this case, the limit problem (75) is equivalent to the following two problems:

$$\left\{ \begin{array}{l} \xi_1^\varepsilon \in U_1^\varepsilon(0,L) \\ EI_1^\varepsilon \displaystyle\int_0^L (\xi_1^\varepsilon)''(\chi_1^\varepsilon - \xi_1^\varepsilon)'' dx_3 \geq \int_0^L F_1^\varepsilon(\chi_1^\varepsilon - \xi_1^\varepsilon) dx_3 \\ - \displaystyle\int_0^L M_1^\varepsilon(\chi_1^\varepsilon - \xi_1^\varepsilon)' dx_3, \text{ for all } \chi_1^\varepsilon \in U_1^\varepsilon(0,L), \end{array} \right. \tag{82}$$

$$\left\{ \begin{array}{l} \xi_2^\varepsilon \in H_0^2(0,L) \\ EI_2^\varepsilon \displaystyle\int_0^L (\xi_2^\varepsilon)''(\chi_2^\varepsilon)'' dx_3 = \int_0^L F_2^\varepsilon \chi_2^\varepsilon dx_3 \\ - \displaystyle\int_0^L M_2^\varepsilon(\chi_2^\varepsilon)' dx_3, \text{ for all } \chi_2^\varepsilon \in H_0^2(0,L). \end{array} \right. \tag{83}$$

We observe that (83) is the usual variational model for bending in the direction Ox_2 and (82) is the *classical one-dimensional obstacle problem* already advanced in (10). *In this way we have mathematically justified this classical model as the first order approximation of the three-dimensional Signorini problem for an elastic rod when the boundary of contact is assumed to be plane and normal to one inertia axis.*

Acknowledgments *This work is part of the European Project "New materials, adaptive systems and their nonlinearities. Modelling, control and numerical* *simulation" (Smart Systems, HPRN-CT-2002-00284) and the Spanish Project "Contacto de materiales viscoplásticos y viscoelásticos: formulación matemática y análisis numérico"(BFM2003-05357).*

References

[1] A. Bermúdez and J. M. Viaño, Une justification des équations de la thermo-élasticité de poutres a section variable par des méthodes asymptotiques, *RAIRO Anal. Numér.*, **18** (1984) 347–376.

[2] H. Brezis and G. Stampacchia, Sur la régularité de la solution d'inéquations elliptiques, *Bull. Soc. Math. France*, **96** (1968) 153–180.

[3] G. Cimatti, The constrained elastic beams, *Meccanica*, **8** (1973) 119–124.

[4] A. Cimetière, G. Geymonat, H. Le Dret, A. Raoult and Z. Tutek, Asymptotic theory and analysis for displacement and stress distributions in nonlinear elastic straight slender rods, *J. Elasticity*, **19** (1988) 111-161.

[5] G. Duvaut and J. L. Lions, *Les Inéquations en Mécanique et en Physique.* Dunod, Paris (1972).

[6] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems.* Springer-Verlag, New York (1984).

[7] H. Irago and J. M. Viaño, Error estimation in the Bernoulli–Navier model for elastic rods, *Asymptotic Anal.*, **21** (1999) 71–87.

[8] J.L. Lions, *Perturbations Singulières dans les Problèmes aux Limites et en Contrôle Optimale*, Lecture Notes in Math., Springer, Berlin (1973).

[9] J.C. Paumier, Le problème de Signorini dans la théorie des plaques minces de Kirchhoff-Love, *C. R. Acad. Sci. Paris*, **335** (2002) 567–570.

[10] L. Trabucho and J. M. Viaño, Mathematical modelling of rods. In: P. G. Ciarlet and J. L. Lions (eds.), *Handbook of Numerical Analysis*, Vol. IV. North-Holland, Amsterdam (1996) 487–973.

Received: 1 December, 2004

Departamento de Matemática Aplicada
Universidade de Santiago de Compostela
15782 Santiago de Compostela, Spain
E-mail: `maviano@usc.es`